Graphs, topologies and simple games*

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Abstract

We study the existence of connected coalitions in a simple game restricted by a partial order. First, we define a topology compatible with the partial order in the set of players. Second, we prove some properties of the covering and comparability graphs of a finite poset. Finally, we analize the core and obtain sufficient conditions for the existence of winning coalitions such that contains dominant players in simple games restricted by the connected subspaces of a finite topological space.

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1. Introduction

A simple game is a cooperative game in which every coalition is either winning or losing, with nothing in between. These games covers direct majority rule, weighted voting, bicameral or multicameral legislatures, committees and veto situations. For simple games it is generally assumed that there are no restrictions on cooperation and hence, every subset of players is a feasible coalition. However, in many social and economic situations, this model does not work. Axelrod (1970) defines a linear order relation, *policy order*, in the set of players and introduces the axiom of formation of connected coalitions, which are really convexes with respect to the order. Faigle and Kern (1992) proposed a model in which cooperation among players is restricted to some family of subsets of players, the *feasible* coalitions of the game. Their idea is to restrict the allowable coalitions by using underlying partially ordered sets. The purpose of this paper is to study the existence of winning and connected coalitions in situations where the preferences for communication among the players are modeled by a partial order. Furthermore, we study in a finite topological space, the *domination* situations given by Peleg (1981) and Einy (1985).

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2. Topology and Order

Alexandroff (1937), has studied spaces endowed with the finest topology compatible with an order. In a poset $(P, \leq)$, the *topology of Alexandroff* $A(P, \leq)$ is the set of all upper sets of P. That is, $U \subseteq P$ is open if and only if $U = \uparrow U$, where $\uparrow U := \{y \in P : \exists x \in U, \ x \leq y\}$. Then, $A(P, \leq)$ is the finest topology where the sets $\downarrow(x) := \{y \in P : y \leq x\}$, are closed.

Moreover, there exists the lowest topology such that the down sets $\downarrow(x)$ are closed, and it is the upper interval topology $\Phi(P, \leq)$ (see Johnstone (1982)).

The *specialization ordering* on a topological space X is defined by $x \leq y$ if and only if $x \in \overline{\{y\}}$, i.e., $\overline{\{x\}} \subseteq \overline{\{y\}}$. This relation is a partial order if and only if the space X satisfies the axiom T_0, that is, $\overline{\{x\}} = \overline{\{y\}}$ implies $x = y$.

Definition 2.1. *A topology in the poset $(P, \leq)$ is compatible with the order if the specialization ordering induced by the topology coincides with the partial order of the poset.*

A topology Ω in $(P, \leq)$ is compatible if and only if $\Phi(P, \leq) \subseteq \Omega \subseteq A(P, \leq)$. If the poset $(P, \leq)$ is finite, then $A(P, \leq) = \Phi(P, \leq)$ and is the unique T_0 topology compatible with the order (see Johnstone, p. 248). In what follows, we assume that every finite poset is endowed with this T_0 topology and we denote this topological space by *FTS*.

A subspace S of a topological space is *connected* if there do not exist a partition of S into two disjoint nonempty open sets in S.

Let $(P \leq)$ be a poset and let $x, y \in P$, with $x \leq y$. We consider the interval $[x, y] := \{z \in P : x \leq z \leq y\}$. The *cover relation* is defined by: $y \succ x$ if and only if the interval $[x, y] = \{x, y\}$.

For a poset $(P, \leq)$ we denote by $C(P)$ its *covering graph*, that is, the graph whose vertices are the elements of P and whose edges are those pairs $\{x, y\}$ for which $x \succ y$ or $y \succ x$. Then, the covering graph is the undirected Hasse diagram of $(P, \leq)$.

