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A Shapley measure of power in hierarchies

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ABSTRACT

A method for measuring positional power in hierarchies is proposed. Inspired by models of cooperative TU-games with restricted cooperation, such as permission structures, we model hierarchies by means of a certain kind of set games, which we have called authorization operators. We then define and characterize a value for authorization operators that allows us to quantify the power of each agent. This power is decomposed into two terms: sovereignty and influence. Sovereignty describes the autonomy of an agent. Influence indicates their capacity to block the actions of others.

1. Introduction

Numerous studies have been carried out on positional power in relational structures. Several of these studies are based on cooperative game theory. For instance, Molinero et al. [9] used the Shapley–Shubik and Banzhaf indices to measure power in social networks. In this paper, concepts of cooperative game theory will be employed to define a measure of power in hierarchies. Our starting point is the model for games with restricted cooperation introduced by Gilles et al. [6]. They introduced the concept of permission structure, which is a mapping that assigns to each agent a subset of direct subordinates. We aim to quantify the positional power of each agent in a permission structure. Since a permission structure is given by a digraph, the use of a measure of power in digraphs, such as that introduced by Herings et al. [7], could be contemplated. However, this measure would be unsuitable because, depending on the interpretation of the superior-subordinate relationship, a permission structure can represent different hierarchies. For instance, in the conjunctive approach, introduced by Gilles et al. [6] and generalized by Gallardo et al. [5], it is assumed that all agents need permission from all their superiors, whereas in the disjunctive approach, van den Brink [2] assumes that all agents need permission from just one of their predecessors. Other interpretations could also be considered. For instance, it could be assumed that all agents need permission from the majority of their predecessors. It can therefore be concluded that the same digraph can represent different hierarchical structures. That is, a digraph does not fully describe a hierarchy. It was hence decided to model hierarchies by means of another mathematical structure. To this end, we took into consideration that Gilles, Owen and van den Brink defined a set function that assigns to each coalition the set of players that are allowed to cooperate within that coalition. In fact, this set function fully describes the hierarchical structure. Bearing this in mind, we decided to model hierarchies by using a certain kind of set functions, called authorization operators, which are the crisp version of the fuzzy authorization operators introduced by Gallardo et al. [4]. Our next goal was to define a value for authorization operators. This value should

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hold information on the positional power in the structure. Since set functions are being dealt with here, the theory of set games introduced by Hoede [8] could be applied. A set game assigns to each coalition a subset of a set $\mathcal{U}$. In our case, $\mathcal{U}$ is the same set of agents. A value for set games assigns to each player a subset of $\mathcal{U}$. Aarts et al. [1] introduced a value for set games, which, within our framework, assigns, to each agent, the set of agents that depend on this agent. However, this value is not sensitive enough to distinguish how strongly an agent depends on another agent. In order to quantify positional power, it was therefore necessary to take a different approach. This led us to define a value for authorization structures as a correspondence that assigns, to each authorization structure, a mapping from the set of agents to the set of fuzzy coalitions. In this paper, a particular value for authorization structures is defined. It is shown that this value can be used to analyze positional power in any hierarchical structure. Moreover, this value allows us to evaluate the dependency relationships in these structures. This is used to introduce the concepts of sovereignty and influence, which are the two components of positional power.

The paper is organized as follows. In Section 2, several basic definitions and results concerning cooperative games, permission structures and set games are recalled. In Section 3, authorization structures, the concept of value for authorization structures and a particular value called the Shapley authorization correspondence are introduced. The indices of sovereignty and influence are also defined. In Section 4, the Shapley authorization correspondence and the indices of sovereignty and influence are characterized. In Section 5, some examples are provided. Finally, in Section 6, we draw conclusions and consider certain ideas for future research.

