Math Meth Oper Res (2001) 53:333–348

Mathematical Methods
of Operations Research

The Shapley value for games on matroids: The static model

J. M. Bilbao[1], T. S. H. Driessen[2], A. Jiménez Losada, and E. Lebrón

[1] Matemática Aplicada II, Escuela Superior de Ingenieros, Camino de los Descubrimientos, 41092 Sevilla, Spain (E-mail: mbilbao@cica.es, http://www.esi2.us.es/~mbilbao/)
[2] Faculty of Mathematical Sciences, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands (E-mail: T.S.H.Driessen@math.utwente.nl)

Manuscript received: July 2000/Final version received: September 2000

Abstract. In the classical model of cooperative games, it is considered that each coalition of players can form and cooperate to obtain its worth. However, we can think that in some situations this assumption is not real, that is, all the coalitions are not feasible. This suggests that it is necessary to rise the whole question of generalizing the concept of cooperative game, and therefore to introduce appropriate solution concepts. We propose a model for games on a matroid, based in several important properties of this combinatorial structure and we introduce the probabilistic Shapley value for games on matroids.

Key words: Cooperative game, matroid, Shapley value

1. Introduction

A *cooperative game* is a pair (N, v) of a finite set N of players and a characteristic function $v : 2^N \to \mathbb{R}$, such that $v(\emptyset) = 0$. A subset S of N is called a coalition. This paper is concerned with cooperative games in which the cooperation among players is partial. We will consider that there are two rules of cooperation between players:

- If a coalition may form, then every subset is also feasible, since if the players that take part in the formation of a coalition have common interests, then every subset of these players has at least the same common interests.
- Given two feasible coalitions with different number of players, there is a player of the largest that he can join with the smallest making a feasible coalition.

For this reason, we will define the feasible coalitions by using combinatorial geometries called *matroids*. The set systems called matroids were introduced by Whitney [10] as an abstraction of linear independence and the cyclic structure of graphs. The origin of the actual matroid theory is the work of Tutte [7] and it has numerous applications in combinatorics and optimization theory. We refer the reader to Welsh [9] and Korte, Lovász and Schrader [4] for a detailed treatment of matroids.

Let us outline the contents. Section 2 treats the essential notions on matroids, such as its properties, its rank function and its basic coalitions (being maximal feasible coalitions with respect to inclusion of sets). For the sake of the game theoretic approach, the rank function of a matroid is interpreted as a classical cooperative game and next, the game theoretic solution concept called core is defined as the set of optimal solutions of a certain linear programming problem in which the rank function of the matroid is involved. Edmonds [2] showed that the core coincides with the convex hull of the incidence vectors corresponding to basic coalitions of the matroid.

Section 3 introduces the concept of a cooperative game on a matroid as a real-valued function on the matroid itself. In other words, the characteristic function of this type of a cooperative game is defined only for feasible coalitions arising from the matroid. Similar, but different versions already do exist, see Faigle [3] and Nagamochi, Zeng, Kabutoya and Ibaraki [5]. The main part of Section 3 deals with the axiomatic development of the solution theory for games on matroids. As a matter of fact, we are concerned with the linearity axiom (in the variable being the characteristic function of the game), the monotonicity axiom (solutions should allocate nonnegative payoffs to players whenever the utility of coalitions increases in accordance with the inclusion of coalitions), as well as the dummy player axiom (non-important players receive their natural solutions). The solutions that satisfy these three axioms are characterized as the so-called quasi-probabilistic values. Such a solution for any individual player may be interpreted as some expected outcome based on the player's marginal contributions for joining the feasible coalitions of the induced contraction matroid. Another equivalence theorem states that an individual solution is a quasi-probabilistic value if and only if the solution is decomposable as the weighted sum of certain solutions for induced subgames defined on power sets associated with the basic coalitions of the matroid. The relevant weights are interpreted as a probabilty distribution over the set of basic coalitions of the matroid.

Section 4 introduces the solution concept called Shapley value for games on matroids, meant to be a generalization of the known Shapley value for classical cooperative games. The axiomatic approach taken here involves, besides the linearity and substitution axioms, a probabilistic version of the efficiency and dummy player property. In this framework, it is supposed that basic coalitions are formed randomly according to a fixed probability distribution over the set of basic coalitions of the matroid. As a result of this axiomatic approach, two explicit formulas for the probabilistic Shapley value of games on matroids are presented and discussed.

2. Essential notions on matroids

A *matroid* is a pair $(N, \mathscr{M})$ consisting of a finite set N and a set $\mathscr{M}$ of subsets of N with $\varnothing \in \mathscr{M}$ and satisfying the following two properties:

(M1) If $S \in \mathscr{M}$ and $T \subseteq S$, then $T \in \mathscr{M}$.
(M2) If $S, T \in \mathscr{M}$ with $|S| = |T| + 1$, then there exists $i \in S \backslash T$ such that $T \cup \{i\} \in \mathscr{M}$.

The *rank function* $r : 2^N \to \mathbb{Z}_+$ of a matroid $\mathscr{M}$ on N is defined by

$$r(X) := \max\{|S| : S \subseteq X, S \in \mathscr{M}\} \quad \text{for all } X \subseteq N. \tag{1}$$

Notice that $S \in \mathscr{M}$ if and only if $r(S) = |S|$. The following two theorems (see Korte et al. [4]) axiomatize matroids in terms of their rank functions.

Theorem 2.1. *A function $r : 2^N \to \mathbb{Z}_+$ is the rank function of a matroid on N if and only if, for all X, $Y \subseteq N$, the following holds:*

(R1 $0 \leq r(X) \leq |X|$.
(R2) $r(X) \leq r(Y)$ *whenever* $X \subseteq Y$.
(R3) $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.

Theorem 2.2. *A function $r : 2^N \to \mathbb{Z}_+$ is the rank function of a matroid on N if and only if, for all $X \subseteq N$ and all $i, j \in N \backslash X$, the following holds:*

(R1′) $r(\varnothing) = 0$.
(R2′) $r(X) \leq r(X \cup \{i\}) \leq r(X) + 1$.
(R3′) *If* $r(X \cup \{i\}) = r(X \cup \{j\}) = r(X)$, *then* $r(X \cup \{i, j\}) = r(X)$.

