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Probabilistic Values on Convex Geometries *

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A game on a convex geometry is a real-valued function defined on the family $\mathcal{L}$ of the closed sets of a closure operator which satisfies the finite Minkowski-Krein-Milman property. If $\mathcal{L}$ is the Boolean algebra 2^N then we obtain an n-person cooperative game. We will extend the work of Weber on probabilistic values to games on convex geometries. As a result, we obtain a family of axioms that give rise to several probabilistic values and a unique Shapley value for games on convex geometries.

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1 Introduction

Let N be a finite set of n players. A cooperative game is essentially a function $v : 2^N \longrightarrow \mathbb{R}$, with $v(\emptyset) = 0$, which describes, for every subset of players, the maximal gain or minimal cost that these players can achieve if they decide to form a coalition. In this model it is generally assumed that there are no restrictions on cooperation and hence, every subset of players is a feasible coalition.

However, in many social and economic situations, this model does not apply. Examples are provided by local public goods which are supplied by local communities, social and sports clubs, labor unions, political parties, and other institutions. Situations like these were studied by Myerson (1977, 1980) and this line of research was continued by Owen (1986), van den Nouweland and Borm (1991), and Borm, Owen and Tijs (1992).

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Faigle and Kern (1992) proposed a model in which cooperation among players is restricted to some family of subsets of players, the *feasible* coalitions of the game. Their idea is to restrict the allowable coalitions by using underlying partially ordered sets.

In our model, we will define the feasible coalitions by using a combinatorial theory of convexity called convex geometries, which includes the model of Faigle and Kern. We propose that the feasible coalitions must verify the following axioms:

- If S and T are feasible coalitions then $S \cap T$ is a feasible coalition as well.
- If S is a feasible coalition and $S \neq N$, then there is a player j not in S such that $S \cup \{j\}$ is feasible.

The idea in this model is that if the players form coalitions because they agree on certain propositions, then intersections of feasible coalitions should also be feasible, since they agree on a larger set of propositions. On the other hand, every player i can always make common cause with any player j through mediators, because that way they will achieve their objective.

Let us briefly outline the contents. In the next section, convex geometries are defined and some of their properties described. We introduce games on convex geometries and consider a few examples, which show how convex geometries have already arisen in the works of the authors mentioned above.

In the last three sections we extend the work of Weber (1988) on probabilistic values to games on convex geometries. We introduce *values* for these games and observe in detail the axioms which characterize such values. In particular, compatible-order values are considered. These form a family of values which associate a set of imputations with each game. Finally, we show how to derive the Shapley value from a set of axioms.

2 Convex Geometries

Convex geometries are a combinatorial abstraction of convex sets introduced by Edelman and Jamison [3].

Definition 1
Let N be a finite set of cardinality n. A family $\mathcal{L}$ of subsets of N is a convex geometry if it satisfies the following properties:

(C1) $\emptyset \in \mathcal{L}$, *and* $\mathcal{L}$ *is closed under intersections,*

(C2) *If* $S \in \mathcal{L}$ *and* $S \neq N$, *then there exists* $j \in N \setminus S$ *such that* $S \cup j \in \mathcal{L}$.

Property (C1) implies that intersections of feasible coalitions should also be feasible, since the players agree on a profile of cooperation. In the model of *conference structures* by Myerson [10], two players are connected if they can be coordinated by meeting in separate conferences which have some members in common to serve as intermediaries. In our model, the coalitions of intermediaries are in the cooperation structure.

We call the sets in a convex geometry *convex sets*. A *maximal chain* of $\mathcal{L} \subseteq 2^N$ is an ordered collection of convex sets that is not contained in any larger chain. From property (C2) and by induction, Edelman and Jamison [3] showed that every maximal chain contains $n+1$ convex sets

$$\emptyset = S_0 \subset S_1 \subset \cdots \subset S_{n-1} \subset S_n = N,$$

and the cardinal $|S_k| = k$, for all $k = 0, 1, \ldots, n$. Thus, the *hierarchical situations* by Moulin [8], when users pay their incremental costs according to an ordering of N, can be modeled by convex geometries.

The family $\mathcal{L}$ gives rise to the operator $- : 2^N \longrightarrow 2^N$ defined by

$$A \longmapsto \overline{A} := \bigcap \{S \in \mathcal{L} \mid A \subseteq S\}.$$

It is easy to check that the operator $-$ has the properties of a closure operator, i.e., $A \subseteq \overline{A}$, $\overline{\overline{A}} = \overline{A}$, $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$, for all $A, B \subseteq N$, and the additional condition that $\emptyset = \overline{\emptyset}$. For $A \subseteq N$ an element $i \in A$ is an *extreme point* of A if $i \notin \overline{A \setminus i}$. For a convex set $S \in \mathcal{L}$ this is equivalent to $S \setminus i \in \mathcal{L}$. Let $ex(S)$ be the set of all extreme points of S. The convex geometries are the abstract closure spaces satisfying the finite Minkowski-Krein-Milman property: *Every convex set is the closure of its extreme points* [3].

3 Games on convex geometries

A *cooperative game* is a pair (N, v) where N is a finite set of n elements and $v : 2^N \longrightarrow \mathbb{R}$, is a function such that $v(\emptyset) = 0$. The *players* are the elements of N and the *coalitions* are the elements $S \subseteq N$ of the Boolean algebra 2^N.

Definition 2
A game on a convex geometry $\mathcal{L}$ is a function $v : \mathcal{L} \longrightarrow \mathbb{R}$, with $v(\emptyset) = 0$.

The coalitions are the convex sets of $\mathcal{L}$ and the players are the elements $i \in N$. Let $\Gamma(\mathcal{L})$ be the vector space of all games on the convex geometry $\mathcal{L} \subseteq 2^N$.