2. Preliminaries

2.1. Cooperative TU-games

A *transferable utility cooperative game or TU-game*is a pair (N, v) where N is a set of cardinality $n \in \mathbb{N}$ and $v : 2^N \to \mathbb{R}$ is a function with $v(\emptyset) = 0$. The elements of N are called *players*, the subsets $E \subseteq N$ are called *coalitions*and $v(E)$ is the *worth* of E. For each coalition E, the worth of E can be interpreted as the maximal gain or minimal cost that players in this coalition can achieve by themselves. Often, a TU-game (N, v) is identified with the function v. The family of all the games with set of players N is denoted by $\mathcal{G}^N$. This set is a $(2^n - 1)$-dimensional real vector space. A game $v \in \mathcal{G}^N$ is said to be monotonic if $v(E) \leqslant v(F)$ for every $E \subseteq F \subseteq N$. And v is superadditive if $v(E) + v(F) \leqslant v(E \cup F)$ for every $E, F \subseteq N$ with $E \cap F = \emptyset$.

A payoff vector for a game on the set of players N is a vector $x \in \mathbb{R}^N$. A *value*on$\mathcal{G}^N$ is a function $\psi : \mathcal{G}^N \longrightarrow \mathbb{R}^N$ that assigns a payoff vector to each game. Numerous values have been defined for several families of games in the literature. The *Shapley value*, introduced in [10], is the most important of these values. The Shapley value $Sh(v) \in \mathbb{R}^N$ of a game $v \in \mathcal{G}^N$ is a weighted average of the marginal contributions of each player to the coalitions. It is defined as

$$Sh_i(v) = \sum_{\{E \subseteq N : i \in E\}} p_E (v(E) - v(E \setminus \{i\})), \quad \text{for all } i \in N, \tag{1}$$

where $p_E = \dfrac{(n - |E|)!(|E| - 1)!}{n!}$, for every $E \subseteq N$.

In this paper, cooperative games that only take the values 0 and 1 are often used: these are called {0, 1}-games. A {0, 1}-game $v \in \mathcal{G}^N$ is said to be a simple game if $v(N) = 1$ and satisfies monotonicity. Shapley et al. [11] considered the Shapley value restricted to simple games, which is called the Shapley–Shubik index. Dubey [3] proved that the Shapley–Shubik index restricted to superadditive simple games is characterized by the following properties:

ANONYMITY. If $v \in \mathcal{G}^N$ is a superadditive simple game, π is a permutation of N and $i \in N$, then $\phi_i(\pi v) = \phi_{\pi(i)}(v)$, where $\pi v(E) = v(\pi(E))$ for every $E \subseteq N$.

NULL PLAYER. If $v \in \mathcal{G}^N$ is a superadditive simple game and i is a null player in v (i.e., $v(E \cup \{i\}) = v(E)$ for all $E \subseteq N$), then $\phi_i(v) = 0$.

TRANSFER. If $v, w \in \mathcal{G}^N$ are superadditive simple games, then $\phi(v) + \phi(w) = \phi(v \vee w) + \phi(v \wedge w)$, where $(v \vee w)(E) = \max(v(E), w(E))$ and $(v \wedge w)(E) = \min(v(E), w(E))$ for every $E \subseteq N$.

EFFICIENCY. If $v \in \mathcal{G}^N$ is a superadditive simple game, then $\sum_{i \in N} \phi_i(v) = 1$.

2.2. Permission structures

A *permission structure* on a finite set of players N is a mapping S: $N \to 2^N$. Given $i \in N$, the players in $S(i)$ are called *successors* of i in S. The transitive closure of S is denoted by $\hat{S}$, that is, $j \in \hat{S}(i)$ if and only if there exists a sequence $\{i_p\}_{p=0}^q$ such that $i_0 = i$, $i_q = j$ and $i_p \in S(i_{p-1})$ for all $1 \le p \le q$. The set of *predecessors* of i in S is denoted by $P_S(i) = \{j \in N : i \in S(j)\}$. The set of *superiors* of i in S is $\hat{P}_S(i) = \{j \in N : i \in \hat{S}(j)\}$. The collection of all permission structures on N is denoted by $\mathcal{S}^N$. A permission structure on N can be identified with a directed graph whose vertex set is N and whose edge set is $\{(i, j) \in N^2 : j \in S(i)\}$.