The *comparability graph* of the poset $(P, \leq)$ is the graph $G = (P, E)$ with $\{x, y\}$ in E whenever $x < y$ or $y < x$. Note that the transitive closure of the covering graph of P is its comparability graph. We consider the following subsets:

$$\Gamma^+(x) = \{y \in P : y \succ x\}, \ \ \Gamma^-(x) = \{y \in P : y \prec x\}, \ \ \Gamma(x) = \Gamma^+(x) \cup \Gamma^-(x).$$

Proposition 2.1. *Let $(P, \leq)$ be an FTS. Then:*

1. *$A \subseteq P$ is open if and only if $\Gamma^+(x) \subseteq A$, for all $x \in A$.*
2. *$B \subseteq P$ is closed if and only if $\Gamma^-(x) \subseteq B$, for all $x \in B$.*
3. *$C \subseteq P$ is closed and open (clopen) if and only if $\Gamma(x) \subseteq C$, for all $x \in C$.*

Proof. (1) Let $x \in A$ be any element. Since A is open we have $\uparrow(A) = A$ hence $\uparrow(x) \subseteq A$. Then $\Gamma^+(x) \subseteq \uparrow(x) \subseteq A$.

Conversely, we only need to show that $\uparrow A \subseteq A$. If $y \in \uparrow A$ then there exists $x \in A$ with $x \leq y$. Assume that $x < y$, we can obtain a path from x to y,

$$x \prec z_1 \prec \cdots \prec z_p \prec y.$$

Thus $z_1 \in \Gamma^+(x)$ and so $z_1 \in A$ and by induction $y \in \Gamma^+(z_p) \subseteq A$.

The proofs of properties (2) and (3) are similar. □

Some notable elements in a poset can be characterized using their topological properties.

- $x \in P$ is maximal $\Leftrightarrow \Gamma^+(x) = \emptyset \Leftrightarrow \{x\}$ is open.
- $x \in P$ is minimal $\Leftrightarrow \Gamma^-(x) = \emptyset \Leftrightarrow \{x\}$ is closed.
- $x \in P$ is maximal and minimal $\Leftrightarrow \Gamma(x) = \emptyset \Leftrightarrow \{x\}$ is clopen.

A subset C of a poset is a *chain* if $\{x, y\} \subseteq C$, $x \neq y$ imply $x < y$ or $y < x$. Then, the subset $C \subseteq P$ is a chain if and only if for every pair $x, y \in C$, the subspace $\{x, y\}$ is connected.

A subset A of a poset is an *antichain* if $\{x, y\} \subseteq A$, $x \leq y$ if and only if $x = y$. Then, the subset $A \subseteq P$ is an antichain if and only if the only connected subspaces are the sets $\{x\}$, $x \in A$.

The following theorem summarizes the properties of the connected subspaces in a finite topological space. This result was showed for the comparability graph of a poset by Khalimsky et al. (1990) and by Préa (1992).

Theorem 2.2. *Let $(P, \leq)$ be an FTS. Then:*

1. *P is a connected topological space if and only if the covering graph $C(P)$ is connected.*
2. *S is a connected subspace of P if and only if the covering graph $C(S)$ of the induced subposet S is connected.*
3. *The components of the finite topological space P coincides with the components of the covering graph.*

Proof. (1) Given $x \in P$, we consider the set

$$C(x) := \{y \in P \,:\, \text{there is a path } x - y\}.$$

Let us show that $A = C(x) \cup \{x\}$ is a clopen set. First, by the definition

$$\Gamma(x) \subseteq C(x) \subseteq A.$$

Next, given $y \in A$ with $x \neq y$, there is a path from x to y. Moreover, there is a path from x to any of the elements of $\Gamma(y)$ obtained by adding the vertex of $\Gamma(y)$ or deleting y. Finally,

$$\Gamma(y) \subseteq C(x) \subseteq A.$$

Therefore $A \neq \emptyset$ is clopen and P is a connected topological space, and so $A = P$.

Conversely, suppose that P is not connected and $|P| > 1$. Then $P = P_1 \cup P_2$, where P_1 and P_2 are nonempty clopen disjoint sets. Take $x_1 \in P_1$ and $x_2 \in P_2$, since the covering graph is connected, there is a path from x_1 to x_2. This path must contain two adjacent vertices

$$y_1 \in P_1,\, y_2 \in P_2, \text{ such that } y_1 \prec y_2 \text{ or } y_2 \prec y_1.$$

Then, we have

$$y_1 \in \overline{\{y_2\}} \subseteq \overline{P_2} = P_2 \text{ or } y_2 \in \overline{\{y_1\}} \subseteq \overline{P_1} = P_1.$$

It follows that $y_1 \in P_1 \cap P_2$ or $y_2 \in P_1 \cap P_2$, which is a contradiction.