Example: A graph $G = (N, E)$ is connected if any two vertices can be joined by a path. A maximal connected subgraph of G is a *component* of G. A *cutvertex* is one whose removal increases the number of components, and a *bridge* is an edge with the same property. A graph is *2-connected* if it is connected, has at least 3 vertices and contains no cutvertex. A subgraph B of a graph G is a *block* of G if either B is a bridge or else it is a maximal 2-connected subgraph of G. A graph G is a *block graph* if every block is a complete graph. The block graphs are denoted by *cycle-complete* graphs in van den Nouweland and Borm [11].

A *communication situation* is a triple (N, G, v), where (N, v) is a game and $G = (N, E)$ is a graph. This concept was first introduced in Myerson [9], and investigated in Owen [12] and Borm, Owen and Tijs [2]. If $G = (N, E)$ is a connected block graph and

$$\mathcal{L} = \{S \subseteq N \mid (S, E(S)) \text{ is a connected subgraph of } G\},$$

then we have [3, Theorem 3.7] that $\mathcal{L}$ is a convex geometry.

Example: Edelman [5] studied *voting games* such that the feasible coalitions are the convex sets of the chain $\{1 < 2 < \cdots < n\}$ defined by the policy order left-right. A subset S of a partially ordered set (poset) $(P, \leq)$ is *convex* whenever $a \in S,\ b \in S$ and $a \leq b$ imply that the interval $[a, b] = \{x \in P \mid a \leq x \leq b\} \subseteq S$. If $(P, \leq)$ is finite, then each element is between a maximal and a minimal one and $ex(C)$ is the union of the maximal and minimal elements of C. The family of the convex sets $Co(P)$ is a convex geometry [3].

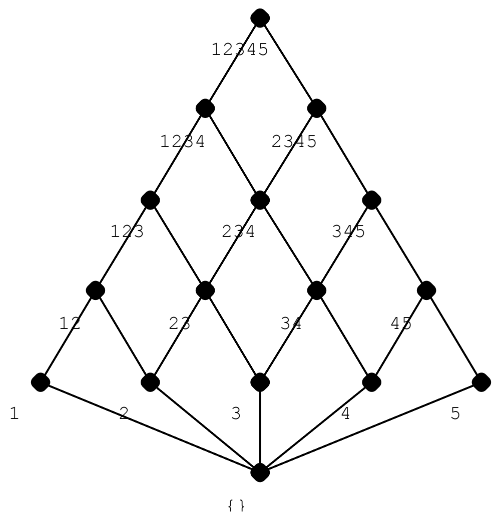

Figure1: The convex geometry $Co(\{1 < 2 < 3 < 4 < 5\})$

Example: Let $(P, \leq)$ be a poset. For any $X \subseteq P$,

$$X \longmapsto \overline{X} := \{y \in P \mid y \leq x \text{ for some } x \in X\},$$

defines a closure operator on $(P, \leq)$. Its closed sets are the *order ideals* (down sets) of $(P, \leq)$, and we use $J(P)$ to denote this collection. Since the union and intersection of order ideals is again an order ideal, it follows that $J(P)$ is a convex geometry closed under set-union and $ex(S)$ is the set of all maximal points $Max(S)$ of the subposet $S \in J(P)$. When $(P, \leq)$ is finite, there is a 1–1 correspondence between antichains of $(P, \leq)$ and order ideals. Then the games $(\mathcal{C}, v)$ and $(\mathcal{A}, c)$ of Faigle and Kern [6] [7], where $\mathcal{C}$ is the family of down sets of $(P, \leq)$ and $\mathcal{A}$ is the set of antichains of a *hierarchy,* are games on convex geometries.

4 Probabilistic values for games

We will follow the work of Weber [14] to obtain an axiomatic development of the *probabilistic values* for games on convex geometries. We consider the following games on $\mathcal{L}$. For any $T \in \mathcal{L}$, $T \neq \emptyset$ the *upper game*, denoted $\zeta_T : \mathcal{L} \longrightarrow \mathbb{R}$, is defined by

$$\zeta_T(S) := \begin{cases} 1, & \text{if } T \subseteq S \\ 0, & \text{otherwise.} \end{cases}$$

The *identity game* δ_T is defined by

$$\delta_T(S) := \begin{cases} 1, & \text{if } S = T \\ 0, & \text{if } S \neq T. \end{cases}$$

The collections of these games are two different bases of the vector space $\Gamma(\mathcal{L})$. Faigle and Kern [6] observed that

$$\zeta_T = \sum_{\{S \in \mathcal{L} \mid S \supseteq T\}} \delta_S.$$

Definition 3
A probabilistic value for $i \in N$ is a function $\Phi_i : \Gamma(\mathcal{L}) \longrightarrow \mathbb{R}$, which satisfies

$$\Phi_i(v) = \sum_{\{S \in \mathcal{L} \mid i \in ex(S)\}} p_S^i \left[v(S) - v(S \setminus i)\right], \quad p_S^i \geq 0, \qquad \sum_{\{S \in \mathcal{L} \mid i \in ex(S)\}} p_S^i = 1.$$

Remark 1
If we take $T = S \setminus i$, then we obtain the following expression

$$\Phi_i(v) = \sum_{\{T \in \mathcal{L} \,|\, i \notin T,\, T \cup i \in \mathcal{L}\}} p_T^i \left[v(T \cup i) - v(T) \right].$$

In this definition we only consider the coalitions $S \in \mathcal{L}$ such that $i \in S$ and $S \setminus i \in \mathcal{L}$. The Shapley value for games on $J(P)$ defined by Faigle and Kern [6], and its generalization to convex geometries by Bilbao and Edelman [1] are probabilistic values.

We will introduce a number of properties for values for games on convex geometries. First, we consider the linearity property.

