A DUAL APPROACH TO COMPROMISE VALUES

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Abstract. The aim of this paper is to analyse the *Tijs value* as a solution concept for cost allocation problems. Given a cost game c, we define its *dual τ-value* as $\tau(c^*)$, if the dual game c^* is quasi-balanced. Then we show that the dual τ-value coincides, for a wide class of cost games, with the *alternate cost avoided* (ACA) allocation proposed in the 1930's by the Tennessee Valley Authority. It turns out that the center of the imputation set, the egalitarian nonseparable contribution and the ACA allocation are collinear.

1. Introduction

A transferable utility game is a pair (N, v), where N is a finite set and $v : 2^N \to \mathbb{R}$, is a function with $v(\emptyset) = 0$. The elements of $N = \{1, \ldots, n\}$ are called *players*, the subsets $S \in 2^N$ *coalitions* and $v(S)$ is the *worth* of S. The worth of a coalition is interpreted as the maximal profit or minimal cost for the players in their own coalition. We will consider *profit games* if $v(S)$ measures the profit of the coalition S and *cost games* $c : 2^N \to \mathbb{R}$ if the function measures the cost $c(S)$ incurred by S. We will use a shorthand notation and write i for the set $\{i\}$, $S \cup i$ for $S \cup \{i\}$, and $S \setminus i$ for $S \setminus \{i\}$.

In a game (N, v), a vector $x \in \mathbb{R}^n$ is called *efficient* if it distributes the worth $v(N)$ among the players, i.e., $\sum_{i \in N} x_i = v(N)$. The set of all efficient vectors is called the *preimputation set* and is denoted by $I^*(v)$. The *imputation set* of a profit game (N, v) is defined by

$$I(v) = \{x \in I^*(v) : \; x_i \geq v(i) \; \text{ for all } i \in N\}.$$

Note that $I(v) \neq \emptyset$ if and only if $v(N) \geq \sum_{i \in N} v(i)$. If (N, c) is a cost game, then the set of (efficient) vectors $x \in \mathbb{R}^n$ with $\sum_{i \in N} x_i = c(N)$ is denoted by $I^*(c)$, and its imputation set is

$$I(c) = \{x \in I^*(c) : \; x_i \leq c(i) \; \text{ for all } i \in N\}.$$

Therefore, $I(c) \neq \emptyset$ if and only if $c(N) \leq \sum_{i \in N} c(i)$. Assuming that the coalition N of all players will be formed, the following solution concept

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will prescribe a distribution of the cost $c(N)$ or the profit $v(N)$ among the players.

Definition 1.1. *The core of a cost game c and the core of a profit game v are respectively defined by*

$$\begin{aligned} Core(c) &= \{x \in I^*(c) : x(S) \le c(S) \ \text{for all } S \subseteq N\}, \\ Core(v) &= \{x \in I^*(v) : x(S) \ge v(S) \ \text{for all } S \subseteq N\}, \end{aligned}$$

where $x(S) = \sum_{i \in S} x_i$ *and* $x(\emptyset) = 0$.

Games with a nonempty core are called *balanced games*.

Definition 1.2. *The dual game* (N, v^*) *of* (N, v) *is defined by the dual function* $v^* : 2^N \to \mathbb{R}$, *where* $v^*(S) = v(N) - v(N \setminus S)$, *for all* $S \subseteq N$.

Note that $(v^*)^* = v$. We remark that if (N, v) is a profit (cost) game and (N, v^*) is a cost (profit) game, then $Core(v) = Core(v^*)$.

The *Tijs value* of a game is a feasible compromise between the upper and the lower vectors for the game, introduced by Tijs [4, 6, 7]. The *upper vector* of the game (N, v) is the vector $M(v) \in \mathbb{R}^n$, where

$$M_i(v) = v(N) - v(N \setminus i),$$

for all $i \in N$. The component $M_i(v)$ is called the utopia payoff for player i in the grand coalition. The remainder of $i \in S$ if the coalition S forms and all other players in S obtain their utopia payoff is

$$R^v(S, i) = v(S) - \sum_{j \in S \setminus i} M_j(v).$$

The *lower vector* is the vector $m(v) \in \mathbb{R}^n$, defined by

$$m_i(v) = \max_{\{S \subseteq N : i \in S\}} R^v(S, i).$$

Definition 1.3. *A game* (N, v) *is called quasi-balanced if it satisfies:*

1. $m(v) \le M(v)$,
2. $\sum_{i \in N} m_i(v) \le v(N) \le \sum_{i \in N} M_i(v)$.