In the setting of the two theorems above, the function r determines uniquely the corresponding matroid through $\mathscr{M} = \{S \subseteq N : r(S) = |S|\}$. Elements of a given matroid are called *feasible sets* and further, a maximal feasible set (with respect to inclusion of sets) is called a *basic set*. Property (M1) implies that all the subsets of any basic set are feasible sets too and thus, $2^B \subseteq \mathscr{M}$ for every basic set B of the matroid $\mathscr{M}$. It is known that all the basic sets have the same cardinality and thus, $|B| = r(N)$ for every basic set B of the matroid $\mathscr{M} \subseteq 2^N$. Throughout this work we suppose that $\bigcup_{S \in \mathscr{M}}\{i : i \in S\} = N$.

In addition, the basic sets are of particular interest to determine the optimal solutions of a certain linear programming problem arising from the rank function of a matroid. Since this paper aims to develop the solution theory for cooperative games on matroids, we start to interpret the rank function $r : 2^N \to \mathbb{Z}_+$ of a matroid as a classical cooperative game (N, r) with player set N. The rank function indicates the maximal feasible cooperation level between the players of a coalition. In this context, the solution set of the relevant LP-problem agrees with the well-known game-theoretic concept called core. The *core* of the game (N, r) is defined to be

$$Core(N, r) := \{x \in \mathbb{R}^N : x(N) = r(N), x(S) \leq r(S) \text{ for all } S \subseteq N\},$$

where $x(S) := \sum_{i \in S} x_i$ and $x(\varnothing) = 0$. For every set $S \subseteq N$, we define the *incidence vector* $e^S \in \mathbb{R}^N$ such that $(e^S)_i := 1$ for all $i \in S$ and $(e^S)_i := 0$ otherwise. The following theorem has been showed by Edmonds [2] and provides one interpretation of the core of a cooperative game induced by the rank function of a matroid.

Theorem 2.3. *Let $r : 2^N \to \mathbb{Z}_+$ be the rank function of a matroid $\mathscr{M} \subseteq 2^N$ and $\mathscr{B}(\mathscr{M})$ the set of basic coalitions of $\mathscr{M}$. Then*

$$Core(N,r) = conv\{e^B : B \in \mathscr{B}(\mathscr{M})\}.$$

Proof. To prove that $e^B \in Core(N,r)$ for all $B \in \mathscr{B}(\mathscr{M})$, consider first that $\sum_{j \in N}(e^B)_j = |B| = r(N)$ for every $B \in \mathscr{B}(\mathscr{M})$. Moreover, $\sum_{j \in S}(e^B)_j = |B \cap S| \leq r(S)$ for all $S \subseteq N$. Since the core is convex, $\mathrm{conv}\{e^B : B \in \mathscr{B}(\mathscr{M})\} \subseteq Core(N,r)$. To prove the reverse inclusion, it suffices to show that the vertices of the core belong to the set $\{e^B : B \in \mathscr{B}(\mathscr{M})\}$. In view of Theorem 2.1 (R3), the rank game (N,r) is submodular and hence, by Driessen [1] (the greedy algorithm for LP-problems with a submodular objective-function), the vertices of the $Core(N,r)$ are determined by the marginal worth vectors, the components of which are composed of the marginal contributions $r(S \cup \{i\}) - r(S)$, $S \subseteq N \backslash \{i\}$, of player $i \in N$, in the rank game (N,r).

Together with Theorem 2.2 (R2′), this implies that any marginal worth vector $y = (y_i)_{i \in N}$ of the rank game (N,r) satisfies $y_i \in \{0,1\}$ for all $i \in N$. That is, $y = e^S$ for some $S \subseteq N$. From $y \in Core(N,r)$ we deduce $|S| = \sum_{j \in S} y_j \leq r(S)$, whereas, by construction, $r(S) \leq |S|$. Thus, $r(S) = |S|$, or equivalently, $S \in \mathscr{M}$. Moreover, from the efficiency of y we deduce $r(N) = \sum_{j \in N} y_j = |S|$. Finally, from $S \in \mathscr{M}$ and $|S| = r(N)$, we conclude that $S \in \mathscr{B}(\mathscr{M})$ and hence $y = e^B$ for some $B \in \mathscr{B}(\mathscr{M})$. □

Let $S \in \mathscr{M}$. The *contraction* $\mathscr{M}/S$ of S from $\mathscr{M}$ is the new matroid

$$\mathscr{M}/S := \{T \in \mathscr{M} : T \cap S = \varnothing \text{ and } T \cup S \in \mathscr{M}\}.$$

Then the contraction of a feasible coalition S is a matroid formed with the feasible coalitions of the initial matroid that do not include any members of S, whereas its union with S is still a feasible coalition in the initial matroid. In the particular case of a individual coalition $S = \{i\}$, $i \in N$, we use $\mathscr{M}/i$ instead of $\mathscr{M}/\{i\}$.

Example 2.1. Given the set $N = \{1, \ldots, n\}$ and any number k, $1 \leq k \leq n$, we define the uniform matroid $U_k^n := \{S \subseteq N : |S| \leq k\}$. Coalitions of cardinality k are the basic coalitions and its rank function $r : 2^N \to \mathbb{Z}_+$ is given by $r(X) = \min\{k, |X|\}$ for all $X \subseteq N$. With reference to the uniform matroid U_2^3, as depicted in figure 1, the core of the corresponding rank game is the convex hull of the vectors $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$.

Example 2.2. For any $i, j \in N$, with $i \neq j$, we define the opposition matroid $M_n(i\|j) := \{S \subseteq N : \{i,j\} \nsubseteq S\}$ being the largest matroid which excludes coalitions containing both players i and j. There are two basic coalitions $N\backslash\{i\}$ and $N\backslash\{j\}$. Its rank function $r : 2^N \to \mathbb{Z}_+$ is given by $r(X) = |X|$ if $X \in M_n(i\|j)$ and $r(X) = |X| - 1$ otherwise. Players 2 and 3 are called istmus players because they belong to every basic coalition of $M_4(1\|4)$.

Example 2.3. Given a graph $G = (V,E)$, where V is the vertex set and E is the edge set, the graphic matroid $M(G)$ consists of all subsets of E that contain no cycle of G. Maximal forests of G are the basic coalitions of $M(G)$ and its rank

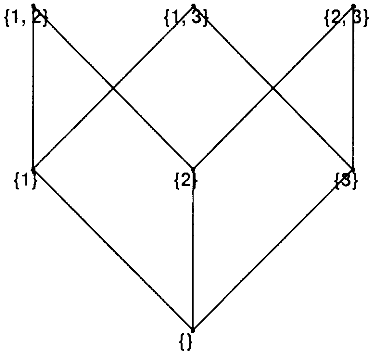

Fig. 1. The uniform matroid U_2^3

function is given by $r(X) = |V(X)| - k(X)$ for all $X \subseteq E$, where $V(X)$ is the vertex set of the spanning forest of X and $k(X)$ its number of connected components.