Linearity axiom. For all $\alpha, \beta \in \mathbb{R}$, and $v, w \in \Gamma(\mathcal{L})$ we have

$$\Phi_i(\alpha v + \beta w) = \alpha \Phi_i(v) + \beta \Phi_i(w).$$

Theorem 4
Let $\Phi_i : \Gamma(\mathcal{L}) \longrightarrow \mathbb{R}$ be a value for i which satisfies the linearity axiom. Then there exists a unique set of coefficients $\left\{ a_S^i \,|\, S \in \mathcal{L},\ S \neq \emptyset \right\}$ such that

$$\Phi_i(v) = \sum_{S \in \mathcal{L}} a_S^i v(S).$$

Proof
The collection $\{\delta_S \,|\, S \in \mathcal{L},\ S \neq \emptyset\}$ is a basis of $\Gamma(\mathcal{L})$. Let $a_S^i = \Phi_i(\delta_S)$, for all $S \in \mathcal{L}$. Then $v = \sum_{S \in \mathcal{L}} v(S)\delta_S$, and the linearity axiom implies

$$\Phi_i(v) = \sum_{S \in \mathcal{L}} \Phi_i(\delta_S) v(S) = \sum_{S \in \mathcal{L}} a_S^i v(S).$$

□

We introduce the concept of *dummy player*.

Definition 5
The player $i \in N$ is a dummy in the game $v \in \Gamma(\mathcal{L})$ if, for every convex $T \in \mathcal{L}$ such that $i \notin T$ and $T \cup i \in \mathcal{L}$, we have

$$v(T \cup i) - v(T) = \begin{cases} v(\{i\}), & \textit{if } \{i\} \in \mathcal{L} \\ 0, & \textit{otherwise.} \end{cases}$$

Remark 2
A different definition has been suggested by Faigle and Kern [6].

We need some properties of the dummy players in the upper and identity games.

Proposition 6
Let $\mathcal{L}$ be a convex geometry and let $S \in \mathcal{L}$ be a non empty convex set. Then

1. *If $i \notin ex(S)$ then i is a dummy player in the upper game ζ_S.*
2. *If $i \in S \setminus ex(S)$ then i is a dummy player in the identity game δ_S.*
3. *If $i \notin S$ and $S \cup i \notin \mathcal{L}$ then i is a dummy player in the identity game δ_S.*

Proof
Let $S \in \mathcal{L}$, $S \neq \emptyset$ and let $i \in N$ be such that $i \notin ex(S)$. Note that if $\{i\} \in \mathcal{L}$ then $S \neq \{i\}$, hence $\zeta_S(\{i\}) = 0$ and $\delta_S(\{i\}) = 0$.

1. Let $i \notin ex(S)$. We suppose that there exists $T \in \mathcal{L}$, and $i \notin T$ such that $T \cup i \in \mathcal{L}$ and satisfies $\zeta_S(T \cup i) \neq \zeta_S(T)$. Then $\zeta_S(T \cup i) = 1$ and $\zeta_S(T) = 0$, hence $S \subseteq T \cup i$ and $S \nsubseteq T$. Thus $S \setminus i = S \cap T \in \mathcal{L}$, and we obtain $i \in ex(S)$, which is a contradiction. Hence, $\zeta_S(T \cup i) - \zeta_S(T) = 0$.

2. Let $i \in S \setminus ex(S)$. If $T \in \mathcal{L}$, $i \notin T$ and $T \cup i \in \mathcal{L}$, we have $T \cup i \neq S$ since $i \notin ex(S)$ and $T \neq S$ since $i \in S$. Then, it is obvious that $\delta_S(T \cup i) = \delta_S(T) = 0$.

3. Let $i \notin S$ and $S \cup i \notin \mathcal{L}$. If $T \in \mathcal{L}$, $i \notin T$ and $T \cup i \in \mathcal{L}$, it is clear that $\delta_S(T \cup i) = \delta_S(T) = 0$. □

Dummy axiom. If the player $i \in N$ is a dummy in $v \in \Gamma(\mathcal{L})$, then

$$\Phi_i(v) = \begin{cases} v(\{i\}), & \text{if } \{i\} \in \mathcal{L} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 7
Let $\Phi_i : \Gamma(\mathcal{L}) \longrightarrow \mathbb{R}$ be a value for i defined by $\Phi_i(v) = \sum_{S \in \mathcal{L}} a_S^i v(S)$ which satisfies the dummy axiom. Then for every game v,

$$\Phi_i(v) = \sum_{\{S \in \mathcal{L} \,|\, i \in ex(S)\}} a_S^i \left[v(S) - v(S \setminus i) \right].$$

Moreover, if $\{i\} \in \mathcal{L}$ then $\displaystyle\sum_{\{S \in \mathcal{L} \,|\, i \in ex(S)\}} a_S^i = 1.$

Proof
Let $E_i : \Gamma(\mathcal{L}) \longrightarrow \mathbb{R}$ be defined for $i \in N$ by

$$E_i(v) := \sum_{\{S \in \mathcal{L} \,|\, i \in ex(S)\}} a_S^i \left[v(S) - v(S \setminus i) \right].$$

The operators E_i and Φ_i are linear and the upper games form a basis of $\Gamma(\mathcal{L})$. Then it will be enough to show that $\Phi_i(\zeta_T) = E_i(\zeta_T)$, for every convex set $T \in \mathcal{L}$, $T \neq \emptyset$. Let us fix $i \in N$ and $T \in \mathcal{L}$, and consider two cases.

First, if $i \notin ex(T)$ then Proposition 6 implies that i is a dummy player in the game ζ_T. Then $\zeta_T(S) - \zeta_T(S \setminus i) = 0$, for all $S \in \mathcal{L}$, such that $i \in ex(S)$ because $\zeta_T(\{i\}) = 0$, when $\{i\} \in \mathcal{L}$. Therefore, the definition of E_i yields $E_i(\zeta_T) = 0$. Furthermore, the dummy axiom implies that $\Phi_i(\zeta_T) = 0$.