The family of quasi-balanced games contains the family of balanced games as a subset (see Tijs [4]). For a quasi-balanced game v the Tijs value, denoted by $\tau(v)$, is the unique preimputation (efficient vector) on the closed interval $[m(v), M(v)]$ in $\mathbb{R}^n$. So we have

$$\tau(v) = m(v) + \lambda \left(M(v) - m(v) \right),$$

where $\lambda \in \mathbb{R}$ is determined by

$$\sum_{i \in N} \tau_i(v) = v(N).$$

2. Cost allocations problems

The Tennessee Valley Authority was a development project to construct dams and reservoirs along the Tennessee River. A method used by civil engineers to allocate the costs of the project is known as the *alternate cost avoided* (ACA) allocation.

We now describe the version of this method given by Young [8]. Let (N, c) be a cost game. For each project $i \in N$ we define its *separable cost*

$$s_i = c(N) - c(N \setminus i).$$

The *alternate cost* for i is the cost $c(i)$ and the difference

$$r_i = c(i) - s_i$$

is the *alternate cost avoided.* Let $s(N) = \sum_{i \in N} s_i$ and $r(N) = \sum_{i \in N} r_i$. Then the ACA allocation assigns to each $i \in N$ the cost allocation given by the formula

$$ACA_i(c) = \begin{cases} s_i + \dfrac{r_i}{r(N)}(c(N) - s(N)), & \text{if } r(N) \neq 0, \\ s_i, & \text{otherwise.} \end{cases} \tag{2.1}$$

Notice that every i pays the sum of its separable cost and a proportion of the *nonseparable cost* $c(N) - s(N)$. Notice further that $r_i \geq 0$ for all $i \in N$ if c is *subadditive*, i.e., for all $S, T \in 2^N$ such that $S \cap T = \emptyset$, we have $c(S \cup T) \leq c(S) + c(T)$.

Definition 2.1. *Let (N, c) be a cost game. The associated savings game (N, v) is defined by $v(S) = \sum_{i \in S} c(i) - c(S)$, for all $S \subseteq N$.*

The worth $v(S)$ is the cost savings obtained from cooperation between the players. Driessen [2, Chapter IV] obtained the relationship between two separable cost allocation methods and the Tijs value of the associated savings game.

If the cost game c is subadditive, then for every $S \subset N$ and $i \in N \setminus S$, we have that $c(S \cup i) \leq c(S) + c(i)$. It follows that

$$v(S \cup i) - v(S) = c(i) - (c(S \cup i) - c(S)) \geq 0.$$

We conclude that its savings game is monotonic and hence nonnegative.

Let (N, c) be a cost game. Then its savings game (N, v) is monotonic if and only if for all $T \subseteq S \subseteq N$,

$$\sum_{i \in T} c(i) - c(T) \leq \sum_{i \in S} c(i) - c(S) \Longleftrightarrow c(S) \leq c(T) + \sum_{i \in S \setminus T} c(i). \tag{2.2}$$

The monotonicity of the savings game implies, for all $i \in N$,

$$s_i = c(N) - c(N \setminus i) \leq c(i), \text{ or } r_i \geq 0.$$

Tijs and Driessen [5] introduced the *reverse τ-value*, $\tau^r(c)$, of a cost game (N, c) defined as $\tau^r(c) := -\tau(-c)$ if $-c$ is quasi-balanced. They proved that the reverse τ-value of a cost game c coincides with the ACA allocation if the cost game is such that $m_i^r(c) = c(i)$ for all $i \in N$, i.e., if $-c$ is semi-convex.

3. The dual Tijs value

Given a cost game c, we define its *dual τ-value* as $\tau^*(c) := \tau(c^*)$, if the dual game c^* is quasi-balanced. First, we study the properties of the upper and the lower vectors for the game (N, c^*).

Theorem 3.1. *Let (N,c) be a cost game such that its savings game is monotonic. Then the components of the upper and the lower vectors of c^* satisfy*

1. $M_i(c^*) = c(i)$ *for all* $i \in N$.
2. $m_i(c^*) = c(N) - c(N \setminus i)$ *for all* $i \in N$.
3. $m_i(c^*) \leq M_i(c^*)$ *for all* $i \in N$, *and* $c^*(N) \leq \sum_{i \in N} M_i(c^*)$.

