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A UNIFIED APPROACH TO RESTRICTED GAMES

ABSTRACT. There have been two main lines in the literature on restricted games: the first line was started by Myerson (1977) that studied graph-restricted games an the second one was initiated by Faigle (1989). The present paper provides a unified way to look on the literature and establishes connections between the two different lines on restricted games. The strength and advantages of this unified approach becomes clear in the study of the inheritance of the convexity from the game to the restricted game where an interesting result by Nouweland and Borm (1991) on the convexity of graph-restricted games is turned into a direct consequence of the corresponding result by Faigle (1989), by means of this relation.

KEY WORDS: Restricted games, Core, Convex games, Convex geometries

1. INTRODUCTION

A cooperative game with transferable utility is a pair (N, v) where N is a finite set and v is a real-valued function $v : 2^N \to \mathbb{R}$, such that $v(\emptyset) = 0$. The elements of $N = \{1, 2, \ldots, n\}$ are called *players*, the subsets $S \in 2^N$ *coalitions* and $v(S)$ is the *worth* of the coalition S.

A game is *superadditive* if $v(S \cup T) \geqslant v(S) + v(T)$, for all $S, T \in 2^N$ such that $S \cap T = \emptyset$. If the function v is supermodular, i.e., for all $S, T \in 2^N$

$$v(S \cup T) + v(S \cap T) \geqslant v(S) + v(T),$$

then we say that the game (N, v) is *convex*. Obviously, if (N, v) is convex it is superadditive too.

In cooperative game theory it is generally assumed that there are no restrictions on communication. However, this classical model seems to be inappropriate in modelling certain situations. For that, Myerson (1977) introduced a communication graph $G = (N, E)$, whose vertex set N is formed by the players and the edge set E is given by bilateral agrements among the players. The game restricted by a communication graph is called *graph-restricted game*. This line

Theory and Decision **50:** 333–345, 2001.

of research was continued by Owen (1986), van den Nouweland and Borm (1991), Borm et al. (1992), and Potters and Reijnierse (1995). There is a model of cooperative sequencing games, that are arisen from one-machine sequencing situations with ready times (see Hamers et al. (1995)).

However, any situations that are derived from a partial cooperation can not be represented by a graph, which gives us the relationship among them. In this case, Faigle (1989), proposed a new model to analyze the partial cooperation by combinatorial methods. In this model, it is defined the game on a system $\mathcal{F}$ of player coalitions called *feasible coalitions*, i.e., $v : \mathcal{F} \to \mathbb{R}, v(\emptyset) = 0$. A game restricted by $\mathcal{F}$ is defined as an extension of the above game on all the coalitions S of N which can be expressed as a union of disjoint fesible coalitions. This idea, that was named *partitioning games* by Kaneko and Wooders (1982), was studied by Kuipers (1994).

In our paper both models of partial cooperation are unified. With this goal, in Section 2 *feasible coalition systems*, that are only slightly less general than in Faigle (1989), are introduced. Restricted games for these systems are defined on 2^N. Section 3 provides the formal definition of *partition systems* and the properties of games restricted by these systems. In Sections 4 and 5, the conditions under which the convexity is inherited from the underlying game to the restricted game are investigated. Finally, in Section 6 some observations, that are arisen when the hypothesis of superadditivity in the game (N, v) is not assumed, are incorporated.

2. FEASIBLE COALITION SYSTEMS

Throughout the Sections 2, 3, 4 and 5 (N, v) will denote a superadditive game.

DEFINITION 1. *A feasible coalition system is a pair* $(N, \mathcal{F})$, $\mathcal{F} \subseteq 2^N$ *that satisfies the following property:* $\emptyset \in \mathcal{F}$ *and* $\{i\} \in \mathcal{F}$ *for all* $i \in N$.