Now, if we suppose that $i \in ex(T)$ then $i \in T$ and hence $\zeta_T(S \setminus i) = 0$. Thus, we obtain the equivalence $\zeta_T(S) - \zeta_T(S \setminus y) = 1$ if and only if $S \in \mathcal{L}$ and $S \supseteq T$.

Observe that

$$\begin{aligned} E_i(\zeta_T) &= \sum_{\{S \in \mathcal{L} \,|\, i \in ex(S),\, S \supseteq T\}} a_S^i \\ &= \sum_{\{S \in \mathcal{L} \,|\, i \in ex(S),\, S \supseteq T\}} \Phi_i(\delta_S) \\ &= \Phi_i \left(\sum_{\{S \in \mathcal{L} \,|\, S \supseteq T\}} \delta_S \right) \\ &= \Phi_i(\zeta_T), \end{aligned}$$

where the last but one equation follows from $\Phi_i(\delta_S) = 0$, when $i \in S \setminus ex(S)$.

Finally, if $\{i\} \in \mathcal{L}$ then i is a dummy in the game $\zeta_{\{i\}}$. Let T be a convex set such that $i \notin T$ and $T \cup i \in \mathcal{L}$. Note that $\zeta_{\{i\}}(T \cup i) = 1,\ \zeta_{\{i\}}(T) = 0,\ \zeta_{\{i\}}(\{i\}) = 1$. Because Φ_i satisfies the dummy axiom, it follows from the above equation that

$$\begin{aligned} \sum_{\{S \in \mathcal{L} \,|\, i \in ex(S)\}} a_S^i &= \sum_{\{S \in \mathcal{L} \,|\, i \in ex(S)\}} \Phi_i(\delta_S) \\ &= \sum_{\{S \in \mathcal{L} \,|\, i \in S\}} \Phi_i(\delta_S) \\ &= \Phi_i \left(\sum_{\{S \in \mathcal{L} \,|\, i \in S\}} \delta_S \right) \\ &= \Phi_i(\zeta_{\{i\}}) = \zeta_{\{i\}}(\{i\}) = 1. \end{aligned}$$

□

We consider a convex geometry with $\{i\} \notin \mathcal{L}$ for some $i \in N$. In this case, note that

$$\sum_{\{S \in \mathcal{L} \,|\, i \in ex(S)\}} a_S^i = \sum_{\{S \in \mathcal{L} \,|\, i \in S\}} a_S^i = \Phi_i \left(\zeta_{\overline{\{i\}}} \right),$$

where the upper game on the closure of $\{i\}$ satisfies

$$\zeta_{\overline{\{i\}}}(T) = \begin{cases} 1, & \text{if } \overline{\{i\}} \subseteq T \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} 1, & \text{if } i \in T \\ 0, & \text{otherwise.} \end{cases}$$

We obtain the following result by the application of Theorems 4 and 7.

Theorem 8
Let $\Phi_i : \Gamma(\mathcal{L}) \longrightarrow \mathbb{R}$ be a value for i which satisfies the linearity and dummy axioms. Then for every game v, there is a collection $\left\{a_S^i \mid S \in \mathcal{L},\ i \in ex(S)\right\}$ such that

$$\Phi_i(v) = \sum_{\{S \in \mathcal{L} \,|\, i \in ex(S)\}} a_S^i \left[v(S) - v(S \setminus i)\right].$$

Moreover, if $\Phi_i\left(\zeta_{\overline{\{i\}}}\right) = 1$ then $\displaystyle\sum_{\{S \in \mathcal{L} \,|\, i \in ex(S)\}} a_S^i = 1.$

Remark 3
For a convex geometry which satisfies $\{i\} \in \mathcal{L}$, for all $i \in N$, condition $\Phi_i\left(\zeta_{\overline{\{i\}}}\right) = 1$ is not necessary, because it follows from the dummy axiom.

Definition 9
A game $v \in \Gamma(\mathcal{L})$ is monotonic if $v(S) \leq v(T)$ for all $S, T \in \mathcal{L}$ such that $S \subseteq T$.

Monotonicity axiom. If $v \in \Gamma(\mathcal{L})$ is monotonic then $\Phi_i(v) \geq 0$.

Theorem 10
Let $\Phi_i : \Gamma(\mathcal{L}) \longrightarrow \mathbb{R}$ be a value for i defined, for every game v, by

$$\Phi_i(v) = \sum_{\{S \in \mathcal{L} \,|\, i \in ex(S)\}} a_S^i \left[v(S) - v(S \setminus i)\right].$$

If the value Φ_i satisfies the monotonicity axiom, then $a_S^i \geq 0$, for all $S \in \mathcal{L}$ such that $i \in ex(S)$.

Proof
For any $T \in \mathcal{L}$, we consider the game

$$\widehat{\zeta}_T(S) := \begin{cases} 1, & \text{if } T \subset S \\ 0, & \text{otherwise.} \end{cases}$$

The game $\widehat{\zeta}_T$ is monotonic, hence $\Phi_i\left(\widehat{\zeta}_T\right) \geq 0$. On the other hand, for every $T \in \mathcal{L}$ such that $i \notin T$ and $T \cup i \in \mathcal{L}$, we obtain the identity

$$\Phi_i\left(\widehat{\zeta}_T\right) = \sum_{\{S \in \mathcal{L} \,|\, i \in ex(S)\}} a_S^i \left[\widehat{\zeta}_T(S) - \widehat{\zeta}_T(S \setminus i)\right] = a_{T \cup i}^i.$$

□

It is easy to check that every probabilistic value satisfies the previous axioms. Thus, we have the following theorems.