Equivalences (2) and (3) follows from (1). □

A graph is a *rooted tree* (see Aigner, (1988)) if it is connected, there is a vertex x_r such that $\Gamma^-(x_r) = \emptyset$ and for each vertex $x \neq x_r$ we have $|\Gamma^-(x)| = 1$.

A bijective map $f : (P, \leq) \to (P', \leq')$ between finite topological spaces is a homeomorphism if and only if for all $x, y \in P$, $x \leq y \Leftrightarrow f(x) \leq' f(y)$ (see Johnstone (1982)).

A topological space X is *strongly connected* if every nonempty closed subset is connected (see Hoffmann (1981)). If X is a T_0-space, then this definition is equivalent to the specialization ordering is down-directed. Therefore, $(P, \leq)$ is a strongly connected FTS if and only if $P = \uparrow(x_r)$, where x_r is the infimum of P.

Proposition 2.3. *Let $(P, \leq)$ be an FTS. The following assertions are equivalent:*

1. *The covering graph of P is a rooted tree.*
2. *$(P, \leq)$ is strongly connected and for every $x \in P$, the closed set $\overline{\{x\}}$ is homeomorphic to a chain.*

Proof. (1) $\Rightarrow$ (2) If P is a rooted tree, there is a vertex x_r such that $x \neq x_r$ implies $x_r < x$, and so $P = \uparrow(x_r)$. Moreover, the closure of any element $x \in P$ is the unique path from x to x_r and it is homeomorphic to a chain.

(2) $\Rightarrow$ (1) If $P = \uparrow(x_r)$, then x_r is the infimum of P, so $\Gamma^-(x_r) = \emptyset$. It suffices to show $|\Gamma^-(x)| = 1$ if $x \neq x_r$. We assume that $|\Gamma^-(x)| > 1$. Then, there are two distinct elements $\{y, z\} \subseteq \Gamma^-(x)$, hence the pair

$$\{y, z\} \subseteq \downarrow(x) = \overline{\{x\}}.$$

But $\overline{\{x\}}$ is homeomorphic to a chain, thus $y \leq z$ or $z \leq y$. Both cases lead to $y = z$, so $|\Gamma^{-}(x)| = 1$. □

To characterize the FTS whose covering graph is a tree, we introduce the following concepts due to Khalimsky et al. (1990).

Definition 2.2. *A finite connected ordered topological space (COTS) is a poset, with at least three points, whose specialization ordering is a zigzag, endowed with the compatible topology.*

Figure 2.1: Covering graph of a finite COTS

Definition 2.3. *A digital arc in a topological space is the range of a homeomorphism from a finite COTS. A topological space X is digitally arc-connected if for every x, $y \in X$, there is a digital arc from x to y.*

Proposition 2.4. *If $(P, \leq)$ is an FTS with $|P| \geq 3$, then the following are equivalent:*

1. *The covering graph (P, E) is a tree.*
2. *$(P, \leq)$ is connected, and given $\{x, y\} \subseteq P$, there is a unique digital arc C with endpoints x and y which contains the supremum and the infimum (if they exist) of their subsets of points.*

Proof. (1) $\Rightarrow$ (2) Since the covering graph $C(P)$ is connected, it is also digitally arc-connected (see Khalimsky et al., Theorem 3.2). Then, there is a digital arc C from x to y, for any $\{x, y\} \subseteq P$. The definition of finite COTS implies that C contains the supremum and the infimum when they exist. Finally, the covering graph is acyclic so C is unique.