The elements of $\mathcal{F}$ are called feasible coalitions. Notice that any nonempty coalition $S \subseteq N$ can be expressed as a disjoint union of feasible coalitions since $S = \bigcup_{i\in S}\{i\}$. The set of all partitions of a nonempty $S \subseteq N$, in nonempty feasible coalitions is denoted by $\mathcal{P}_{\mathcal{F}}(S)$. Moreover, $\mathcal{P}_{\mathcal{F}}(\emptyset) = \{\emptyset\}$.

DEFINITION 2. *Let $(N, \mathcal{F})$ be a feasible coalition system and let (N, v) be a game. A game restricted by $\mathcal{F}$ is the pair $(N, \tilde{v}^{\mathcal{F}})$ defined by*

$$\tilde{v}^{\mathcal{F}} : 2^N \to \mathbb{R}, \quad \tilde{v}^{\mathcal{F}}(S) = \max\left\{\sum_{i \in I} v(T_i) : \{T_i\}_{i \in I} \in \mathcal{P}_{\mathcal{F}}(S)\right\}.$$

This concept is an extension of the definition that has been given by Faigle in the study of games with restricted cooperation.

DEFINITION 3. *Let $(N, \mathcal{F})$ be a feasible coalition system. The $\mathcal{F}$-components of $S \subseteq N$ ae the maximal subsets of S belonging to $\mathcal{F}$.*

Taken into account the above definition it can be checked, in a straightforward way, that for all $S \subseteq N$, we have that $S = \bigcup_{k \in K} T_k$, where $\{T_k\}_{k \in K}$ are the $\mathcal{F}$-components of S. However, the $\mathcal{F}$-components of S might not be a partition of S, as can be seen in the next example.

EXAMPLE 1. Let $N = \{1, 2, 3, 4\}$ be a set of players and let also

$$\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\}$$

be a feasible coalition system. If $S = \{1, 2, 4\}$ then $T_1 = \{1, 2\}$ and $T_2 = \{2, 4\}$ are maximal subsets of S in $\mathcal{F}$ and $T_1 \cap T_2 = \{2\}$. Therefore the $\mathcal{F}$-components of S are not a partition of S.

If $(N, \mathcal{F})$ is a feasible coalition system then we will denote by Π_S the family of the $\mathcal{F}$-components of the coalition S.

THEOREM 1. *Let $(N.\mathcal{F})$ be a feasible coalition system and let (N, v) be a game. If Π_S is a partition of S then $\tilde{v}^{\mathcal{F}}(S) = \sum_{T \in \Pi_S} v(T)$.*

Proof. By assumption, the $\mathcal{F}$-components $\Pi_S = \{T_k\}_{k \in K}$ of S form a partition. For each $\{S_i\}_{i \in I} \in \mathcal{P}_{\mathcal{F}}(S)$ there exists a partition $\{I_k\}_{k \in K}$ of I such that if $i \in I_k$ then $S_i \subseteq T_k$. Therefore

$\sum_{i \in I_k} v(S_i) \leqslant v\left(\bigcup_{i \in I_k} S_i\right) = v(T_k)$, for all $k \in K$, since (N, v) is superadditive. Hence $\sum_{i \in I} v(S_i) \leqslant \sum_{k \in K} v(T_k)$. We may apply the definition of $\tilde{v}^{\mathcal{F}}$ to obtain our conclusion

$$\tilde{v}^{\mathcal{F}}(S) = \max\left\{\sum_{i \in I} v(S_i) : \{S_i\}_{i \in I} \in \mathcal{P}_{\mathcal{F}}(S)\right\} = \sum_{T \in \Pi_S} v(T).$$

□

A distribution of the amount $v(N)$ among the players will be represented by a real-valued function x on the players set N satisfying the *efficiency principle* $\sum_{j \in N} x(j) = v(N)$. Here $x(i)$ which is also denoted by x_i represents the payoff to player i according to the involved payoff function x. We usually identify a real-valued function $x : N \to \mathbb{R}$ with the vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^N$ of real numbers. The vectors $x \in \mathbb{R}^N$ that satisfy the efficiency principle are called efficient payoff vectors or *pre-imputations*. The *core* of a game (N, v) is the set

$$C(v) = \{x \in \mathbb{R}^N : x(N) = v(N), x(S) \geqslant v(S), \text{ for all } S \subset N\},$$

where $x(S) = \sum_{i \in S} x_i$ and $x(\emptyset) = 0$.