Theorem 11
Let $\Phi_i : \Gamma(\mathcal{L}) \longrightarrow \mathbb{R}$ be a value for $i \in N$ such that $\{i\} \in \mathcal{L}$. Then Φ_i satisfies the linearity, dummy, and monotonicity axioms if and only if Φ_i is a probabilistic value.

Theorem 12
Let $\Phi_i : \Gamma(\mathcal{L}) \longrightarrow \mathbb{R}$ be a value for $i \in N$. Then Φ_i satisfies the linearity, dummy, monotonicity axioms and $\Phi_i\left(\zeta_{\overline{\{i\}}}\right) = 1$ if and only if Φ_i is a probabilistic value.

5 Efficiency and compatible-order values

Let Φ be a group value $\Phi : \Gamma(\mathcal{L}) \longrightarrow \mathbb{R}^n,\ v \longmapsto (\Phi_1(v), \ldots, \Phi_n(v))$. If the payoff vector $\Phi(v)$ is a distribution of the available resources of the coalition N, then Φ satisfies the following axiom:

Efficiency axiom. If N is the set of all players of $v \in \Gamma(\mathcal{L})$ then

$$\sum_{i \in N} \Phi_i(v) = v(N).$$

Theorem 13
Let $\Phi : \Gamma(\mathcal{L}) \longrightarrow \mathbb{R}^n$ be a group value that satisfies efficiency axiom. If Φ_i satisfies linearity, dummy, and monotonicity axioms for every $i \in N$ then Φ_i is a probabilistic value.

Proof
Theorem 12 implies that it suffices to show that $\Phi_i\left(\zeta_{\overline{\{i\}}}\right) = 1$, for all $i \in N$. We claim that $ex\left(\overline{\{i\}}\right) = \{i\}$, for each $i \in N$. To verify this claim note that in

a convex geometry $ex\left(\overline{\{i\}}\right) \neq \emptyset$. Suppose $j \in ex\left(\overline{\{i\}}\right)$ for some $j \neq i$, then $\overline{\{i\}} \setminus j \in \mathcal{L}$. Thus, we obtain $i \in \overline{\{i\}} \setminus j$, hence by definition of the closure operator $\overline{\{i\}} \subseteq \overline{\{i\}} \setminus j$, and this is a contradiction. It follows from Proposition 6 that every $j \neq i$ satisfies $\Phi_j\left(\zeta_{\overline{\{i\}}}\right) = 0$, because $j \notin ex\left(\overline{\{i\}}\right)$. Therefore, the efficiency axiom implies that

$$\sum_{j \in N} \Phi_j\left(\zeta_{\overline{\{i\}}}\right) = \Phi_i\left(\zeta_{\overline{\{i\}}}\right) = 1.$$

□

The efficiency axiom implies the following properties for the coefficients of the values that satisfy the linearity and dummy axioms.

Theorem 14
Let $\Phi : \Gamma(\mathcal{L}) \longrightarrow \mathbb{R}^n$ be a group value defined, for every game v and for all $i \in N$, by

$$\Phi_i(v) = \sum_{\{S \in \mathcal{L} \,|\, i \in ex(S)\}} a_S^i \left[v(S) - v(S \setminus i)\right].$$

Then the group value Φ satisfies the efficiency axiom if and only if

$$\sum_{i \in ex(N)} a_N^i = 1, \quad \sum_{i \in ex(S)} a_S^i = \sum_{\{j \notin S \,|\, S \cup j \in \mathcal{L}\}} a_{S \cup j}^j, \text{ for all } S \in \mathcal{L},\ S \neq \emptyset,\ S \neq N.$$

Proof
For any $v \in \Gamma(\mathcal{L})$ we have

$$\begin{aligned} \sum_{i \in N} \Phi_i(v) &= \sum_{i \in N} \sum_{\{S \in \mathcal{L} \,|\, i \in ex(S)\}} a_S^i \left[v(S) - v(S \setminus i)\right] \\ &= \sum_{S \in \mathcal{L}} v(S) \left[\sum_{i \in ex(S)} a_S^i - \sum_{\{j \notin S \,|\, S \cup j \in \mathcal{L}\}} a_{S \cup j}^j\right]. \end{aligned}$$

If the coefficients satisfy the equations of the hypothesis, then $\sum_{i \in N} \Phi_i(v) = v(N)$ and Φ satisfies the efficiency axiom.

Conversely, let us fix a non empty set $T \in \mathcal{L}$, and consider the identity game δ_T. If we apply the previous equation to the game δ_T, then we have

$$\sum_{i \in N} \Phi_i(\delta_T) = \begin{cases} \sum_{i \in ex(N)} a_N^i, & \text{if } T = N \\ \sum_{i \in ex(S)} a_S^i - \sum_{\{j \notin S \,|\, S \cup j \in \mathcal{L}\}} a_{S \cup j}^j, & \text{if } T = S \neq N. \end{cases}$$

Thus, if Φ satisfies the efficiency axiom then the relations are true. □

Edelman and Jamison [3] defined a *compatible ordering* of a convex geometry $\mathcal{L} \subseteq 2^N$ as a total ordering of the elements of N, $i_1 < i_2 < \cdots < i_n$ such that the set $\{i_1, i_2, \ldots, i_j\} \in \mathcal{L}$, for all $1 \leq j \leq n$.

A compatible ordering of $\mathcal{L}$ corresponds exactly to a maximal chain in $\mathcal{L}$ [4, Prop. 2.2]. Let $c([T, S])$ denote the number of maximal chains from T to S, where $T \subset S$ and $c(S) := c([\emptyset, S])$ is the number of maximal chains from $\emptyset$ to $S \neq \emptyset$. We use $c(\mathcal{L})$ to denote the total number of maximal chains and $\mathcal{C}(\mathcal{L})$ to denote the set of all compatible orderings. Given an element $i \in N$ and a compatible ordering $C = (i_1, i_2, \ldots, i_n)$ of $\mathcal{L}$, let $C(i) := \{j \in N \mid j \leq i \text{ in } C\}$.