3. An axiomatic approach to quasi-probabilistic values

A *cooperative game on the matroid* $\mathscr{M}$ is defined to be a real-valued function $v : \mathscr{M} \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. In words, a cooperative game on a matroid represents an evaluation of the potential utility of any feasible coalition, whereas non-feasible coalitions are totally ignored because such coalitions are supposed not to be formed anyhow. For instance, in the context of the uniform matroid U_k^n, the n participants in a trip are only interested in measuring the utility of any group consisting of at most k persons because of the limited number of seats in various identical minibuses to be used for transportation during the trip.

Let $\Gamma(\mathscr{M})$ denote the set of all cooperative games on the matroid $\mathscr{M}$. Clearly, $\Gamma(\mathscr{M})$ is a vector space. An *individual value* for player i on $\Gamma(\mathscr{M})$ is defined to be a function $\psi_i : \Gamma(\mathscr{M}) \to \mathbb{R}$. For every game $v \in \Gamma(\mathscr{M})$, the value $\psi_i(v)$ represents an assessment by player i of his gains from participating in the game v. We consider the following three axioms for an individual value ψ_i on $\Gamma(\mathscr{M})$.

(1) *Linearity:* $\psi_i(\alpha v + \beta w) = \alpha\psi_i(v) + \beta\psi_i(w)$, for all $v, w \in \Gamma(\mathscr{M})$ and all α, $\beta \in \mathbb{R}$.
(2) λ_i*-dummy player property* ($\lambda_i \in [0, 1]$ is the rate of participation by player i): If player i is a *dummy* in the game $v \in \Gamma(\mathscr{M})$, that is $v(S \cup \{i\}) - v(S) = v(\{i\})$ for all $S \in \mathscr{M}/i$, then $\psi_i(v) = \lambda_i v(\{i\})$.
(3) *Monotonicity:* If $v \in \Gamma(\mathscr{M})$ is monotone ($v(S) \leq v(T)$ for all $S, T \in \mathscr{M}$ with $S \subseteq T$) then $\psi_i(v) \geq 0$.

Theorem 3.1. *Let $\mathscr{M} \subseteq 2^N$ be a matroid and $\psi_i : \Gamma(\mathscr{M}) \to \mathbb{R}$ an individual value for player $i \in N$. If $\lambda_i \in [0, 1]$ is the rate of participation by player i, then the following two statements are equivalent.*

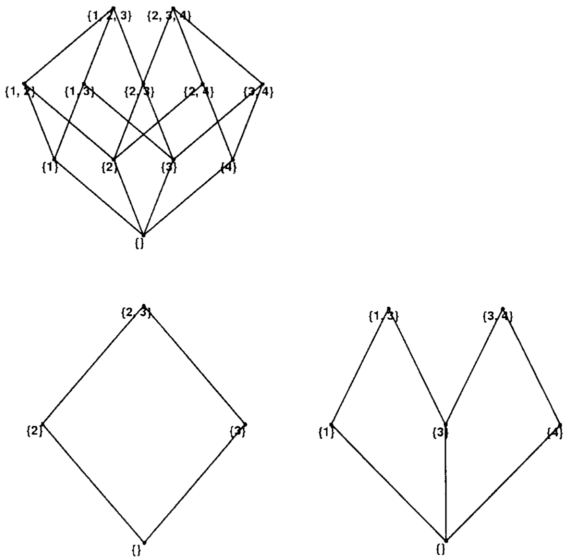

Fig. 2. The matroid $M_4(1\|4)$ and the contractions of 1 and 2

(i) *ψ_i satisfies the linearity, monotonicity, and λ_i-dummy player properties.*
(ii) *There exists a collection $\{p_S^i \geq 0 : S \in \mathcal{M}/i\}$ with $\sum_{S\in\mathcal{M}/i} p_S^i = \lambda_i$ such that, for all $v \in \Gamma(\mathcal{M})$,*

$$\psi_i(v) = \sum_{S\in\mathcal{M}/i} p_S^i[v(S \cup \{i\}) - v(S)]. \tag{2}$$

Proof. The implication $(ii) \Rightarrow (i)$ is obvious. To prove the converse implication, we first define, for every nonempty $T \in \mathcal{M}$, the *unanimity game* $u_T : \mathcal{M} \to \mathbb{R}$ and the *identity game* $\delta_T : \mathcal{M} \to \mathbb{R}$ as follows:

$$u_T(S) := \begin{cases} 1 & \text{if } S \supseteq T \\ 0 & \text{if } S \not\supseteq T \end{cases} \qquad \delta_T(S) := \begin{cases} 1 & \text{if } S = T \\ 0 & \text{if } S \neq T. \end{cases}$$

Note that $u_T = \sum_{\{S\in\mathcal{M}:S\supseteq T\}} \delta_S$ for all nonempty $T \in \mathcal{M}$. Suppose ψ_i satisfies the linearity, monotonicity, and λ_i-dummy player properties. Let $\Phi_i : \Gamma(\mathcal{M}) \to \mathbb{R}$ be defined, for all $v \in \Gamma(\mathcal{M})$, by

$$\Phi_i(v) := \sum_{S\in\mathcal{M}/i} \psi_i(\delta_{S\cup\{i\}})[v(S \cup \{i\}) - v(S)]. \tag{3}$$

We claim $\psi_i = \Phi_i$. Since both values are linear on $\Gamma(\mathcal{M})$ and the collection $\{u_T : T \in \mathcal{M}, T \neq \emptyset\}$ of unanimity games forms a basis of $\Gamma(\mathcal{M})$, it suffices to establish that $\psi_i(u_T) = \Phi_i(u_T)$ for all nonempty $T \in \mathcal{M}$.

First, let $T \in \mathcal{M}$, $T \neq \emptyset$, such that $i \notin T$. Note that $u_T(S \cup \{i\}) = u_T(S)$ for all $S \in \mathcal{M}/i$ and hence, by (3), $\Phi_i(u_T) = 0$ whenever $i \notin T$. Moreover, $u_T(\{i\}) = 0$, so $i \notin T$ is a dummy player in the unanimity game u_T and thus, by the λ_i-dummy player property for ψ_i, we obtain that $\psi_i(u_T) = \lambda_i u_T(\{i\}) = 0$ if $i \notin T$. So far, $\psi_i(u_T) = 0 = \Phi_i(u_T)$ if $i \notin T$.