Bondareva (1963) and Shapley (1967) state that a game (N, v) is *balanced* if and only if it has a nonempty core. A game is *totally balanced* if each subgame is balanced.

THEOREM 2. *Let $(N, \mathcal{F})$ be a feasible coalition system and let (N, v) be a game such that $v(N) = \tilde{v}^{\mathcal{F}}(N)$. Then the following statements hold:*

(a) $C(\tilde{v}^{\mathcal{F}}) = \{x \in \mathbb{R}^N : x(N) = v(N), x(S) \geqslant v(S)$, for all $S \in \mathcal{F}\}$.
(b) $C(v) \subseteq C(\tilde{v}^{\mathcal{F}})$.
(c) If the game (N, v) is balanced then so is the restricted game $(N, \tilde{v}^{\mathcal{F}})$.

Proof. (a) Let $x \in C(\tilde{v}^{\mathcal{F}})$, then $x(N) = \tilde{v}^{\mathcal{F}}(N) = v(N)$ and $x(S) \geqslant \tilde{v}^{\mathcal{F}}(S)$, for all $S \subset N$. If $S \in \mathcal{F}$ then $\{S\} \in \mathcal{P}_{\mathcal{F}}(S)$, hence $x(S) \geqslant \tilde{v}^{\mathcal{F}}(s) \geqslant v(S)$. It remains to prove the inverse inclusion.

Indeed let $S \subseteq N$, and $\{S_k\}_{k\in K} \in \mathcal{P}_{\mathcal{F}}(S)$. Therefore,

$$x(S) = \sum_{i\in S} x_i = \sum_{k\in K}\left(\sum_{i\in S_k} x_i\right) = \sum_{k\in K} x(S_k) \geqslant \sum_{k\in K} v(S_k),$$

and it follows that $x(S) \geqslant \max\{\sum_{i\in I} v(T_i) : \{T_i\}_{i\in I} \in \mathcal{P}_{\mathcal{F}}(S)\} = \tilde{v}^{\mathcal{F}}(S)$.

(b) It follows from (a).

(c) Statement (a) implies the result. □

In general, $C(v) \not\subseteq C(\tilde{v}^{\mathcal{F}})$. The condition $v(N) = \tilde{v}^{\mathcal{F}}(N)$ is also necessary in order to establish the above inclusion, because otherwise $C(v) \cap C(\tilde{v}^{\mathcal{F}}) = \emptyset$.

3. PARTITION SYSTEMS

DEFINITION 4. *A partition system is a feasible coalition system $(N, \mathcal{F})$ such that for all $S \subseteq N$, the family of the $\mathcal{F}$-components Π_S is a partition of S.*

The feasible coalition systems $(N, \mathcal{F})$ with $\mathcal{F} = 2^N$ and $\mathcal{F} = \{\emptyset, \{1\}, \ldots, \{n\}\}$ are respectively the maximal and the minimal partition systems. Theorem 1 implies that every game restricted by a partition system $(N, \mathcal{F})$ satisfies

$$\tilde{v}^{\mathcal{F}}(S) = \sum_{T\in\Pi_S} v(T), \quad \text{for all } S \subseteq N.$$

The characterization of a partition system is given by the next theorem.