A *game with permission structure over* N is a pair (v, S) where $v \in \mathcal{G}^N$ and $S \in \mathcal{S}^N$. Numerous assumptions can be made about how a permission structure affects the possibilities of cooperation in a TU-game. Gilles et al. [6] considered the *conjunctive approach*, in which it is assumed that all players need the permission from all their superiors (if they have any) in the permission structure. The *conjunctive sovereign part* of a coalition E, denoted by $A_C^S(E)$, contains those players that are allowed to cooperate within coalition E if we assume the conjunctive approach:

$$A_C^S(E) = \left\{ i \in E : \hat{P}_S(i) \subseteq E \right\}.$$

Van den Brink [2] considered the *disjunctive approach*, in which it is assumed that players only need the permission from one of their predecessors (if they have any), as long as that predecessor also has permission to cooperate (note that in both the conjunctive and the disjunctive approach the authors assume transitivity in the dependence relationships). The *disjunctive sovereign part* of a coalition E, denoted by $A_D^S(E)$, contains all the players that are allowed to cooperate within coalition E if the disjunctive approach is assumed:

$$A_D^S(E) = \{ i \in E : \text{ there exist } i_0, \dots, i_m \in E \text{ with } i = i_0,$$
$$P_S(i_m) = \emptyset \text{ and } i_{k-1} \in S(i_k) \text{ for all } k = 1, \dots, m \}.$$

2.3. Set games

A *set game* is a triple $(N, v, \mathcal{U})$ where $\mathcal{U}$ is a set, called universe, N is a finite set of players and v is a mapping from 2^N into $2^{\mathcal{U}}$ satisfying $v(\emptyset) = \emptyset$. The worth $v(S)$ of a coalition $S \subseteq N$ can be interpreted as the set of items that can be obtained by coalition S if its members cooperate. A value for set games is a mapping from the collection of all set games into $\left(2^{\mathcal{U}}\right)^N$. Aarts et al. [1] introduced a value for monotonic set games that can be considered as the analogue of the Shapley value for TU-games. It is defined as

$$\varphi_i(v) = \bigcup_{\{E \subseteq N : \, i \in E\}} (v(E) \setminus v(E \setminus \{i\})), \quad \text{for all } i \in N.$$

Note that this value assigns each element of the universe to every player that holds some control over it.

3. Proposed methodology

We aim to introduce structures that model situations in which there are dependency relationships among the agents. A value and several indices for those structures will be defined.

Throughout this and the following section, N denotes a fixed set of cardinality $n \in \mathbb{N}$. The elements of N are called agents.

Whatever approach is assumed when dealing with a permission structure, the dependency relationships are determined by the set function that assigns to each coalition the set of agents that are allowed to cooperate within that coalition. By taking this into consideration, we introduce a kind of set function which can model situations in which certain agents need the authorization of others.

Definition 1. An authorization operator on N is a mapping A: $2^N \to 2^N$ that satisfies the following conditions:

1. $A(E) \subseteq E$ for any $E \subseteq N$,
2. If $E \subset F$ then $A(E) \subseteq A(F)$.

The pair (N, A) is called an authorization structure. The set of all authorization operators on N is denoted by $\mathcal{A}^N$.

If $A \in \mathcal{A}^N$ and $E \subseteq N$, then $A(E)$ can be interpreted as the set of agents that are allowed to act within coalition E. With this in mind, the two conditions considered in the definition of an authorization operator seem to be reasonable.

Given $A, B \in \mathcal{A}^N$, then $A \cup B, A \cap B \in \mathcal{A}^N$ can be defined in the following natural way:

$$(A \cup B)(E) = A(E) \cup B(E),$$
$$(A \cap B)(E) = A(E) \cap B(E),$$

for every $E \subseteq N$.

We aim to define a value for authorization structures. Notice that an authorization operator is specifically a monotonic set game. The value φ for monotonic set games, introduced by Aarts et al. [1], could therefore be considered. However, this value is not practical for our purposes, as explained below.