Secondly, let $T \in \mathscr{M}$ such that $i \in T$. From (3) and the linearity property for ψ_i, we deduce that the following holds:

$$\Phi_i(u_T) = \sum_{S \in \mathscr{M}/i} \psi_i(\delta_{S\cup\{i\}}) u_T(S \cup \{i\}) = \sum_{\{S \in \mathscr{M}: S \supseteq T\}} \psi_i(\delta_S)$$

$$= \psi_i\left(\sum_{\{S \in \mathscr{M}: S \supseteq T\}} \delta_S\right) = \psi_i(u_T).$$

We conclude that $\psi_i(u_T) = \Phi_i(u_T)$ for all nonempty $T \in \mathscr{M}$. Hence, by linearity, $\psi_i = \Phi_i$. Next, we define $p_S^i := \psi_i(\delta_{S\cup\{i\}})$ for all $S \in \mathscr{M}/i$. Further, we have $u_{\{i\}}(S \cup \{i\}) - u_{\{i\}}(S) = 1 = u_{\{i\}}(\{i\})$ for all $S \in \mathscr{M}/i$. That is, player i is a dummy in the unanimity game $u_{\{i\}}$ and thus, by the λ_i-dummy player property for ψ_i and (3), we obtain that $\lambda_i = \lambda_i u_{\{i\}}(\{i\}) = \psi_i(u_{\{i\}}) = \sum_{S \in \mathscr{M}/i} p_S^i$.

It remains to establish that $p_S^i \geq 0$ for all $S \in \mathscr{M}/i$. For every $S \in \mathscr{M}$, $S \neq \emptyset$, we define the monotone game $\zeta_S : \mathscr{M} \to \mathbb{R}$ by $\zeta_S := u_S - \delta_S$, that is $\zeta_S(T) := 1$ if $T \supsetneq S$ and $\zeta_S(T) := 0$ otherwise. Let $S \in \mathscr{M}/i$. By the monotonicity of the game ζ_S, we obtain that $\zeta_S(T \cup \{i\}) - \zeta_S(T) \in \{0, 1\}$ for all $T \in \mathscr{M}/i$ and moreover, $\zeta_S(T \cup \{i\}) - \zeta_S(T) = 1$ if and only if $S = T$. Hence, by (3), $\psi_i(\zeta_S) = p_S^i$ and by the monotonicity property for ψ_i, we conclude that $p_S^i = \psi_i(\zeta_S) \geq 0$ for all $S \in \mathscr{M}/i$. □

In fact, the above theorem generalizes the characterization by Weber [8] of so-called *probabilistic values*, which are defined as a particular case of the new concept, as given by (2), applied to a free matroid.

Definition 3.1. *Let $\mathscr{M} \subseteq 2^N$ be a matroid. An individual value $\psi_i : \Gamma(\mathscr{M}) \to \mathbb{R}$ for player $i \in N$ is said to be a λ_i-quasi-probabilistic value if there exists a collection $\{p_S^i \geq 0 : S \in \mathscr{M}/i\}$ satisfying $\sum_{S \in \mathscr{M}/i} p_S^i = \lambda_i$ such that, for all $v \in \Gamma(\mathscr{M})$,*

$$\psi_i(v) = \sum_{S \in \mathscr{M}/i} p_S^i[v(S \cup \{i\}) - v(S)]. \tag{4}$$

The following equivalence theorem states that, given a matroid $\mathscr{M}$, an individual value can be classified as a quasi-probabilistic value on $\Gamma(\mathscr{M})$ if and only if it is decomposable as a weighted sum of certain probabilistic values on free matroids induced by the basic coalitions of the given matroid. For every $S \in \mathscr{M}$, let $\mathscr{B}_S(\mathscr{M}) := \{B \in \mathscr{B}(\mathscr{M}) : S \subseteq B\}$ represent the set of basic coalitions containing the feasible coalition S. Particularly, we write $\mathscr{B}_i(\mathscr{M})$ corresponding to individual coalitions $S = \{i\}$, $i \in N$.

Theorem 3.2. *Let $\mathscr{M} \subseteq 2^N$ be a matroid. Let $\psi_i : \Gamma(\mathscr{M}) \to \mathbb{R}$ be an individual value for player $i \in N$ and $\lambda_i \in [0, 1]$ the rate of participation by player i. Then the following two statements are equivalent.*

(i) *ψ_i is a λ_i-quasi-probabilistic value.*
(ii) *There exists a probability distribution P^i on the set $\mathscr{B}(\mathscr{M})$ such that*

$$\sum_{B \in \mathscr{B}_i(\mathscr{M})} P^i(B) = \lambda_i,$$

and for every $B \in \mathscr{B}_i(\mathscr{M})$, there exists an individual probabilistic value $\psi_i^B : \Gamma(2^B) \to \mathbb{R}$ such that, for all $v \in \Gamma(\mathscr{M})$,

$$\psi_i(v) = \sum_{B \in \mathscr{B}_i(\mathscr{M})} P^i(B) \psi_i^B(v_B), \tag{5}$$

where v_B is the restriction of the game v to 2^B.

Proof. Fix $i \in N$. Suppose P^i is a (yet unspecified) probability distribution on $\mathscr{B}(\mathscr{M})$ and suppose, for every $B \in \mathscr{B}_i(\mathscr{M})$, that $\psi_i^B : \Gamma(2^B) \to \mathbb{R}$ is an individual probabilistic value defined, for all $w \in \Gamma(2^B)$, by

$$\psi_i^B(w) = \sum_{\{T \subseteq B : i \notin T\}} p_{B,T}^i [w(T \cup \{i\}) - w(T)], \tag{6}$$

where $\{p_{B,T}^i \geq 0 : T \subseteq B, i \notin T\}$ is a set of (yet unspecified) numbers such that $\sum_{\{T \subseteq B : i \notin T\}} p_{B,T}^i = 1$. Then, we obtain, for all $v \in \Gamma(\mathscr{M})$, the following chain of equalities:

$$\begin{aligned}
&\sum_{B \in \mathscr{B}_i(\mathscr{M})} P^i(B) \psi_i^B(v_B) \\
&\quad = \sum_{B \in \mathscr{B}_i(\mathscr{M})} P^i(B) \left(\sum_{\{T \subseteq B : i \notin T\}} p_{B,T}^i [v(T \cup \{i\}) - v(T)] \right) \\
&\quad = \sum_{T \in \mathscr{M}/i} \left(\sum_{B \in \mathscr{B}_{T \cup \{i\}}(\mathscr{M})} P^i(B) p_{B,T}^i \right) [v(T \cup \{i\}) - v(T)] \\
&\quad = \sum_{T \in \mathscr{M}/i} p_T^i [v(T \cup \{i\}) - v(T)],
\end{aligned}$$

where the interrelationship between the coefficients is determined as follows:

$$p_T^i = \sum_{B \in \mathscr{B}_{T \cup \{i\}}(\mathscr{M})} P^i(B) p_{B,T}^i \tag{7}$$

for all $T \in \mathscr{M}/i$. The equality (7) yields the following result:

$$\begin{aligned}
\sum_{S \in \mathscr{M}/i} p_S^i &= \sum_{S \in \mathscr{M}/i} \left(\sum_{B \in \mathscr{B}_{S \cup \{i\}}(\mathscr{M})} P^i(B) p_{B,S}^i \right) \\
&= \sum_{B \in \mathscr{B}_i(\mathscr{M})} P^i(B) \left(\sum_{\{S \subseteq B : i \notin S\}} p_{B,S}^i \right) = \sum_{B \in \mathscr{B}_i(\mathscr{M})} P^i(B).
\end{aligned}$$

The implication $(ii) \Rightarrow (i)$ follows immediately from the definition of p_T^i by (7), provided that $P^i(B)$ and $p_{B,T}^i$ are given for all $B \in \mathscr{B}_{T\cup\{i\}}(\mathscr{M})$, $i \notin T$. In order to prove the converse implication, it suffices to solve the equation (7) for the variables $P^i(B)$ and $p_{B,T}^i$, provided that p_T^i are given for all $T \in \mathscr{M}/i$, and taking into account the two additional restrictions: $\sum_{\{T\subseteq B: i\notin T\}} p_{B,T}^i = 1$ and $\sum_{B\in\mathscr{B}(\mathscr{M})} P^i(B) = 1$. For that purpose, define $b_S := |\mathscr{B}_S(\mathscr{M})|$ for every $S \in \mathscr{M}$ and further,

$$P^i(B) := \sum_{\{S\subseteq B: i\notin S\}} \frac{p_S^i}{b_{S\cup\{i\}}} \quad \text{for all } B \in \mathscr{B}_i(\mathscr{M}),$$

$$p_{B,T}^i := \frac{p_T^i}{b_{T\cup\{i\}} P^i(B)} \quad \text{for all } B \in \mathscr{B}_{T\cup\{i\}}(\mathscr{M}), i \notin T.$$

By construction, the equation (7) is solved and $\sum_{\{T\subseteq B: i\notin T\}} p_{B,T}^i = 1$. Finally, it follows that the remaining restriction $\sum_{B\in\mathscr{B}(\mathscr{M})} P^i(B) = 1$ can be met trivially due to the following chain of (in)equalities:

$$\begin{aligned}\sum_{B\in\mathscr{B}_i(\mathscr{M})} P^i(B) &= \sum_{B\in\mathscr{B}_i(\mathscr{M})} \left(\sum_{\{S\subseteq B: i\notin S\}} \frac{p_S^i}{b_{S\cup\{i\}}} \right) \\ &= \sum_{S\in\mathscr{M}/i} p_S^i \left(\sum_{B\in\mathscr{B}_{S\cup\{i\}}(\mathscr{M})} \frac{1}{b_{S\cup\{i\}}} \right) \\ &= \sum_{S\in\mathscr{M}/i} p_S^i = \lambda_i \leq 1.\end{aligned}$$

This completes the proof. Notice that if $P^i(B) = 0$, for some $B \in \mathscr{B}_i(\mathscr{M})$, then we can use any probabilistic value on 2^B, as definition of Ψ_i^B. □

Definition 3.2. *Let $\mathscr{M} \subseteq 2^N$ be a matroid. A group value $\psi = (\psi_i)_{i\in N}$ on $\Gamma(\mathscr{M})$ is said to be a basic value if there exists a probability distribution P on $\mathscr{B}(\mathscr{M})$ such that every component ψ_i, $i \in N$, is of the following form:*

$$\psi_i(v) = \sum_{B\in\mathscr{B}_i(\mathscr{M})} P(B)\psi_i^B(v_B), \quad \textit{for all } v \in \Gamma(\mathscr{M}), \tag{8}$$

where $\psi_i^B : \Gamma(2^B) \to \mathbb{R}$ is an individual probabilistic value and $B \in \mathscr{B}_i(\mathscr{M})$.

Remark 3.1. Clearly, by Theorem 3.2, every component of a basic group value is a quasi-probabilistic value. The converse statement, however, is not true. That is, if each component of a group value is a quasi-probabilistic value, then the group value is not necessarily a basic value. For instance, consider the opposition matroid

$$M_3(1\|2) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}, \{2,3\}\}$$

and thus, $\mathscr{B}(\mathscr{M}) = \{\{1,3\},\{2,3\}\}$. In addition, consider any group value $\psi = (\psi_i)_{i \in N}$, the components of which are quasi-probabilistic values such that

$$\psi_1(v) = p^1_{\{3\}}[v(\{1,3\}) - v(\{3\})],$$

$$\psi_3(v) = p^3_{\{1\}}[v(\{1,3\}) - v(\{1\})],$$

for all $v \in \Gamma(\mathscr{M})$, where $0 \le p^1_{\{3\}} \le 1$ and $0 \le p^3_{\{1\}} \le 1$. We claim that, for every $0 < p^1_{\{3\}} < 1$ and $p^3_{\{1\}} = 1$, the quasi-probabilistic group value ψ can not be interpreted as a basic value of the form (8). For that purpose, suppose there exists a probability distribution P on $\mathscr{B}(\mathscr{M})$ such that (8) holds. In particular, for all $v \in \Gamma(\mathscr{M})$, the following holds:

$$\begin{aligned}\psi_1(v) &= P(\{1,3\})\psi_1^{\{1,3\}}(v_{\{1,3\}})\\ &= P(\{1,3\})[p^1_{\{1,3\},\{3\}}[v(\{1,3\}) - v(\{3\})] + p^1_{\{1,3\},\varnothing}v(\{1\})];\\ \psi_3(v) &= P(\{1,3\})\psi_3^{\{1,3\}}(v_{\{1,3\}}) + P(\{2,3\})\psi_3^{\{2,3\}}(v_{\{2,3\}})\\ &= P(\{1,3\})[p^3_{\{1,3\},\{1\}}[v(\{1,3\}) - v(\{1\})] + p^3_{\{1,3\},\varnothing}v(\{3\})]\\ &\quad + P(\{2,3\})[p^3_{\{2,3\},\{2\}}[v(\{2,3\}) - v(\{2\})] + p^3_{\{2,3\},\varnothing}v(\{3\})].\end{aligned}$$