Proof. 1. For every $i \in N$,

$$M_i(c^*) = c^*(N) - c^*(N \setminus i) = c(N) - (c(N) - c(i)) = c(i).$$

2. For every $i \in N$,

$$\begin{aligned} m_i(c^*) &= \max_{\{S \subseteq N : i \in S\}} \left\{ c^*(S) - \sum_{j \in S \setminus i} M_j(c^*) \right\} \\ &= \max_{\{S \subseteq N : i \in S\}} \left\{ c(N) - c(N \setminus S) - \sum_{j \in S \setminus i} c(j) \right\} \\ &= c(N) - \min_{\{S \subseteq N : i \in S\}} \left\{ c(N \setminus S) + \sum_{j \in S \setminus i} c(j) \right\} \\ &= c(N) - \min_{T \subseteq N \setminus i} \left\{ c(T) + \sum_{j \in (N \setminus i) \setminus T} c(j) \right\}. \end{aligned}$$

From the monotonicity of the associated savings game and (2.2) we obtain

$$c(N \setminus i) \leq c(T) + \sum_{j \in (N \setminus i) \setminus T} c(j),$$

for all $T \subseteq N \setminus i$. Therefore,

$$\min_{T \subseteq N \setminus i} \left\{ c(T) + \sum_{j \in (N \setminus i) \setminus T} c(j) \right\} = c(N \setminus i),$$

and we conclude that $m_i(c^*) = c(N) - c(N \setminus i)$.
3. The inequality (2.2) implies that $c(N) \leq c(N \setminus i) + c(i)$. From 1 and 2, it follows that $m_i(c^*) = c(N) - c(N \setminus i) \leq c(i) = M_i(c^*)$. Moreover,

$$c^*(N) = c(N) \leq \sum_{i \in N} c(i) = \sum_{i \in N} M_i(c^*). \qquad \square$$

Theorem 3.2. *Let (N,c) be a cost game such that its savings game is monotonic. Then*

1. *The dual game c^* is quasi-balanced if and only if $s(N) \leq c(N)$.*
2. *If $s(N) \leq c(N)$, then the dual Tijs value $\tau(c^*)$ coincides with the allocation $ACA(c)$.*

Proof. 1. It follows from Theorem 3.1.3 that the dual game c^* is quasi-balanced if and only if

$$\sum_{i \in N} (c(N) - c(N \setminus i)) \leq c(N), \text{ or } s(N) \leq c(N).$$

2. For every $i \in N$, the component i of the dual Tijs value is

$$\tau_i(c^*) = s_i + \lambda (c(i) - s_i) = s_i + \lambda r_i,$$

where $\lambda \in \mathbb{R}$ is such that

$$\sum_{i \in N} \tau_i(c^*) = c(N).$$

Since $r_i \geq 0$ we have

$$r(N) = 0 \Longleftrightarrow r_i = 0 \text{ for all } i \in N \Longleftrightarrow \tau(c^*) = M(c^*) = m(c^*).$$

Otherwise,

$$c(N) = \sum_{i \in N} (s_i + \lambda r_i) = s(N) + \lambda r(N)$$

implies that

$$\lambda = \frac{c(N) - s(N)}{r(N)}.$$

Thus, the components of the dual Tijs value satisfy

$$\tau_i(c^*) = \begin{cases} s_i, & \text{if } r(N) = 0, \\ s_i + \dfrac{r_i}{r(N)} (c(N) - s(N)), & \text{if } r(N) > 0, \end{cases}$$

which coincides with the formula (2.1) of the ACA allocation. □

4. Collinearity of cost allocation rules

The center of the imputation set (CIS) and the egalitarian nonseparable contribution ($ENSC$) are well known cost allocation rules (cf. [1], [2], [3]).

Definition 4.1. *Let (N,c) be a cost game. For each $i \in N$, the CIS and the $ENSC$ allocations are given by*

$$\begin{aligned} CIS_i(c) &= c(i) + \frac{1}{n}\left(c(N) - \sum_{i \in N} c(i)\right), \\ ENSC_i(c) &= s_i + \frac{1}{n}(c(N) - s(N)). \end{aligned}$$

Obviously $CIS(c) = ENSC(c^*)$ and $ENSC(c) = CIS(c^*)$. In the next theorem we compute the reverse τ-value of the dual of a cost game (N, c), that is, the vector $\tau^r(c^*) = -\tau(-c^*)$. This vector is the unique efficient compromise between the two vectors $M^r(c^*)$ and $m^r(c^*)$.