Given A an authorization structure on N, $\varphi(A)$ assigns, to each agent $i \in N$, the set $\varphi_i(A)$ of agents that need authorization from i to act within, at least, one coalition. Therefore, $\varphi(A)$ indicates whether an agent depends on another agent, but it fails to describe the strength of the dependency relationships. This can be illustrated with the following example. Consider the authorization operators $B, C \in \mathcal{A}^{\{1,2,3\}}$ defined by

$$B(E) = \begin{cases} E \setminus \{1\}, & \text{if } E \neq \{1,2,3\}, \\ \{1,2,3\}, & \text{if } E = \{1,2,3\}, \end{cases} \quad \text{and} \quad C(E) = \begin{cases} \emptyset, & \text{if } E = \{1\}, \\ E, & \text{if } E \neq \{1\}, \end{cases}$$

for every $E \subseteq \{1, 2, 3\}$. It is clear that agent 1 depends on agent 2 more strongly in $(\{1, 2, 3\}, B)$ than in $(\{1, 2, 3\}, C)$. However, it is verified that $\varphi(B) = \varphi(C)$.

Since the quantification of positional power is required, a more sensitive value needs to be defined. To this end, we propose to consider values that assign, to each authorization operator A on N and each agent $i \in N$, a fuzzy subset of N. The levels of this fuzzy subset are aimed to indicate how strongly each agent depends on i in (N, A). This leads to the following definition.

Definition 2. A value for authorization structures is a mapping $\Psi : \mathcal{A}^N \to \left([0,1]^N\right)^N$.

We proceed with the definition of a particular value for authorization structures. Suppose $A \in \mathcal{A}^N$. For each $j \in N$, we can consider

$$\begin{aligned} A_j : 2^N &\to \{0,1\} \\ E &\to A_j(E) = |A(E) \cap \{j\}|. \end{aligned}$$

Notice that A_j is a superadditive {0, 1}-game. In order to evaluate how strongly agent j depends on each agent in N, the Shapley value of the game A_j is calculated. The following value for authorization structures can now be defined.

Definition 3. The Shapley authorization correspondence assigns, to each authorization operator $A \in \mathcal{A}^N$, the mapping $\Phi(A) \in ([0, 1]^N)^N$ defined as

$$\Phi_i(A) = \left(Sh_i(A_j)\right)_{j\in N}, \quad \text{for every } i \in N,$$

where Sh denotes the Shapley value.

Note that the number $\Phi_{ij}(A)$ is greater than zero if and only if there exists a coalition $E \subseteq N$ such that $j \in A(E)\backslash A(E\backslash\{i\})$, that is, if j needs to be authorized by i within, at least, one coalition. In fact, by taking into account the meaning of the Shapley value, the number $\Phi_{ij}(A)$ indicates how strongly agent j depends on i in (N, A). This leads to the following definitions.

Definition 4. Let $A \in \mathcal{A}^N$ and $i, j \in N$ with $i \neq j$. The number $\Phi_{ij}(A)$ is called *the influence of i over j in* (N, A).

Definition 5. Let $A \in \mathcal{A}^N$ and $i \in N$. The sum of the influence of i over each of the remaining agents is called the influence index of i in (N, A) and is denoted by $infl_i(A)$, that is,

$$infl_i(A) = \sum_{j\in N\backslash\{i\}} \Phi_{ij}(A).$$

Definition 6. Let $A \in \mathcal{A}^N$ and $i \in N$. The number $\Phi_{ii}(A)$ is called the sovereignty index of i in (N, A) and is denoted by $sov_i(A)$.

Definition 7. Let $A \in \mathcal{A}^N$ and $i \in N$. The addition of the influence and sovereignty indices of i is called the power index of i in (N, A) and is denoted by $pow_i(A)$, that is,

$$pow_i(A) = sov_i(A) + infl_i(A).$$

4. Results

The value and the indices introduced in the previous section will now be characterized.

4.1. A characterization of the Shapley authorization correspondence

We aim to give a characterization of the Shapley authorization correspondence, but first two concepts need to be introduced.