(2) $\Rightarrow$ (1) If the FTS $(P, \leq)$ is connected, then its covering graph is connected. Assume that the covering graph has a cycle C_k, $k > 3$. Then there are two different digital arcs (which are obtained deleting intermediate chains) and we obtain a contradiction. □

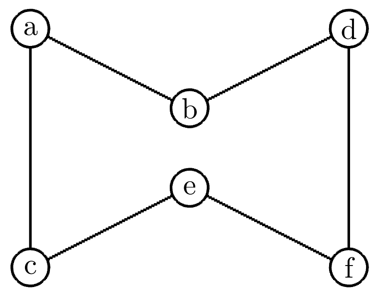

Figure 2.2: Covering graph with a cycle

3. Topological Simple Games

A simple game on a finite set N is a function $v : 2^N \to \{0, 1\}$, with $v(\emptyset) = 0$, and such that $v(S) \leq v(T)$ whenever $S \subseteq T$. The elements of N are called players and the elements of 2^N coalitions. Any coalition $S \subseteq N$ is winning if $v(S) = 1$ or losing if $v(S) = 0$. A simple game is *proper* if $v(S) = 1$ implies $v(N \setminus S) = 0$, for all $S \subseteq N$, i.e.,

$$v(S) + v(N \setminus S) \leq 1 \text{ for all } \quad S \subseteq N.$$

Definition 3.1. *Let (N, t) be a finite topological space satisfying the axiom T_0 and let v be a simple game on N. The associated topological simple game $\left(N, v^t\right)$, denoted TSG, is*

$$v^t(S) := \max\{v(T) : T \textit{ is a connected subspace of } (S, t)\}.$$

Note that $\left(v^t\right)^t = v^t$, and so a simple game is a topological simple game if and only if $v^t = v$.

If $G = (N, E)$ is the comparability graph of the specialization ordering of (N, t) and $S \subseteq N$, the subgraph of G induced by S is the comparability graph of the induced subposet S. Therefore, the following statements are equivalents:

1. The subspace S is connected in the topological T_0-space (N, t).
2. The covering graph of the induced subposet S is connected.
3. The comparability graph of the induced subposet S is connected.

We note that if v is a proper simple game the its associated topological simple game v^t is proper and hence

$$v^t(S) = \sum\{v(T) : T \subseteq S \text{ is a maximal connected subgraph of } G\}.$$

Remark 3.1. *Let G be the comparability graph of the specialization ordering of t. Thus, the topological simple game v^t is a Γ-component additive game by Potters and Reijnierse (1995).*

Example. Let $N = \{1, \ldots, n\}$ and consider the collection $\mathcal{F}_n$ of all the connected subspaces of a finite COTS (N, t), that is,

$$\mathcal{F}_n = \{[i, j] : 1 \leq i \leq j \leq n\} \cup \{\emptyset\},$$

where $[i, j] = \{i, i+1, \ldots, j-1, j\}$. We introduce a special class of simple games called *weighted voting games*. The symbol $[q;\ w_1, \ldots, w_n]$ will be used, where q is the quota needed for a coalition to win, and w_i is the number of votes of player i. Then, the above symbol represents the simple game v defined by

$$v(S) = \begin{cases} 1, & \text{if } w(S) \geq q \\ 0, & \text{if } w(S) < q, \end{cases}$$

where $w(S) = \sum_{i \in S} w_i$. Then $\left(N, v^t\right)$ is a topological simple game which corresponds to a voting situation in a unidimensional policy order.

The *core* of a game (N, v) is the set

$$C(v) = \{x \in \mathbb{R}^n : x(N) = v(N),\ x(S) \geq v(S) \text{ for all } S \subseteq N\},$$

where $x(S) = \sum_{i \in S} x_i$ and $x(\emptyset) = 0$. A simple game has a nonempty core if and only if the set

$$\mathcal{V} = \bigcap_{\{S \subseteq N : v(S) = 1\}} S \neq \emptyset.$$

First, we obtain a characterization of the core of v^t by using only connected coalitions.

Proposition 3.1. *Let (N, v) be a simple game and let (N, t) be an FTS with comparability graph $G = (N, E)$. If $\mathcal{F}$ is the collection of the connected subgraphs of G, and $v(N) = v^t(N)$ then*

$$C\left(v^t\right) = \{x \in \mathbb{R}^n : x(N) = v(N),\ x(S) \geq v(S) \text{ for all } S \in \mathcal{F}\}.$$

Proof. If $x \in C\left(v^t\right)$ then $x(N) = v^t(N) = v(N)$ and $x(S) \geq v^t(S)$, for all $S \subseteq N$. Hence $x(S) \geq v^t(S) = v(S)$, for all $S \in \mathcal{F}$.