THEOREM 3. *A feasible coalition system $(N, \mathcal{F})$ is a partition system if and only if every $A \in \mathcal{F}$, $B \in \mathcal{F}$ with $A \cap B \neq \emptyset$ implies $A \cup B \in \mathcal{F}$.*

Proof. Suppose $A \cup B \notin \mathcal{F}$ for some pair $\{A, B\} \subset \mathcal{F}$ with $A \cap B \notin \emptyset$. Then, there exist two $\mathcal{F}$-components T_1, T_2 of $A \cup B$ such that $T_1 \supseteq A$ and $T_2 \supseteq B$. Hence, $T_1 \cap T_2 \supseteq A \cap B \neq \emptyset$, and this contradicts that the $\mathcal{F}$-components of $A \cup B$ form a partition, since $(N, \mathcal{F})$ is a partition system.

Conversely, if $(N, \mathcal{F})$ is a feasible coalition system, let $\Pi_S = \{T_1, \ldots, T_p\}$ be the family of $\mathcal{F}$-components of S. If Π_S is not a partition of S then $T_i \cap T_j \neq \emptyset$, and hence $T_i \cup T_j \in \mathcal{F}$, which contradicts the maximality of T_i and T_j. □

EXAMPLE 2. A *communication situation* is a triple (N, G, v), where (N, v) is a game and $G = (N, E)$ is a graph. This concept was first introduced in Myerson (1977), and investigated in Owen (1986), Borm et al. (1992). The pair $(N, \mathcal{F})$ with

$$\mathcal{F} = \{S \subseteq N : (S, E(S)) \text{ is a connected subgraph of } G\},$$

is a partition system. In this case, the game $\tilde{v}^{\mathcal{F}}$ is called a Γ-component additive game by Potters and Reijnierse (1995).

THEOREM 4. *Let $(N, \mathcal{F})$ be a partition system. If (N, v) is totally balanced, then so is $(N, \tilde{v}^{\mathcal{F}})$.*

Proof. We have to show that for all $S \subseteq N, S \neq \emptyset$ the induced subgames $(S, \tilde{v}_S^{\mathcal{F}})$ are balanced. If $S \in \mathcal{F}$, the subgame (S, v_S) is balanced and $v(S) = \tilde{v}^{\mathcal{F}}(S)$ then, according to part (c) of Theorem 2, it follows that so is the game $(S, \tilde{v}_S^{\mathcal{F}})$. If $S \notin \mathcal{F}$, let $\Pi_S = \{S_1, S_2, \ldots, S_k\}$ be the partition of S in $\mathcal{F}$-components, then by hypothesis $C(v_{S_t}) \notin \emptyset$, for all $t = 1, \ldots, k$.

To prove that the subgame $(S, \tilde{v}_S^{\mathcal{F}})$ is balanced, we consider the vectors

$$x^{S_1} \in C(v_{S_1}), \ldots, x^{S_k} \in C(v_{S_k}).$$

Note that every element $i \in S$ belongs to a unique $\mathcal{F}$-component of S. Hence, if $i \in S_p \subset S$ and $S_p \in \Pi_S$, then we can associate $i \mapsto x_i^{S_p}$, where $x_i^{S_p}$ is the component of the player i in the vector $x^{S_p} \in C(v_{S_p})$. It defines the vector $y \in \mathbb{R}^{|S|}$ such that $y_i = x_i^{S_p}$. The following equations imply that this vector is feasible in the induced subgame $(S, \tilde{v}_S^{\mathcal{F}})$,

$$y(S) = \sum_{i \in S} y_i = \sum_{i \in S} x_i^{S_p} = \sum_{Sp \in \Pi_S} \left(\sum_{i \in S_p} x_i^{S_p} \right)$$
$$= \sum_{Sp \in \Pi_S} x^{S_p}(S_p) = \sum_{Sp \in \Pi_S} v(S_p) = \tilde{v}_S^{\mathcal{F}}(S).$$