Definition 8. Let $A \in \mathcal{A}^N$ and $i, j \in N$. Agent j has veto power over agent i in (N, A) if $i \in A(N)\backslash A(N\backslash\{j\})$. Let us denote

$$V_i(A) = \{j \in N : j \text{ has veto power over } i \text{ in } (N, A)\}.$$

Definition 9. Let $A \in \mathcal{A}^N$ and $i, j \in N$. Agent i depends partially on agent j in (N, A) if there exists $E \subseteq N$ such that $i \in A(E)\backslash A(E\backslash\{j\})$. Let us denote

$$P_i(A) = \{j \in N : i \text{ depends partially on } j \text{ in } (N, A)\}.$$

In order to characterize the Shapley authorization correspondence, the properties stated below are considered. In the statement of these properties $\Psi : \mathcal{A}^N \to \left([0,1]^N\right)^N$ is a value for authorization structures.

The first property states that for each agent i that is allowed to act when coalition N is formed, there is a unit of positional power to be distributed among all the agents.

Efficiency. If $A \in \mathcal{A}^N$, then

$$\sum_{k\in N} \Psi_k(A) = \mathbf{1}_{A(N)}$$

where $\mathbf{1}_{A(N)}$ denotes the indicator vector of $A(N)$, that is,

$$\left(\mathbf{1}_{A(N)}\right)_j = \begin{cases} 1, & \text{if } j \in A(N), \\ 0, & \text{if } j \in N \setminus A(N). \end{cases}$$

The following property states that if an agent j does not depend partially on an agent i, then i receives no positional power from j.

Zero dependence. If $A \in \mathcal{A}^N$, then $\Psi_{ij}(A) = 0$ for every $i \in N \backslash P_j(A)$.

Suppose that we have an authorization structure on N, $T \subseteq N$ and $i \in T$ such that coalition T cannot authorize agent i. Assume now that coalition T acquires the power to authorize agent i. In this case, it is reasonable to presume that all the agents in T benefit equally from that change. This is what the fairness property states.

Fairness. Let $A \in \mathcal{A}^N$, $T \in 2^N \backslash \{\emptyset\}$ and $i \in T$. Consider $A^{T,i} \in \mathcal{A}^N$ defined by

$$A^{T,i} : 2^N \to 2^N$$
$$E \to A^{T,i}(E) = \begin{cases} A(E), & \text{if } T \nsubseteq E, \\ A(E) \cup \{i\}, & \text{if } T \subseteq E. \end{cases}$$

Then,

$$\Psi_{ji}\left(A^{T,i}\right) - \Psi_{ji}(A) = \Psi_{ki}\left(A^{T,i}\right) - \Psi_{ki}(A), \qquad \text{for all } j, k \in T.$$

The following result provides a characterization of the Shapley authorization correspondence.

Theorem 1. *A value for authorization structures is equal to the Shapley authorization correspondence if and only if it satisfies the properties of efficiency, zero dependence and fairness.*

Proof. Firstly, we will prove that the Shapley authorization correspondence satisfies the three properties stated in the theorem.

Efficiency. Let $A \in \mathcal{A}^N$. For every $j \in N$ it is verified that

$$\left(\sum_{k\in N} \Phi_k(A)\right)_j = \sum_{k\in N} \Phi_{kj}(A) = \sum_{k\in N} Sh_k(A_j) = A_j(N) = |A(N) \cap \{j\}| = \left(\mathbf{1}_{A(N)}\right)_j.$$

Zero dependence. Let $A \in \mathcal{A}^N$ and $i, j \in N$ such that $i \notin P_j(A)$. Therefore,

$$\begin{aligned} \Phi_{ij}(A) = Sh_i(A_j) &= \sum_{\{E\subseteq N:\, i\in E\}} p_E\left(A_j(E) - A_j(E \setminus \{i\})\right) \\ &= \sum_{\{E\subseteq N:\, i\in E\}} p_E\left(|A(E) \cap \{j\}| - |A(E \setminus \{i\}) \cap \{j\}|\right) \\ &= \sum_{\{E\subseteq N:\, i\in E\}} p_E |(A(E) \setminus A(E \setminus \{i\})) \cap \{j\}| = 0. \end{aligned}$$