Conversely, let $x \in \mathbb{R}^n$ such that $x(N) = v(N)$, and $x(S) \geq v(S)$, for all $S \in \mathcal{F}$. Then, for all $S \subseteq N$,

$$x(S) = \sum_{i \in S} x_i = \sum_k x(T_k) \geq \sum_k v(T_k) = \max_k v(T_k) = v^t(S),$$

where $\{T_k\}$ is the partition of S in its maximal connected subgraphs. □

The vectors $\{e_i\}_{i=1}^n$ are the vectors of the canonical basis of $\mathbb{R}^n$. The *indicator function* $\mathbf{1}_S : N \to \{0,1\}$ for the subset $S \subseteq N$ is given by

$$\mathbf{1}_S(i) = \begin{cases} 1, & \text{if } i \in S \\ 0, & \text{otherwise.} \end{cases}$$

In the following theorem, we will give results concerning the structure of the core for topological simple games. Let $G = (N, E)$ be the comparability graph of the specialization ordering of (N, t) and let $\mathcal{F}$ be the collection of the connected subgraphs of G. We consider now the set

$$\mathcal{V}_{\mathcal{F}} = \bigcap_{\{S \in \mathcal{F} \,:\, v(S)=1\}} S.$$

Theorem 3.2. *Let (N, v^t) be a TSG with $v^t(N) = 1$ and let $\mathcal{F}$ be the collection of the connected subgraphs of its comparability graph. Then $\mathcal{V}_{\mathcal{F}} \neq \emptyset$ if and only if $C\left(v^t\right) \neq \emptyset$. Furthermore,*

$$C\left(v^t\right) = \{x \in \mathbb{R}^n \,:\, x \geq 0,\ x(N) = x(\mathcal{V}_{\mathcal{F}}) = 1\}.$$

Proof. If $\mathcal{V}_{\mathcal{F}} \neq \emptyset$ we take $e_i \in \mathbb{R}^n$ such that $i \in \mathcal{V}_{\mathcal{F}}$. For all $S \in \mathcal{F}$ such that $v(S) = 1$ we have $i \in S$, and hence $e_i(S) \geq v(S)$ for all $S \in \mathcal{F}$. Moreover, since $e_i(N) = 1 = v^t(N)$, Proposition 3.1 implies that $e_i \in C\left(v^t\right)$.

We observe now that $\{i\} \in \mathcal{F}$, and then

$$C\left(v^t\right) = \{x \in \mathbb{R}^n : x \geq 0,\ \ x(N) = x(S) = 1 \text{ for all } S \in \mathcal{W}_{\mathcal{F}}\},$$

where $\mathcal{W}_{\mathcal{F}} = \{S \in \mathcal{F} : v(S) = 1\}$. To obtain the converse, if $C\left(v^t\right)$ is nonempty we have that the linear system

$$\sum_{j=1}^n x_j = 1, \quad Ax = 1, \quad x_j \geq 0, \quad j = 1, \ldots, n$$

where $A = (\mathbf{1}_S)_{S \in \mathcal{W}_{\mathcal{F}}}$ has a solution $x \neq 0$. We claim that $\mathcal{V}_{\mathcal{F}} \neq \emptyset$, because $\bigcap_{S \in \mathcal{W}_{\mathcal{F}}} S = \emptyset$ implies that every column of the matrix A has at least one entry equal to 0. We take the sum of equations $\langle \mathbf{1}_S, x \rangle = 1$, for all $S \in \mathcal{W}_{\mathcal{F}}$ and obtain

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = |\mathcal{W}_{\mathcal{F}}|, \text{ with } \alpha_j < |\mathcal{W}_{\mathcal{F}}|,\ 1 \leq j \leq n.$$

Therefore, $(|\mathcal{W}_{\mathcal{F}}| - \alpha_1)\, x_1 + \cdots + (|\mathcal{W}_{\mathcal{F}}| - \alpha_n)\, x_n = 0$, and this is a contradiction. □

Corollary 3.3. *If (N, v^t) is a TSG with $v^t(N) = 1$, then*

$$C\left(v^t\right) = \operatorname{conv}\{e_i \,:\, i \in \mathcal{V}_{\mathcal{F}}\}.$$