From the two expressions for ψ_1, we deduce $p^1_{\{1,3\},\varnothing} = 0$. In fact, we apply (7) by looking at $0 = p^1_{\varnothing} = P(\{1,3\})p^1_{\{1,3\},\varnothing}$, where $P(\{1,3\}) \neq 0$ because of the assumption $p^1_{\{3\}} > 0$. Thus, $p^1_{\{1,3\},\{3\}} = 1 - p^1_{\{1,3\},\varnothing} = 1$ and consequently, (7) yields

$$p^1_{\{3\}} = P(\{1,3\})p^1_{\{1,3\},\{3\}} = P(\{1,3\}).$$

Finally, from the two expressions for ψ_3 (or once again (7)), we deduce that $1 = p^3_{\{1\}} = P(\{1,3\})p^3_{\{1,3\},\{1\}}$. The resulting equality, however, is in contradiction with $0 \le p^3_{\{1,3\},\{1\}} \le 1$ and $0 < P(\{1,3\}) < 1$. Therefore, the given quasi-probabilistic group value is not a basic value.

4. The Shapley value for games on matroids: the static model

Recall that the solution part of classical cooperative games on the free matroid $\mathscr{M} = 2^N$ is based on the assumption that the grand coalition N forms and thus, solution concepts aim to prescribe equitable divisions of the associated worth $v(N)$ among the players of any game (N, v). In the context of any non-trivial matroid $\mathscr{M} \neq 2^N$, the fullest measure of cooperation among players is supposed to take place within feasible coalitions that are as large as possible (with respect to inclusion of sets). Thus, for every basic coalition $B \in \mathscr{B}(\mathscr{M})$, the family 2^B of feasible coalitions is called a *cooperation area* for the members of B only. The basic coalitions of a matroid, however, are not necessarily disjoint and so, two basic coalitions with at least one mutual member can not be formed at the same time. This section is devoted to a model based on a probabilistic approach to the formation of basic coalitions. In other words, this *static model* involves probability distributions over the various cooperation areas (basic coalitions) and according to this random process, each feasible coalition interacts within its relevant cooperation areas with certain probabilities. Since it is supposed that a basic coalition forms randomly, we will deal with an arbitrary probability distribution over the set of basic coalitions denoted by

$$\mathscr{P}(\mathscr{M}) = \left\{ P \in (\mathbb{R}_+)^{\mathscr{B}(\mathscr{M})} : \sum_{B \in \mathscr{B}(\mathscr{M})} P(B) = 1 \right\}.$$

Definition 4.1. *Let $\mathscr{M} \subseteq 2^N$ be a matroid and let $P \in \mathscr{P}(\mathscr{M})$ a probability distribution over $\mathscr{B}(\mathscr{M})$. For every $S \in \mathscr{M}$, the probabilistic participation influence $w^P(S)$ of S within the cooperation areas of $\mathscr{M}$ (with respect to P) is given by the sum of the probabilities of basic coalitions containing S,*

$$w^P(S) := \sum_{B \in \mathscr{B}_S(\mathscr{M})} P(B) \quad \text{for all } S \in \mathscr{M}. \tag{9}$$

Particularly, for all $P \in \mathscr{P}(\mathscr{M})$, we call $w^P := (w^P(\{i\}))_{i \in N} \in \mathbb{R}^N$ the *probabilistic participation influence vector* (with respect to P), the components of which are the probabilistic participation influences of individuals within the cooperation areas of the matroid. The next result asserts that the set consisting of all probabilistic participation influence vectors coincides with the core of the rank game (induced by the rank function of the matroid $\mathscr{M}$). In other words, every core-allocation of the rank game represents in a natural and unique manner the (probabilistic) rate of participation by individuals within the cooperation areas of the matroid. Obviously, the probabilistic rate of participation by any *istmus player* (who belongs to every basic coalition) equals one.

Proposition 4.1. *Let $\mathscr{M} \subseteq 2^N$ be a matroid and $r : 2^N \to \mathbb{Z}_+$ its rank function. Then $Core(N, r) = \{w^P : P \in \mathscr{P}(\mathscr{M})\}$.*

Proof. By Theorem 2.3, every $x \in Core(N, r)$ can be written as a convex combination of the incidence vectors e^B, $B \in \mathscr{B}(\mathscr{M})$. Then, there exist nonnegative numbers $P(B)$, $B \in \mathscr{B}(\mathscr{M})$, with $\sum_{B \in \mathscr{B}(\mathscr{M})} P(B) = 1$ such that $x = \sum_{B \in \mathscr{B}(\mathscr{M})} P(B) e^B = w^P$ or equivalently, $x_i = \sum_{B \in \mathscr{B}(\mathscr{M})} P(B) = w^P(\{i\})$, for all $i \in N$. □

Now we are in a position to state our main theorem concerning the extension of the well-known *Shapley value* for classical games to games on matroids. In an axiomatic way we introduce a basic value on $\Gamma(\mathscr{M})$ constructed by the classical Shapley values on $\Gamma(2^B)$, for all $B \in \mathscr{B}(\mathscr{M})$, and a probability distribution $P \in \mathscr{P}(\mathscr{M})$:

$$Sh_i^P(v) = \sum_{B \in \mathscr{B}_i(\mathscr{M})} P(B) Sh_i^B(v_B), \quad \text{for all } v \in \Gamma(\mathscr{M}) \text{ and all } i \in N,$$

where $Sh^B = (Sh_i^B)_{i \in B}$ is the classical Shapley value on $\Gamma(2^B)$, $B \in \mathscr{B}(\mathscr{M})$. The axiomatic approach to the extended Shapley value involves four axioms, of which the linearity is formulated in a classical manner, the substitution axiom is applied to some substitutes in unanimity games, whereas the dummy player and efficiency axioms are formulated with reference to a given probability distribution over the cooperation areas of the considered matroid.