Fairness. Let $A \in \mathcal{A}^N$, $T \in 2^N \backslash \{\emptyset\}$ and $i \in T$. Take $j \in T$. Therefore,

$$\begin{aligned} \Phi_{ji}\left(A^{T,i}\right) - \Phi_{ji}(A) &= Sh_j\left(A_i^{T,i}\right) - Sh_j(A_i) \\ &= \sum_{\{E\subseteq N:\, j\in E\}} p_E\left(A_i^{T,i}(E) - A_i^{T,i}(E \setminus \{j\})\right) \\ &\quad - \sum_{\{E\subseteq N:\, j\in E\}} p_E\left(A_i(E) - A_i(E \setminus \{j\})\right). \end{aligned}$$

By taking into consideration that if $T \nsubseteq F \subseteq N$ then $A^{T,i}(F) = A(F)$, the difference above is therefore equal to

$$\sum_{\{E\subseteq N:\, T\subseteq E\}} p_E\left(A_i^{T,i}(E) - A_i(E)\right).$$

It suffices to notice that the number obtained does not depend on the agent $j \in T$ chosen.

The uniqueness of the Shapley authorization correspondence will now be proved. Let $\Psi : \mathcal{A}^N \to \left([0, 1]^N\right)^N$ be a value for authorization structures that satisfies the properties of efficiency, zero dependence and fairness. In order to show that

$$\Psi(A) = \Phi(A), \qquad \text{for every } A \in \mathcal{A}^N, \tag{2}$$

we first define

$$m(A) = \sum_{F\subseteq N} |A(F)|, \qquad \text{for every } A \in \mathcal{A}^N.$$

Eq. (2) will be proved by induction on $m(A)$.

1. Base case. Let $A \in \mathcal{A}^N$ be such that $m(A) = 0$. It is clear that for every $i, j \in N$, $i \notin P_j(A)$ holds. Through the property of zero dependence it can be concluded that $\Psi_{ij}(A) = \Phi_{ij}(A) = 0$ for every $i, j \in N$.
2. Inductive step. Let $A \in \mathcal{A}^N$ be such that $m(A) > 0$. We must prove that $\Psi_{ij}(A) = \Phi_{ij}(A)$ for every $i, j \in N$. Take $j \in N$. From the property of zero dependence, it is known that

$$\Psi_{ij}(A) = 0, \quad \text{for every } i \in N \setminus P_j(A), \tag{3}$$

$$\Phi_{ij}(A) = 0, \quad \text{for every } i \in N \setminus P_j(A). \tag{4}$$

Suppose $i \in P_j(A)$. There exists $E \subseteq N$ such that $j \in A(E) \setminus A(E \setminus \{i\})$. Take $T \subseteq E$ minimal with $j \in A(T)$. It is clear that $i \in T$. We define

$$B : 2^N \to 2^N$$
$$S \to B(S) = \begin{cases} A(S), & \text{if } S \neq T, \\ A(T) \setminus \{j\}, & \text{if } S = T. \end{cases}$$

It is straightforward to verify that $B \in \mathcal{A}^N$ and $B^{T,j} = A$. By using the fairness property, the following is obtained:

$$\Psi_{ij}(A) - \Psi_{ij}(B) = \Psi_{jj}(A) - \Psi_{jj}(B),$$
$$\Phi_{ij}(A) - \Phi_{ij}(B) = \Phi_{jj}(A) - \Phi_{jj}(B).$$

Since $m(B) = m(A) - 1$, it is known from the induction hypothesis that $\Psi(B) = \Phi(B)$. From this fact and the two equalities above,

$$\Psi_{ij}(A) - \Phi_{ij}(A) = \Psi_{jj}(A) - \Phi_{jj}(A).$$

Therefore, it is proved that

$$\Psi_{ij}(A) - \Phi_{ij}(A) = \Psi_{jj}(A) - \Phi_{jj}(A), \quad \text{for every } i \in P_j(A). \tag{5}$$