Proof. Since $e_i \in C\left(v^t\right)$ for all $i \in \mathcal{V}_{\mathcal{F}}$, the convexity of the core implies that the convex hull of these vectors is a subset of the core. To prove the reverse inclusion, let x be a vector of $C\left(v^t\right)$. If $i \notin \mathcal{V}_{\mathcal{F}}$ then $x_i = 0$, since there is at least one $S \in \mathcal{W}_{\mathcal{F}}$ such that $i \notin S$ and $x(S) = x(N) = 1$. □

Example. Let $N = \{1, \ldots, n\}$ be a set of players. Let us consider the weighted voting game $v = [q;\ w_1, \ldots, w_n]$, given by

$$w_1 = \cdots = w_n = 1 \text{ and } q = \left\lceil \frac{n+1}{2} \right\rceil,$$

where $\lceil x \rceil$ is the least integer $\geq x$. If (N, t) is a finite COTS then the collection of the connected subspaces is $\mathcal{F}_n = \{[i, j] : 1 \leq i \leq j \leq n\} \cup \{\emptyset\}$. We observe that

$$\mathcal{V}_{\mathcal{F}_n} = \left\{ \left\lfloor \frac{n+1}{2} \right\rfloor, \left\lceil \frac{n+1}{2} \right\rceil \right\},$$

and hence we may apply Corollary 3.3 and obtain

$$C\left(v^t\right) = \begin{cases} \{e_{k+1}\} & \text{if } n = 2k+1, \\ \operatorname{conv}\{e_k, e_{k+1}\} & \text{if } n = 2k. \end{cases}$$

Note the power of the central players with respect to the policy order.

Given a game (N, v) and a coalition $S \subseteq N$, the *subgame* $(S, v\mid_S)$ is obtained by restricting v to 2^S. Propositions 2.3 and 2.4 imply the next properties of topological simple games.

Proposition 3.4. *Let (N, v^t) be a TSG whose covering graph of specialization ordering of t is a rooted tree. Then:*

1. *Every coalition containing the root is connected.*
2. *For all $i \in N$, the subgame $\left(\overline{\{i\}},\ v^t\mid_{\overline{\{i\}}}\right)$ satisfies $v^t\mid_{\overline{\{i\}}} = v\mid_{\overline{\{i\}}}$.*

Proposition 3.5. *Let (N, v^t) be a TSG whose covering graph of specialization ordering of t is a tree. Then:*

1. *For every $\{i, j\} \subseteq N$ with $i < j$, the subgame $\left([i, j],\ v^t\mid_{[i,j]}\right)$ satisfies $v^t\mid_{[i,j]} = v\mid_{[i,j]}$.*
2. *If $\{i, j\} \subseteq N$ is an antichain, there is a subgame C such that it is a COTS with the induced topology.*

Note that under the respective hypothesis, if $i < j$ then we obtain topological subgames which are ordinary games in the intervals $[i, j]$. For an antichain, the subgame is defined in the connected coalitions of a COTS, and these coalitions coincide with the convex coalitions for linear orderings studied by Axelrod, Peleg and Einy (see Einy, Definition 3.2).

Now, we analyze the relation between domination and connectivity. We need the following concepts by Einy and Peleg.

Definition 3.2. *Let (N, v) be a simple game and let $S \subseteq N$. A player $i \in S$ weakly dominates S if $v\left(B \cup (S \setminus \{i\})\right) = 1$ implies $v\left(B \cup \{i\}\right) = 1$, for every B such that $B \cap S = \emptyset$.*

In this case, we denote $\{i\} \dashv S \setminus \{i\}$. Let (N, v^t) be a topological simple game and we define:

$$\begin{aligned} \mathcal{H}_i &= \{S \subseteq N : i \in S,\ v(S) = 1, \text{ and } \{i\} \dashv S \setminus \{i\}\}. \\ \mathcal{C}_i &= \{S \subseteq N : i \in S,\ v(S) = 1, \text{ and } S \text{ is connected in } (N, t)\}. \end{aligned}$$

Einy (Propositions 5.8 and 5.9) studied the compatibility of Axelrod's hypothesis (*only connected coalitions with respect to a linear order are formed*) with several hypothesis about winning coalitions which are dominated by a player. We obtain sufficient conditions for the existence of topological simple games with winning and connected coalitions containing a player such that this player weakly dominates these coalitions.