Theorem 4.1. *Let (N, c) be a cost game such that its savings game (N, v) is nonnegative. Then we have*

1. $M_i^r(c^*) = c(i)$ *for all* $i \in N$.
2. $m_i^r(c^*) = c(i) + \left(c(N) - \sum_{j\in N} c(j)\right)$ *for all* $i \in N$.
3. *The reverse τ-value of the dual game c^* coincides with the allocation $CIS(c)$.*

Proof. 1. $M_i^r(c^*) = -M_i(-c^*) = c^*(N) - c^*(N \setminus i) = c(i)$, for all $i \in N$.
2. The components of the lower vector satisfy

$$\begin{aligned} m_i^r(c^*) &= -m_i(-c^*) \\ &= -\max_{\{S\subseteq N:\, i\in S\}} \left\{ -c^*(S) - \sum_{j\in S\setminus i} M_j(-c^*) \right\} \\ &= -\max_{\{S\subseteq N:\, i\in S\}} \left\{ c(N\setminus S) - c(N) + \sum_{j\in S\setminus i} c(j) \right\} \\ &= c(N) - \max_{T\subseteq N\setminus i} \left\{ c(T) + \sum_{j\in (N\setminus i)\setminus T} c(j) \right\}. \end{aligned}$$

For every $T \subseteq N \setminus i$ we have that

$$\begin{aligned} c(T) + \sum_{j\in(N\setminus i)\setminus T} c(j) &= c(T) + \sum_{j\in N\setminus i} c(j) - \sum_{j\in T} c(j) \\ &= \sum_{j\in N\setminus i} c(j) - v(T) \\ &\leq \sum_{j\in N\setminus i} c(j), \end{aligned}$$

since the savings game (N, v) is nonnegative. Moreover, this upper bound is attained in $T = \emptyset$ and hence

$$m_i^r(c^*) = c(N) - \sum_{j\in N\setminus i} c(j) = c(i) + \left(c(N) - \sum_{j\in N} c(j) \right),$$

for all $i \in N$.
3. First, we show that the game $-c^*$ is quasi-balanced. By hypothesis $v(N) \geq 0$ and hence $m_i^r(c^*) = c(i) - v(N) \leq c(i) = M_i^r(c^*)$, for all $i \in N$.

Furthermore,

$$\sum_{i\in N} c(i) - c(N) \geq 0 \text{ implies } c(N) \leq \sum_{i\in N} M_i^r(c^*).$$

Finally,

$$\begin{aligned}\sum_{i\in N} m_i^r(c^*) &= \sum_{i\in N} c(i) + nc(N) - n\sum_{i\in N} c(i)\\ &= c(N) + (n-1)\left(c(N) - \sum_{i\in N} c(i)\right)\\ &= c(N) - (n-1)\,v(N) \leq c(N).\end{aligned}$$

Therefore, the reverse τ-value is defined and equals the unique efficient vector between the upper and lower vectors with components $M_i^r(c^*) = c(i)$ and $m_i^r(c^*) = c(i) - v(N)$. Then

$$\tau_i^r(c^*) = c(i) - v(N) + \lambda v(N) = c(i) + (\lambda - 1)\,v(N).$$

The efficiency implies that

$$c(N) = \sum_{i\in N} \tau_i^r(c^*) = \sum_{i\in N} c(i) + n(\lambda-1)\,v(N).$$

From this we conclude that for all $i \in N$,

$$\tau_i^r(c^*) = c(i) + \frac{1}{n}\left(c(N) - \sum_{i\in N} c(i)\right),$$

which coincides with the CIS allocation. □

In order to obtain the *collinearity* of the three allocations rules CIS, $ENSC$ and ACA, it is useful to recall some properties of orthogonal projections of $\mathbb{R}^n$ onto a hyperplane. The *inner product* of vectors x and y is the number $\langle x,y\rangle = \sum_{i=1}^n x_i y_i$. Then a hyperplane H in $\mathbb{R}^n$ is a set of the form $H = \{x \in \mathbb{R}^n : \langle m,x\rangle = \alpha\}$, where m is a nonzero vector normal to H and α is a real number. The orthogonal projection $P_H : \mathbb{R}^n \to H$ onto H is given by

$$P_H(q) = q + \frac{\alpha - \langle m,q\rangle}{\|m\|^2}\, m,$$

for each $q \in \mathbb{R}^n$. Note that

(P1) $P_H(q) = q$ if and only if $q \in H$,

(P2) P_H is an affine map, i.e., $P_H(\lambda q + \eta r) = \lambda P_H(q) + \eta P_H(r)$ for all $q, r \in \mathbb{R}^n$ and $\lambda, \eta \in \mathbb{R}$ with $\lambda + \eta = 1$.