By using the property of efficiency, (3), (4) and (5), it can be written that

$$0 = \sum_{i \in N} \Psi_{ij}(A) - \sum_{i \in N} \Phi_{ij}(A) = \sum_{i \in P_j(A)} (\Psi_{ij}(A) - \Phi_{ij}(A)) = |P_j(A)|(\Psi_{jj}(A) - \Phi_{jj}(A)).$$

Hence, either $P_j(A) = \emptyset$ or $\Psi_{jj}(A) = \Phi_{jj}(A)$. In any case, it is clear from (3–5) that

$$\Psi_{ij}(A) = \Phi_{ij}(A), \quad \text{for every } i \in N.$$

Since an arbitrary $j \in N$ had been chosen, it has therefore been proved that $\Psi(A) = \Phi(A)$. □

4.2. A characterization of the sovereignty index

In order to characterize the sovereignty index, the properties stated below are considered. In the statement of these properties, ψ is a mapping from $\mathcal{A}^N$ into $\mathbb{R}^N$.

Firstly, it is reasonable to think that if an agent is not authorized to act within any coalition, then this agent has no sovereignty.

Inactive agent property. If $A \in \mathcal{A}^N$ and $i \in N \setminus A(N)$, then

$$\psi_i(A) = 0.$$

Given an authorization structure (N, A) and an agent $i \in A(N)$, if our aim is to define a number $\psi_i(A)$ that measures the autonomy of i, perhaps the first values that would be considered are $\frac{1}{|V_i(A)|}$ and $\frac{1}{|P_i(A)|}$. Notice that both are extreme measures of autonomy, in the sense that, with the first number, all the dependency relationships that are not veto relationships would be ignored, whereas with the second number, all the dependency relationships would be equally valued. The following property distinguishes the indices that lie between these two values.

$P - V$ **bounds for sovereignty.** If $A \in \mathcal{A}^N$ and $i \in A(N)$, then

$$\frac{1}{|P_i(A)|} \leqslant \psi_i(A) \leqslant \frac{1}{|V_i(A)|}.$$

The following property, based on the transfer property of the Shapley–Shubik index, distinguishes the indices that satisfy the condition that equal changes in the authorization structures produce equal changes in the values of the index.

Transfer property. Let $A, \hat{A}, B, \hat{B} \in \mathcal{A}^N$ be such that $A(E) \setminus \hat{A}(E) = B(E) \setminus \hat{B}(E)$ and $\hat{A}(E) \setminus A(E) = \hat{B}(E) \setminus B(E)$ for all $E \subseteq N$. Then,

$$\psi(A) - \psi(\hat{A}) = \psi(B) - \psi(\hat{B}).$$

In the following theorem it is proved that these properties uniquely determine the sovereignty index. The following well known result will be employed.

Lemma 2. *If $T \in 2^N \setminus \{\emptyset\}$, then*

$$\sum_{\{E \subseteq N:\, T \subseteq E\}} p_E = \frac{1}{|T|},$$

where the numbers p_E are the coefficients of the Shapley value.

Theorem 3. *A mapping $\psi : \mathcal{A}^N \to \mathbb{R}^N$ is equal to the sovereignty index if and only if it satisfies the properties of inactive agent, $P-V$ bounds for sovereignty and transfer.*

Proof. Firstly, it will be proved that the sovereignty index satisfies the properties stated in the theorem.

INACTIVE AGENT PROPERTY. It is trivial to check that if $i \notin A(N)$ then $sov_i(A) = 0$.

$P-V$ BOUNDS FOR SOVEREIGNTY. Let $A \in \mathcal{A}^N$ and $i \in A(N)$.

$$\begin{aligned} sov_i(A) &= \Phi_{ii}(A) = Sh_i(A_i) = \sum_{\{E \subseteq N:\, i \in E\}} p_E[A_i(E) - A_i(E \setminus \{i\})] \\ &= \sum_{\{E \subseteq N:\, i \in E\}} p_E A_i(E) = \sum_{\{E \subseteq N:\, i \in A(E)\}} p_E. \end{aligned} \tag{6}$$

It will be first shown that $\frac{1}{|P_i(A)|} \leqslant sov_i(A)$.