If $T \subseteq S$ then $\tilde{v}_S^{\mathcal{F}}(T) = \sum_{k=1}^{h} v(T_k)$, where $\Pi_T = \{T_1, T_2, \ldots, T_h\}$. Given $T_k \in \Pi_T$ there exists a unique j such that $T_k \subseteq S_j$ and $x^{S_j} \in C(v_{Sj})$. Therefore,

$$v(T_k) = v_{Sj}(T_k) \leqslant x^{Sj}(T_k) = \sum_{i \in T_k} x_i^{S_j}.$$

From this, we conclude that

$$\tilde{v}_S^{\mathcal{F}}(T) = \sum_{k=1}^{h} v(T_k) \leqslant \sum_{k=1}^{h} \left(\sum_{i \in T_k} x_i^{Sj} \right) = \sum_{i \in T} x_i^{Sj} = \sum_{i \in T} y_i = y(T).$$

□

4. INTERSECTING SYSTEMS

The convexity is not transmitted, in a general way, to the corresponding restricted game. Intersecting systems are a special class of partition systems, in which the convexity is hereditary.

EXAMPLE 3. Let (N, G, v) be a communication situation where the set of players is $N = \{1, 2, 3, 4\}$ and the worth function is $v(S) = |S| - 1$ if $S = \emptyset$, $v(\emptyset) = 0$. Note that v is a supermodular function. Let G be the graph diagrammed in Figure 1 and let $\mathcal{F}$ be the family of connected subgraphs of G. $(N, \mathcal{F})$ is a partition system and it is easy to see that v is convex while $\tilde{v}^{\mathcal{F}}$ is not convex. If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ then

$$\tilde{v}^{\mathcal{F}}(A \cup B) + \tilde{v}^{\mathcal{F}}(A \cap B) = 3 \ngeq \tilde{v}^{\mathcal{F}}(A) + \tilde{v}^{\mathcal{F}}(b) = 2 + 2.$$

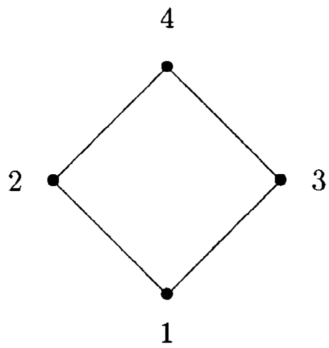

Figure 1. The four cycle.

We will show that the convexity of the game (N, v) is transmitted to the restricted game when we consider a special kind of partition system. Grötschel et al. (1988) introduce the following concept.

DEFINITION 5. *A feasible coalition system* $(N, \mathcal{F})$ *is an intersecting system if*

$$S, T \in \mathcal{F} \text{ with } S \cap T \neq \emptyset \Longrightarrow S \cap T \in \mathcal{F}, S \cup T \in \mathcal{F}. \quad (1)$$

Theorem 3 implies that if $(N, \mathcal{F})$ is an intersecting system then $(N, \mathcal{F})$ is a partition system which is stable for any intersections of the coalitions.

Let $\mathcal{F} \subseteq 2^N$ be a family of coalitions and let (N, v) be a game. Faigle (1989) introduces the family $\tilde{\mathcal{F}}$ of the coalitions that can be expressed as disjoint unions of elements of $\mathcal{F}$. That is if $A \in \tilde{\mathcal{F}}$ then $A = A_1 \cup \ldots \cup A_p$, where $A_i \in \mathcal{F}$, $i = 1, \ldots, p$, and $A_i \cap A_j = \emptyset$ for $i \neq j$. Faigle defines the game

$$\tilde{v} : \tilde{\mathcal{F}} \to \mathbb{R}, \qquad \tilde{v}(A) = \max \left\{ \sum_{i \in I} v(A_i) : \{A_i\}_{i \in I} \in \mathcal{P}_{\mathcal{F}}(A) \right\},$$

and proves (Faigle 1989, Lemma 11) that if $\mathcal{F}$ satisfies property (1) and the game v satisfies $v(A) + v(B) \leq v(A \cup B) + v(A \cap B)$, for all $A \in \mathcal{F}$, $B \in \mathcal{F}$, then

$$\tilde{v}(A) + \tilde{v}(B) \leqslant \tilde{v}(A \cup B) + \tilde{v}(A \cap B), \text{ for all } A \in \tilde{\mathcal{F}}, B \in \tilde{\mathcal{F}}.$$

The main difference between the definition given by Faigle and the one we consider is that in general $\tilde{\mathcal{F}} \neq 2^N$. In the following theorem, we consider a special partition system, which is stable for the intersection in order to obtain a result quite similar to the one in Faigle (1989).