Definition 15
Let $\{p_C \mid C \in \mathcal{C}(\mathcal{L})\}$ be a probability distribution. A compatible-order value on $\Gamma(\mathcal{L})$ is a function $\Omega : \Gamma(\mathcal{L}) \longrightarrow \mathbb{R}^n$, $v \longmapsto (\Omega_1(v), \ldots, \Omega_n(v))$, defined by

$$\Omega_i(v) = \sum_{C \in \mathcal{C}(\mathcal{L})} p_C \left[v\left(C(i)\right) - v\left(C(i) \setminus i\right)\right], \quad \text{for every } i \in N.$$

The relationship between compatible-order values and probabilistic values that satisfy the efficiency axiom is a matter of the following theorems.

Theorem 16
Let $\Omega = (\Omega_1, \ldots \Omega_n)$ be a compatible-order value on $\Gamma(\mathcal{L})$. Then Ω satisfies the efficiency axiom and each component Ω_i is a probabilistic value for every $i \in N$.

Proof
For every $i \in N$, and $v \in \Gamma(\mathcal{L})$ we have

$$\begin{aligned} \Omega_i(v) &= \sum_{C \in \mathcal{C}(\mathcal{L})} p_C \left[v\left(C(i)\right) - v\left(C(i) \setminus i\right)\right] \\ &= \sum_{\{S \in \mathcal{L} \,|\, i \in ex(S)\}} \left(\sum_{\{C \in \mathcal{C}(\mathcal{L}) \,|\, C(i) = S\}} p_C \right) \left[v(S) - v(S \setminus i)\right]. \end{aligned}$$

The coefficients satisfy

$$\sum_{\{S \in \mathcal{L} \,|\, i \in ex(S)\}} \left(\sum_{\{C \in \mathcal{C}(\mathcal{L}) \,|\, C(i) = S\}} p_C \right) = \sum_{C \in \mathcal{C}(\mathcal{L})} p_C = 1,$$

hence Ω_i is a probabilistic value. Furthermore, for every game $v \in \Gamma(\mathcal{L})$ we have

$$\begin{aligned}\sum_{i \in N} \Omega_i(v) &= \sum_{i \in N} \sum_{C \in \mathcal{C}(\mathcal{L})} p_C \left[v\left(C(i)\right) - v\left(C(i) \setminus i\right)\right] \\ &= \sum_{C \in \mathcal{C}(\mathcal{L})} p_C \left(\sum_{i \in N} \left[v\left(C(i)\right) - v\left(C(i) \setminus i\right)\right] \right) \\ &= \sum_{C \in \mathcal{C}(\mathcal{L})} p_C \left[v(N) - v(\emptyset)\right] \\ &= v(N).\end{aligned}$$

□

Theorem 17
Let $\Phi : \Gamma(\mathcal{L}) \longrightarrow \mathbb{R}^n$ be a group value that satisfies the efficiency axiom such that Φ_i is a probabilistic value for all $i \in N$. Then Φ is a compatible-order value.

Proof
The hypothesis implies that

$$\Phi_i(v) = \sum_{\{S \in \mathcal{L} \,|\, i \in ex(S)\}} p_S^i \left[v(S) - v(S \setminus i)\right], \quad \text{for all } i \in N.$$

For any $T \in \mathcal{L}$ and $i \notin T$ such that $T \cup i \in \mathcal{L}$, we define

$$A(i,T) = \begin{cases} \dfrac{p_{T \cup i}^i}{\displaystyle\sum_{\{j \notin T \,|\, T \cup j \in \mathcal{L}\}} p_{T \cup j}^j}, & \text{if the denominator is not equal to } 0 \\ 0, & \text{otherwise.} \end{cases}$$

For any compatible ordering $C \in \mathcal{C}(\mathcal{L})$ such that the associated maximal chain is $i_1 < i_2 < \cdots < i_n$, we define

$$p_C = p_{\{i_1\}}^{i_1} \, A\left(i_2, \{i_1\}\right) \, A\left(i_3, \{i_1, i_2\}\right) \cdots A\left(i_n, \{i_1, i_2, \ldots, i_{n-1}\}\right),$$

where $C = (i_1, i_2, \ldots, i_n)$ and the last factor is 1. The collection $\{p_C \,|\, C \in \mathcal{C}(\mathcal{L})\}$ satisfies

$$\begin{aligned}\sum_{C \in \mathcal{C}(\mathcal{L})} p_C &= \sum_{\{i_1 \,|\, \{i_1\} \in \mathcal{L}\}} \; \sum_{\{i_2 \notin \{i_1\} \,|\, \{i_1, i_2\} \in \mathcal{L}\}} \cdots \sum_{i_n \notin \{i_1, \ldots, i_{n-1}\}} p_{(i_1, \ldots, i_n)} \\ &= \sum_{\{i_1 \,|\, \{i_1\} \in \mathcal{L}\}} p_{\{i_1\}}^{i_1} \\ &= \sum_{i \in N} \Phi_i\left(\widehat{\zeta}_{\emptyset}\right) \\ &= 1,\end{aligned}$$

where the last equation follows from the efficiency axiom applied to the game $\widehat{\zeta_\emptyset}$, defined by $\widehat{\zeta_\emptyset}(S) = 1$, if $S \neq \emptyset$. Hence $\{p_C \mid C \in \mathcal{C}(\mathcal{L})\}$ is a probability distribution, and the compatible-order value associated is

$$\Omega_i(v) = \sum_{\{S \in \mathcal{L} \mid i \in ex(S)\}} \left(\sum_{\{C \in \mathcal{C}(\mathcal{L}) \mid C(i) = S\}} p_C \right) [v(S) - v(S \setminus i)], \quad \text{for all } i \in N.$$