Theorem 4.2. *Let $\mathscr{M} \subseteq 2^N$ a matroid and let $P \in \mathscr{P}(\mathscr{M})$ a probability distribution over $\mathscr{B}(\mathscr{M})$. There exists a unique group value $\psi = (\psi_i)_{i \in N}$ on $\Gamma(\mathscr{M})$ that satisfies the following four axioms:*

(1) *Linearity: For every $i \in N$, $\psi_i(\alpha v + \beta w) = \alpha\psi_i(v) + \beta\psi_i(w)$ for all $v, w \in \Gamma(\mathscr{M})$ and $\alpha, \beta \in \mathbb{R}$.*
(2) *Substitution apply to unanimity games: For each $T \in \mathscr{M}$ we have $\psi_i(u_T) = \psi_j(u_T)$ for every pair $i, j \in T$.*
(3) *P-dummy player property: $\psi_i(v) = w^P(\{i\})v(\{i\})$ for every dummy player i in the game $v \in \Gamma(\mathscr{M})$.*
(4) *Probabilistic efficiency: $\sum_{i \in N} \psi_i(v) = \sum_{B \in \mathscr{B}(\mathscr{M})} P(B)v(B)$, for all $v \in \Gamma(\mathscr{M})$.*

This unique group value $Sh^P = (Sh_i^P)_{i \in N}$ is called the probabilistic Shapley value on $\Gamma(\mathscr{M})$ and we present two explicit formulas for it:

$$Sh_i^P(v) = \sum_{T \in \mathscr{M}/i} \frac{w^P(T \cup \{i\})}{(r(N) - |T|)\binom{r(N)}{|T|}} [v(T \cup \{i\}) - v(T)] \tag{10}$$

$$Sh_i^P(v) = \sum_{B \in \mathscr{B}_i(\mathscr{M})} P(B) Sh_i^B(v_B) \tag{11}$$

for all $v \in \Gamma(\mathscr{M})$ and all $i \in N$, where $Sh^B = (Sh_i^B)_{i \in B}$ represents the classical Shapley value on $\Gamma(2^B)$, $B \in \mathscr{B}(\mathscr{M})$.

Proof. In order to prove the uniqueness part, suppose a group value ψ satisfies the linearity, substitution applied to unanimity games, probabilistic efficiency, and P-dummy player properties. Consider, for every $T \in \mathscr{M}$, $T \neq \emptyset$, the unanimity game $u_T : \mathscr{M} \to \mathbb{R}$ as defined at the beginning of the proof of Theorem 3.1. Recall that every $i \notin T$ is a dummy player in the game u_T and thus, by the P-dummy player property for ψ_i, it holds that $\psi_i(u_T) = w^P(\{i\})u_T(\{i\}) = 0$ whenever $i \notin T$. Further, by the substitution property for ψ, $\psi_i(u_T) = \psi_j(u_T)$ for every pair $i,j \in T$. Thirdly, by the probabilistic efficiency property for ψ, it holds $\sum_{i \in N} \psi_i(u_T) = \sum_{B \in \mathscr{B}(\mathscr{M})} P(B)u_T(B) = \sum_{B \in \mathscr{B}_T(\mathscr{M})} P(B) = w^P(T)$. So far, we obtain that the group value ψ for every unanimity game u_T, $T \in \mathscr{M}$, $T \neq \emptyset$, is uniquely determined by:

$$\psi_i(u_T) = \begin{cases} \dfrac{w^P(T)}{|T|}, & \text{if } i \in T \\ 0, & \text{if } i \notin T. \end{cases} \tag{12}$$

Because the set $\{u_T : T \in \mathscr{M}, T \neq \emptyset\}$ of unanimity games forms a basis of $\Gamma(\mathscr{M})$ and ψ is supposed to be linear, we conclude that the group value ψ is uniquely determined on $\Gamma(\mathscr{M})$ by the four axioms involved.

In order to prove the existence part, we first show that formula (10) agrees with the alternative formula (11). Recall that, for every $B \in \mathscr{B}(\mathscr{M})$, the classical Shapley value $Sh^B = (Sh_i^B)_{i \in B}$ on $\Gamma(2^B)$ is given as follows:

$$Sh_i^B(v') = \sum_{\{T \subseteq B : i \notin T\}} \frac{1}{(|B| - |T|)\binom{|B|}{|T|}} [v'(T \cup \{i\}) - v'(T)],$$

for all $v' \in \Gamma(2^B)$ and all $i \in B$.

By the above equation and $|B| = r(N)$ for every $B \in \mathscr{B}(\mathscr{M})$, we arrive, for every $v \in \Gamma(\mathscr{M})$ and all $i \in N$, at the following chain of equalities:

$$\begin{aligned}
&\sum_{B \in \mathscr{B}_i(\mathscr{M})} P(B) Sh_i^B(v_B) \\
&= \sum_{B \in \mathscr{B}_i(\mathscr{M})} P(B) \sum_{\{T \subseteq B : i \notin T\}} \frac{1}{(|B| - |T|)\binom{|B|}{|T|}} [v(T \cup \{i\}) - v(T)] \\
&= \sum_{T \in \mathscr{M}/i} \left(\sum_{B \in \mathscr{B}_{T \cup \{i\}}(\mathscr{M})} P(B) \right) \frac{[v(T \cup \{i\}) - v(T)]}{(|B| - |T|)\binom{|B|}{|T|}} \\
&= \sum_{T \in \mathscr{M}/i} \frac{w^P(T \cup \{i\})}{(r(N) - |T|)\binom{r(N)}{|T|}} [v(T \cup \{i\}) - v(T)].
\end{aligned}$$

Thus, (10) is fully equivalent with (11). Clearly, by (10), the probabilistic Shapley value Sh^P is linear on $\Gamma(\mathscr{M})$. By (12) the relevant substitution property holds because the probabilistic Shapley value for a unanimity game, u_S, is the equitable allocation of the probabilistic participation influence of S between the players of S.