THEOREM 5. *Let $(N, \mathcal{F})$ be an intersecting system. If the game (N, v) is convex then so is $(N, \tilde{v}^{\mathcal{F}})$.*

Proof. The family $(N, \mathcal{F})$ is a partition system whence satisfies $\tilde{\mathcal{F}} = 2^N$. We deduce that for all $S \subseteq N$,

$$\tilde{v}(S) = \sum_{T \in \Pi_S} v(T) = \max \left\{ \sum_{i \in I} v(S_i) : \{S_i\}_{i \in I} \in \mathcal{P}_{\mathcal{F}}(S) \right\} = \tilde{v}(S).$$

Then to obtain the result it is enough to apply Faigle's Lemma. □

5. PARTITION CONVEX GEOMETRIES

A special class of intersecting systems is formed by the *partition convex geometries*. First we define *convex geometries*, a notion developed to abstract convexity by Edelman and Jamison (1985). Let $\mathcal{F}$ be a family of subsets of a finite set N, that we call *convex sets*, $\cap$-stable and that contains N and the empty set. $\mathcal{F}$ is called an *alignment* of N. If $\mathcal{F}$ is an alignment of N, the intersection of all convex sets that contain $S \subseteq N$ is the *convex hull* of S that is denoted by $\mathcal{F}(S)$.

A point $i \in S$, where $S \in \mathcal{F}$ is an *extreme point* of S if the set $S \setminus \{i\} \in \mathcal{F}$. The set of all extreme points of S is denoted by $ex(S)$. The system $(N, \mathcal{F})$ is a convex geometry if $\mathcal{F}$ is an alignment of N that verifies the finite Minkowski–Krein-Milman property: any convex set is the convex hull of its extreme points.

DEFINITION 6. *A partition convex geometry is an intersecting system which is also a convex geometry.*

If $(N, \mathcal{F})$ is a partition convex geometry then $(N, \mathcal{F})$ is a partition system. Therefore, it is possible to consider the restricted game $\tilde{v}^{\mathcal{F}}(S) = \sum_{T \in \Pi_S} v(T)$, where Π_S is the partition of S in maximal convex sets.

EXAMPLE 4. A graph $G = (N, E)$ is connected if any two vertices can be joined by a path. A maximal connected subgraph of G is a *component* of G. A *cutvertex* is a vertex whose removal increases the number of components, and a *bridge* is an edge with the same property. A graph is *2-connected* if it is connected, has at least 3 vertices and contains no cutvertex. A subgraph B of a graph G is a *block* of G if either B is a bridge or else it is a maximal 2-connected subgraph of G.

A graph G is a *block graph* if every block is a complete graph. The block graphs are called *cycle-complete* graphs in van den Nouweland and Borm (1991). If G is a disjoint union of trees, then G is a block graph. Jamison (1985, Theorem 3.7) showed: $G = (N, E)$ *is a connected block graph* if and only if $(N, \mathcal{F})$, where $\mathcal{F}$ is the collection of subsets of N which induce connected subgraphs, is a convex geometry. Since, in every graph the union of connected subgraph with nonempty intersection is a connected subgraph, we have that $(N, \mathcal{F})$ is a partition convex geometry.