Thus, $\Phi_i = \Omega_i$ if for every $S \in \mathcal{L}$ and $i \in ex(S)$, the coefficients satisfy

$$p_S^i = \sum_{\{C \in \mathcal{C}(\mathcal{L}) \mid C(i) = S\}} p_C.$$

Any maximal chain $C_1 \in \mathcal{C}\left([\emptyset, S \setminus i]\right)$ and $C_2 \in \mathcal{C}\left([S, N]\right)$ can be concatenated to make a chain $C_1 \cup \{i\} \cup C_2 \in \mathcal{C}(\mathcal{L})$. Then we obtain, with $s = |S|$,

$$\begin{aligned}
&\sum_{\{C \in \mathcal{C}(\mathcal{L}) \mid C(i) = S\}} p_C \\
= \;& \sum_{i_{s-1} \in ex(S \setminus i)} \sum_{i_{s-2} \in ex(S \setminus \{i, i_{s-1}\})} \cdots \sum_{i_1 \in ex(S \setminus \{i, i_{s-1}, \ldots, i_2\})} \sum_{\{i_{s+1} \notin S \mid S \cup i_{s+1} \in \mathcal{L}\}} \\
&\cdots \sum_{\{i_n \notin S \cup \{i_{s+1}, \ldots, i_{n-1}\} \mid S \cup \{i_{s+1}, \ldots, i_n\} \in \mathcal{L}\}} p_{(i_1, \ldots, i_{s-1}, i, i_{s+1}, \ldots, i_n)} \\
= \;& A(i, S \setminus i) \sum_{i_{s-1} \in ex(S \setminus i)} A\left(i_{s-1}, S \setminus \{i, i_{s-1}\}\right) \cdots \\
&\sum_{i_1 \in ex(S \setminus \{i, i_{s-1} \ldots, i_2\})} p_{\{i_1\}}^{i_1} \sum_{\{i_{s+1} \notin S \mid S \cup i_{s+1} \in \mathcal{L}\}} A(i_{s+1}, S) \cdots \\
&\sum_{\{i_n \notin S \cup \{i_{s+1}, \ldots, i_{n-1}\} \mid S \cup \{i_{s+1}, \ldots, i_n\} \in \mathcal{L}\}} A\left(i_n,\, S \cup \{i_{s+1}, \ldots, i_{n-1}\}\right).
\end{aligned}$$

The first s factors are equal to

$$\begin{aligned}
&\frac{p_S^i}{\displaystyle\sum_{\{j \notin S \setminus i \mid (S \setminus i) \cup j \in \mathcal{L}\}} p_{(S \setminus i) \cup j}^j} \\
&\cdot \sum_{i_{s-1} \in ex(S \setminus i)} \left(\frac{p_{S \setminus i}^{i_{s-1}}}{\displaystyle\sum_{\{j \notin S \setminus \{i, i_{s-1}\} \mid (S \setminus \{i, i_{s-1}\}) \cup j \in \mathcal{L}\}} p_{(S \setminus \{i, i_{s-1}\}) \cup j}^j} \right) \\
&\cdots \sum_{i_1 \in ex(S \setminus \{i, i_{s-1}, \ldots, i_2\})} p_{\{i_1\}}^{i_1} \\
&= p_S^i,
\end{aligned}$$

because the equations of Theorem 14 imply that the denominator of a factor is equal to the next numerator. Furthermore, the following factor for $s+1 \leq k \leq n$, with $T_k = S \cup \{i_{s+1}, \ldots, i_k\}$, satisfies

$$\sum_{\{i_k \notin T_{k-1} \,|\, T_{k-1} \cup i_k \in \mathcal{L}\}} A(i_k, T_{k-1}) =$$

$$\sum_{\{i_k \notin T_{k-1} \,|\, T_{k-1} \cup i_k \in \mathcal{L}\}} \left(\frac{p^{i_k}_{T_{k-1} \cup i_k}}{\sum_{\{j \notin T_{k-1} \,|\, T_{k-1} \cup j \in \mathcal{L}\}} p^{j}_{T_{k-1} \cup j}} \right) = 1.$$

Therefore, the result holds. □

6 Axioms for the Shapley value

A set $U \in 2^N$ is called a *carrier* for a game v if $v(S) = v(S \cap U)$ for all $S \in 2^N$. The carrier axiom states: If U is a carrier of v then $\sum_{i \in U} \Phi_i(v) = v(U)$. In the classical characterization, the *Shapley value* is the only value that satisfies the carrier, symmetry and additivity axioms on the class of the superadditive games [13]. For the class of all games, Weber [14, Theorem 15] considered the linearity, dummy, symmetry, and efficiency axioms, and proved the uniqueness of the Shapley value. This value for the player $i \in N$ is

$$\Phi_i(N, v) = \sum_{\{S \in 2^N \,|\, i \in S\}} \frac{(s-1)!(n-s)!}{n!} \left[v(S) - v(S \setminus i) \right],$$

where $n = |N|$ and $s = |S|$.

Let $S \in \mathcal{L}$ and $i \in S$. We define the *hierarchical strength* $h_S(i)$ of i in S as

$$h_S(i) := \frac{|\{C \in \mathcal{C}(\mathcal{L}) \mid C(i) \cap S = S\}|}{|\mathcal{C}(\mathcal{L})|},$$

i.e., $h_S(i)$ is the average number of compatible orderings of $\mathcal{L}$ in which i is the last member of S in the ordering. Note that $h_S(i)$ satisfies $h_S(i) \neq 0$ if and only if $i \in ex(S)$.

Faigle and Kern [6] proposed the following axiom for games on the lattice $J(P)$ of the order ideals of a poset $(P, \leq)$.