With the aid of (11) and the classical efficiency of the classical Shapley value on free matroids, we are able to establish the probabilistic efficiency property for the probabilistic Shapley value Sh^P as follows:

$$\begin{aligned}
\sum_{i \in N} Sh_i^P(v) &= \sum_{i \in N} \left(\sum_{B \in \mathscr{B}_i(\mathscr{M})} P(B) Sh_i^P(v_B) \right) \\
&= \sum_{B \in \mathscr{B}(\mathscr{M})} P(B) \left(\sum_{i \in B} Sh_i^B(v_B) \right) = \sum_{B \in \mathscr{B}(\mathscr{M})} P(B) v(B),
\end{aligned}$$

for all $v \in \Gamma(\mathscr{M})$. For the sake of the P-dummy player property, suppose player i is a dummy in $v \in \Gamma(\mathscr{M})$. That is, $v(S \cup \{i\}) - v(S) = v(\{i\})$ for all $S \in \mathscr{M}/i$. In particular, for every $B \in \mathscr{B}_i(\mathscr{M})$, we have $v(S \cup \{i\}) - v(S) =$

$v(\{i\})$ for all $S \subseteq B$ with $i \notin S$. In words, player i is a dummy in any subgame v_B associated with any basic coalition containing player i. Since the classical Shapley value on free matroids satisfies the classical dummy player property, we arrive at the P-dummy player property for the probabilistic Shapley value Sh_i^P as follows:

$$Sh_i^P(v) = \sum_{B \in \mathscr{B}_i(\mathscr{M})} P(B) Sh_i^B(v_B) = \sum_{B \in \mathscr{B}_i(\mathscr{M})} P(B) v(\{i\}) = w^P(\{i\}) v(\{i\}).$$

An alternative proof of the P-dummy player property for the probabilistic Shapley value Sh_i^P proceeds as follows. From (10), we derive that, for every dummy player i in a game $v \in \Gamma(\mathscr{M})$, the following holds:

$$\begin{aligned} Sh_i^P(v) &= \sum_{S \in \mathscr{M}/i} \frac{w^P(S \cup \{i\})}{(r(N) - |S|)\binom{r(N)}{|S|}} [v(S \cup \{i\}) - v(S)] \\ &= v(\{i\}) \sum_{S \in \mathscr{M}/i} \frac{w^P(S \cup \{i\})}{(r(N) - |S|)\binom{r(N)}{|S|}} = v(\{i\}) w^P(\{i\}). \end{aligned}$$

In order to prove the last equality, we claim that for all $S \in \mathscr{M}$ and all $k \geq |S|$,

$$\sum_{\{T \in \mathscr{M} : T \supseteq S, |T| = k\}} w^P(T) = \binom{r(N) - |S|}{k - |S|} w^P(S). \tag{13}$$

Indeed, in view of (9), the proof of (13) proceeds as follows,

$$\begin{aligned} \sum_{\{T \in \mathscr{M} : T \supseteq S, |T| = k\}} w^P(T) &= \sum_{\{T \in \mathscr{M} : T \supseteq S, |T| = k\}} \sum_{B \in \mathscr{B}_T(\mathscr{M})} P(B) \\ &= \sum_{B \in \mathscr{B}_S(\mathscr{M})} P(B) \sum_{\{T \in \mathscr{M} : S \subseteq T \subseteq B, |T| = k\}} 1 \\ &= \binom{r(N) - |S|}{k - |S|} w^P(S). \end{aligned}$$

From (13) we deduce

$$\begin{aligned} \sum_{S \in \mathscr{M}/i} \frac{w^P(S \cup \{i\})}{(r(N) - |S|)\binom{r(N)}{|S|}} &= \sum_{k=1}^{r(N)} \frac{\sum_{\{T \in \mathscr{M} : |T| = k, i \in T\}} w^P(T)}{(r(N) - k + 1)\binom{r(N)}{k-1}} \\ &= \sum_{k=1}^{r(N)} \frac{[\binom{r(N)-1}{k-1} w^P(\{i\})]}{(r(N) - k + 1)\binom{r(N)}{k-1}} \\ &= w^P(\{i\}) \sum_{k=1}^{r(N)} \frac{1}{r(N)} = w^P(\{i\}). \end{aligned}$$

This completes the alternative proof of the P-dummy player property for the probabilistic Shapley value Sh_i^P as given by (10). □

Remark 4.1. By (10), the probabilistic Shapley value for games on an arbitrary matroid represents some kind of an expected value, in that the expected payoff to every player is composed of the player's marginal contributions in the game with respect to feasible coalitions contained in the contraction of the matroid to the player. We take into account the probabilistic participation influences of these feasible coalitions enlarged with the player as well as a (classical) probability distribution arising from the belief that the feasible coalition, to which the player joins, is equally likely with probability $r(N)^{-1}$ to be of any cardinality t, $0 \le t \le r(N) - 1$, and that all such coalitions of the same cardinality t are equally likely with probability $\binom{r(N)-1}{t}^{-1}$. To be exact, we claim

$$\sum_{T \in \mathscr{M}/i} \frac{w^P(T \cup \{i\})}{(r(N) - |T|)\binom{r(N)}{|T|}} = w^P(\{i\}) \quad \text{for every } i \in N,$$

(see the last stage of the proof of Theorem 4.2).

Remark 4.2. By the decomposition formula (11), the probabilistic Shapley value for games on a matroid is fully determined by classical Shapley values on free matroids induced by the basic coalitions of the given matroid, taking into account the probabilities of these basic coalitions. The impact of the decomposition formula (11) was already illustrated in the proof of Theorem 4.2 in that known properties (like efficiency and dummy player property) for the classical Shapley value are basic tools to prove similar properties for the extension of the Shapley value to games on an arbitrary matroid. For the ongoing research on the Shapley value for games on matroids, this decomposition formula (11) may be an extremely helpful tool to get knowledge of particular properties.

Remark 4.3. The probabilistic Shapley value for games on matroids as given by (10) does not satisfy the classical substitution property. For instance, consider again the matroid $M_3(1\|2)$ (see Remark 3.1). Given the game $v(\{1\}) = v(\{3\}) = v(\{2\}) = 1, v(\{1,3\}) = v(\{2,3\}) = 2$, the players 1 and 3 are substitutes in the classical sense since $v(S\backslash\{1\}) = v(S\backslash\{3\})$ for all $S \in M_3(1\|2)$ with $\{1,3\} \subseteq S$. By choosing the probabilities $P(\{1,3\}) = P(\{2,3\}) = \frac{1}{2}$, formula (10) yields $Sh_1^P(v) = \frac{1}{2}$, whereas $Sh_3^P(v) = 1$, although the players 1 and 3 are substitutes in the classical sense.

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