Theorem 3.6. *Let (N, v) be a proper simple game with $|N| \geq 3$ and let (N, t) be an FTS such that its covering graph is a tree. If there is a player $i \in N$ such that $v\left(\overline{\{i\}}\right) = 1$ and $v(\{i, j\}) = 1$ for some $j \notin \overline{\{i\}}$, then $\mathcal{H}_i \cap \mathcal{C}_i \neq \emptyset$.*

Proof. Given $\{i, j\} \subset N$, there is a unique connected coalition S' homeomorphic to a COTS with endpoints i and j. Then, $S' \setminus \{j\}$ and $\overline{\{i\}}$ are connected containing i, hence $S = (S' \setminus \{j\}) \cup \overline{\{i\}}$ is connected and $v(S) = 1$ since v is monotone. Therefore, $S \in \mathcal{C}_i$. If we prove that i weakly dominates S, we have $S \in \mathcal{H}_i$, and $\mathcal{H}_i \cap \mathcal{C}_i \neq \emptyset$. Thus, given $B \subseteq N$ with $B \cap S = \emptyset$ and $v\left(B \cup (S \setminus \{i\})\right) = 1$, it is enough to show that $v(B \cup \{i\}) = 1$. If we assume that $j \in B$, then $\{i, j\} \subseteq B \cup \{i\}$ and the hypothesis $v(\{i, j\}) = 1$ implies that $v(B \cup \{i\}) = 1$. Note that if $j \notin B$, then $B \cup (S \setminus \{i\}) \subseteq N \setminus \{i, j\}$. But v is proper and $v(\{i, j\}) = 1$, thus $v(B \cup (S \setminus \{i\})) = 0$, which contradicts the hypothesis. □

The condition of Theorem 3.6 is not necessary, as the following example shows:

Example. Let $v = [6;\ 3, 2, 2, 3]$ be a voting game with four players. In the COTS with covering graph  player 1 satisfies $v\left(\overline{\{1\}}\right) = v\left(\{1, 2\}\right) = 0$. But $S = \{1, 2, 3\} \in \mathcal{C}_1$ and $\{1\} \dashv S \setminus \{1\}$. Hence $S \in \mathcal{H}_1 \cap \mathcal{C}_1$.

The Dilworth's chain decomposition can be interpreted as a Ramsey theorem: *Any ordered set P of size at least $ab+1$ contains either a chain of length $a+1$ or an antichain of size $b+1$* (see Bogart et al.(1990)). We apply the Dilworth's theorem to a society N, with a partial ordering of its members. The minimal (by set inclusion in $N \times N$) ordering is the trivial ordering, i.e., $x \leq y$ in N implies $x = y$ and the maximal ordering is the linear ordering. We suppose that every coalition of three members has got at least one relation and obtain winning chains (coalitions with total cooperation) for majority games.

Theorem 3.7. *Let (N, v^t) be a TSG whose covering graph has no antichain of size three and $v(S) = 1$ if and only if $|S| \geq \lfloor n/2 \rfloor + 1$, where $n = |N|$. Then, if $n = 2k+1$ there is a minimal winning chain and if $n = 2k,\ k \geq 2$, there is a minimal winning chain or $N = C_1 \cup C_2$, where the chains have exactly k elements.*

Proof. If $n = 2k+1$, and take $a = k,\ b = 2$, then there is a chain $S \subseteq N$ with $a+1 = k+1$ elements. Therefore, $|S| = k+1 = \lfloor n/2 \rfloor + 1$, so S is a minimal winning chain. If $n = 2k$, take $a = k-1,\ b = 2$, then $ab+1 = 2k-1$ and $|N| = 2k > ab+1$. Thus, there is a chain S with $a+1 = k$ elements. If S is not maximal, then there is a minimal winning chain with $k+1 = \lfloor n/2 \rfloor + 1$ elements. Otherwise, N is the union of two disjoint chains of k elements. □

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