From (P2) it follows that the orthogonal projection on H maps a line in $\mathbb{R}^n$ onto a line (or a point) in H. This implies:

(P3) If $p, q, r \in \mathbb{R}^n$ are collinear, i.e., there is a line in $\mathbb{R}^n$ containing p, q and r, then also $P_H(p), P_H(q)$ and $P_H(r)$ are collinear. Furthermore, if $p = \lambda q + \eta r$ with $\lambda + \eta = 1$, then $P_H(p) = \lambda P_H(q) + \eta P_H(r)$.

In the next proposition we obtain the orthogonal projections of the vectors $k = (c(i))_{i\in N}$ and $s = (s_i)_{i\in N}$ on the preimputation set $I^*(c)$. For a geometric illustration see Figure 1.

Proposition 4.2. *Let H be the preimputation set $I^*(c)$ of a cost game (N, c) and let k, s be the vectors defined above. Then $P_H(k) = CIS(c)$ and $P_H(s) = ENSC(c)$.*

Proof. Note that $H = \{x \in \mathbb{R}^n : \langle m, x\rangle = c(N)\}$, where $m = (1, \ldots, 1)$. Then

$$P_H(k) = k + \frac{1}{n}\left(c(N) - \sum_{i=1}^{n} c(i)\right)(1, \ldots, 1),$$

which implies

$$(P_H(k))_i = c(i) + \frac{1}{n}\left(c(N) - \sum_{i=1}^{n} c(i)\right) = CIS_i(c),$$

for each $i \in N$. Further,

$$P_H(s) = s + \frac{1}{n}\left(c(N) - \sum_{i=1}^{n} s_i\right)(1, \ldots, 1).$$

Hence, for each $i \in N$, we obtain

$$(P_H(s))_i = s_i + \frac{1}{n}(c(N) - s(N)) = ENSC_i(c). \qquad \square$$

Theorem 4.3. *Let (N, c) be a cost game with $s(N) \le c(N) \le k(N)$. Then the allocations $CIS(c)$, $ENSC(c)$ and $ACA(c)$ are collinear. Moreover, if $s(N) \ne k(N)$ then*

$$ACA(c) = ENSC(c) + \frac{c(N) - s(N)}{r(N)}(CIS(c) - ENSC(c)).$$

Otherwise, $ACA(c) = ENSC(c)$.

Proof. The definition (2.1) of the ACA allocation implies that

$$ACA(c) = s + \frac{c(N) - s(N)}{k(N) - s(N)}(k - s)$$

if $s(N) \ne k(N)$, and $ACA(c) = s$, otherwise. Thus the vector $ACA(c)$ is a convex combination of the vectors $k = (c(i))_{i\in N}$ and $s = (s_i)_{i\in N}$. Further

$ACA(c) \in I^*(c) = H$. In the case that $s(N) \neq k(N)$ the properties (P1), (P2) and (P3) imply that

$$ACA(c) = P_H(ACA(c)) = P_H(s) + \frac{c(N) - s(N)}{k(N) - s(N)} (P_H(k) - P_H(s)).$$

Then, by Proposition 4.2, we obtain

$$ACA(c) = ENSC(c) + \frac{c(N) - s(N)}{r(N)} (CIS(c) - ENSC(c)).$$

If $s(N) = k(N)$ then $s(N) = c(N)$ and so $ACA(c) = ENSC(c)$. □

Remark 4.1. *If $s(N) = k(N)$ and the savings game is monotonic, then the allocations $ACA(c) = CIS(c) = ENSC(c)$.*

Theorem 4.3 suggests us to define the following concept.

Definition 4.2. *The compromise set of a cost game (N, c) is the convex hull of the vectors $CIS(c)$ and $ENSC(c)$.*

In future research, we expect to see new applications of this set, consisting of the line segment with end points $CIS(c)$ and $ENSC(c)$.

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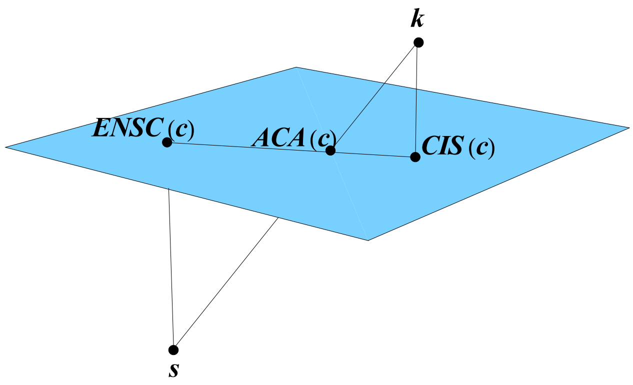

Figure 1. Orthogonal projections on $I^{*}(c)$