Van den Nouweland and Borm (1991, Theorem 1), showed that if (N, G, v) is a communication situation where the graph G is cycle-complete, the corresponding restricted game is convex when v is convex. The previous concepts allow to deduce that this hereditary property for convex games can be obtained as a direct consequence of Theorem 5. Indeed, any block-graph is a finite union of connected block-graphs. Therefore, it is a finite union of intersecting systems and, hence, an intersecting system. Thus, if (N, v) is a convex game and G is a block-graph, then Theorem 5 implies that the graph-restricted game is convex.

6. REMARKS ON THE NON-SUPERADDITIVE GAMES

In the above sections the game (N, v) is assumed superadditive and this implies that if $\mathcal{F}$ is a partition system then $\tilde{v}^{\mathcal{F}}(S) = \sum_{T \in \Pi_S} v(T)$. However, if the game (N, v) is not superadditive, the above equality may not be true. The following example illustrates this fact.

EXAMPLE 5. Let $N = \{1, 2, 3, 4\}$ and let $v : 2^N \to \mathbb{R}$ defined by

$$v(S) = \begin{cases} 3/4, & \text{if } |S| = 1 \\ 1, & \text{if } 2 \leqslant |S| \leqslant 3 \\ 3, & \text{if } S = N. \end{cases}$$

Clearly, (N, v) is not superadditive since

$$v(\{1, 2, 3\}) = 1 \not\geq v(\{1, 2\}) + v(\{3\}) = 7/4.$$

If now, we consider the partition system

$$\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, N\},$$

for the coalition $S = \{2, 3, 4\}$, we have that $\Pi_S = \{\{4\}, \{2, 3\}\}$ and

$$7/4 = \sum_{T \in \Pi_s} v(T) \neq \tilde{v}^{\mathcal{F}}(S) = 9/4.$$

A new type of restricted game can be defined in a partition system by

$$v^{\mathcal{F}}(S) = \sum_{T \in \Pi_s} v(T).$$

In general, we have $v^{\mathcal{F}}(S) \leq \tilde{v}^{\mathcal{F}}(S)$ and the equality is true when (N, v) is superadditive. This concept is a generalization of the Myerson's concept of graph-restricted game. If $v(N) = v^{\mathcal{F}}(N)$ then can be shown that:

(a) $C(v^{\mathcal{F}}) = \{x \in \mathbb{R}^N : x(N) = v(N), x(S) \geq v(S) \text{ for all } S \in \mathcal{F}\}$.
(b) $C(v) \subseteq C(v^{\mathcal{F}})$.
(c) If the game (N, v) is balanced then so is the restricted game $(N, v^{\mathcal{F}})$.

As a consequence, the conclusions of Theorem 4 are true too. Moreover, for $v^{\mathcal{F}}$ the results about the convexity are valid. As we already have noticed, if the game (N, v) is not superadditive, in general, $v^{\mathcal{F}} \neq \tilde{v}^{\mathcal{F}}$. However, the equality between the core of the game $(N, \tilde{v}^{\mathcal{F}})$ and $(N, v^{\mathcal{F}})$ is a consequence of the statement (a) and Theorem 2.

7. FINAL REMARKS

We introduce feasible coalitions systems $\mathbb{FS}$, partition systems $\mathbb{PS}$, intersecting systems $\mathbb{IS}$ and partition convex geometries $\mathbb{PCG}$. These set systems satisfy

$$\mathbb{PCG} \subsetneqq \mathbb{IS} \subsetneqq \mathbb{PS} \subsetneqq \mathbb{FS}.$$

We obtain the relationship between the cores of a cooperative game and the restricted game by a feasible coalitions system and by a partition system. We also show a characterization of the cores of restricted games, as well as the conditions which imply that the restricted game is convex. Intersecting systems are partition systems in which the convexity is hereditary. This means a generalization about the result of van den Nouweland and Borm by using some results that have been given by Faigle.

ACKNOWLEDGEMENTS

We would like to express our gratitude to Peter Borm and an anonymous referee for interesting comments and for their careful reading of the manuscript.

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