Hierarchical strength axiom. For any $S \in J(P)$ and $i, j \in S$,

$$h_S(i)\Phi_j(\zeta_S) = h_S(j)\Phi_i(\zeta_S).$$

They provided [6, Theorems 1 and 2] the following characterization of the Shapley value for a game $v : J(P) \longrightarrow \mathbb{R}$.

There is a unique function $\Phi : v \longmapsto (\Phi_1(v), \ldots, \Phi_n(v))$ satisfying the axioms of linearity, carrier and hierarchical strength. Moreover, for every $i \in P$,

$$\Phi_i(v) = \sum_{\{T \in J(P) \mid i \in Max(T)\}} \frac{e(T \setminus i)\, e(P \setminus T)}{e(P)} \left[v(T) - v(T \setminus i) \right],$$

where $e(\cdot)$ is the number of linear extensions of the corresponding subposets of $(P, \leq)$.

Bilbao and Edelman [1] generalized this formula for the Shapley value for games on convex geometries. In this context, the number of linear extensions e is the number of maximal chains c.

Definition 18
Let $v \in \Gamma(\mathcal{L})$ be a game on a convex geometry. The Shapley value of the game v is given by, for all $i \in N$,

$$\Phi_i(v) := \sum_{\{S \in \mathcal{L} \mid i \in ex(S)\}} \frac{c(S \setminus i)\, c([S, N])}{c(N)} \left[v(S) - v(S \setminus i) \right],$$

where $c([N, N]) = c(\emptyset) = 1$.

Now, we introduce a new axiom, in which the value of the player depends on the number of maximal chains.

Chain axiom. For any $S \in \mathcal{L}$ and $i, j \in ex(S)$,

$$c(S \setminus i)\Phi_j(\delta_S) = c(S \setminus j)\, \Phi_i(\delta_S).$$

By using our previous results, we prove the following characterization for the Shapley value on convex geometries.

Theorem 19
The Shapley value is the unique function $\Phi : \Gamma(\mathcal{L}) \longrightarrow \mathbb{R}^n$ that satisfies the linearity, dummy, efficiency, and chain axioms.

Proof
It is obvious that the Shapley value satisfies the properties. Conversely, it follows from Theorems 8 and 14 that, for every $i \in N$, $\left\{ a_S^i \mid S \in \mathcal{L},\, i \in ex(S) \right\}$ is a collection such that

$$\Phi_i(v) = \sum_{\{S \in \mathcal{L} \mid i \in ex(S)\}} a_S^i \left[v(S) - v(S \setminus i) \right],$$

$$\begin{aligned}
\sum_{i \in ex(N)} a_N^i &= 1, \\
\sum_{i \in ex(S)} a_S^i &= \sum_{\{j \notin S \,|\, S \cup j \in \mathcal{L}\}} a_{S \cup j}^j, \quad \text{for all } S \in \mathcal{L}, \;\; S \neq \emptyset, \;\; S \neq N.
\end{aligned}$$

Therefore, it suffices to show that

$$a_S^i = \frac{c(S \setminus i)\; c([S,N])}{c(N)}, \quad \text{for all } \; S \in \mathcal{L} \; \text{ and } \; i \in ex(S).$$

The coefficients are $a_S^i = \Phi_i(\delta_S)$, hence the chain axiom implies that for all pairs $i, j \in ex(S)$, $a_S^j \, c(S \setminus i) = a_S^i \, c(S \setminus j)$. If we fix $i \in ex(S)$, then we obtain

$$\begin{aligned}
\sum_{j \in ex(S)} a_S^j &= a_S^i + \sum_{\{j \in ex(S) \,|\, j \neq i\}} \frac{c(S \setminus j)}{c(S \setminus i)} \, a_S^i \\
&= \frac{a_S^i}{c(S \setminus i)} \sum_{j \in ex(S)} c(S \setminus j) \\
&= a_S^i \frac{c(S)}{c(S \setminus i)}.
\end{aligned}$$

For $S = N$ we take the first efficiency equation and hence $c(N \setminus i) = a_N^i \, c(N)$, for every $i \in ex(N)$. Thus, $a_N^i = c(N \setminus i)\; c([N,N]) \,/ c(N)$, for all $i \in ex(N)$. We assume the following induction hypothesis: For every $T \in \mathcal{L}$, with $|T| = k \geq 2$ we have

$$a_T^i = \frac{c(T \setminus i)\; c([T,N])}{c(N)}, \quad \text{for all } i \in ex(T).$$

The case $k = n$ has just been proved. Let $S \in \mathcal{L}$, such that $|S| = k - 1 < n$. Then $S \neq N$, and the efficiency equations imply that

$$\begin{aligned}
\sum_{i \in ex(S)} a_S^i &= \sum_{\{j \notin S \,|\, S \cup j \in \mathcal{L}\}} a_{S \cup j}^j \\
&= \sum_{\{j \notin S \,|\, S \cup j \in \mathcal{L}\}} \frac{c(S)\; c([S \cup j, N])}{c(N)} \\
&= \frac{c(S)}{c(N)} \sum_{\{j \notin S \,|\, S \cup j \in \mathcal{L}\}} c([S \cup j, N]) \\
&= \frac{c(S)\, c([S,N])}{c(N)},
\end{aligned}$$

where the second equation follows from the induction hypothesis for $T = S \cup j$. Finally, for every $i \in ex(S)$, the identity

$$a_S^i \frac{c(S)}{c(S \setminus i)} = \frac{c(S)\, c([S,N])}{c(N)} \quad \text{implies} \quad a_S^i = \frac{c(S \setminus i)\, c([S,N])}{c(N)}.$$

□

Remark 4
Let $\{p_C \mid C \in \mathcal{C}(\mathcal{L})\}$ *be a probability distribution such that* $p_C = 1/c(N)$, *for all* $C \in \mathcal{C}(\mathcal{L})$. *If* Ω *is the compatible-order associated, then* Ω *is the Shapley value.*

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