Suppose that $j \in N \setminus P_i(A)$. It is clear that $i \in A(N \setminus \{j\})$. Now, if $k \in N \setminus P_i(A)$, then $i \in A(N \setminus \{j, k\})$. Through iteration, $i \in A(P_i(A))$, and hence $\{E \subseteq N: P_i(A) \subseteq E\} \subseteq \{E \subseteq N: i \in A(E)\}$. Therefore, the sum in (6) is greater or equal to

$$\sum_{\{E \subseteq N:\, P_i(A) \subseteq E\}} p_E,$$

which, from Lemma 2, is equal to $\frac{1}{|P_i(A)|}$.

It will now be proved that $sov_i(A) \leqslant \frac{1}{|V_i(A)|}$.

It is clear that, for every $E \subseteq N$ with $i \in A(E)$, $V_i(A) \subseteq E$. Therefore, the sum in (6) is less or equal to

$$\sum_{\{E \subseteq N:\, V_i(A) \subseteq E\}} p_E,$$

which, from Lemma 2, is equal to $\frac{1}{|V_i(A)|}$.

TRANSFER PROPERTY. Let $A, \hat{A}, B, \hat{B} \in \mathcal{A}^N$ be such that $A(E) \setminus \hat{A}(E) = B(E) \setminus \hat{B}(E)$ and $\hat{A}(E) \setminus A(E) = \hat{B}(E) \setminus B(E)$ for every $E \subseteq N$. Let $i \in N$. It is clear that $A_i - \hat{A}_i = B_i - \hat{B}_i$. Therefore,

$$\begin{aligned} sov_i(A) - sov_i(\hat{A}) &= Sh_i(A_i) - Sh_i(\hat{A}_i) = Sh_i(A_i - \hat{A}_i) \\ &= Sh_i(B_i - \hat{B}_i) = Sh_i(B_i) - Sh_i(\hat{B}_i) \\ &= sov_i(B) - sov_i(\hat{B}). \end{aligned}$$

It will now be shown that these properties uniquely determine the sovereignty index. Suppose that $\psi : \mathcal{A}^N \to \mathbb{R}^N$ satisfies the properties of inactive agent, $P-V$ bounds for sovereignty and transfer. It must be proved that ψ is equal to the sovereignty index.

Let $A \in \mathcal{A}^N$. If $A(N) = \emptyset$, it is known, from the inactive agent property, that $\psi(A) = sov(A) = 0$. Now suppose that $A(N) \neq \emptyset$. For every $T \in 2^N \setminus \{\emptyset\}$ and $i \in T$ we consider

$$C_{T,i} : 2^N \to 2^N$$

$$E \to C_{T,i}(E) = \begin{cases} \{i\}, & \text{if } T \subseteq E, \\ \emptyset, & \text{if } T \nsubseteq E. \end{cases}$$

It is clear that $C_{T,i} \in \mathcal{A}^N$. It is easy to verify that

$$A = \bigcup_{\{(T,i) \in 2^N \times N:\, i \in A(T)\}} C_{T,i}.$$

Therefore, if we want to prove that $\psi = sov$, it is sufficient to show, for every $m \in \mathbb{N}$, $T_1, \ldots, T_m \in 2^N \setminus \{\emptyset\}$ and $i_1, \ldots, i_m \in N$ with $i_k \in T_k$ for all $k = 1, \ldots, m$, that

$$\psi\left(\bigcup_{k=1}^{m} C_{T_k, i_k}\right) = sov\left(\bigcup_{k=1}^{m} C_{T_k, i_k}\right).$$

Let us prove this equality by strong induction on m.

1. Base case. Let $T \in 2^N \backslash \{\emptyset\}$ and $i \in T$. From the inactive agent property,

$$\psi_j(C_{T,i}) = 0, \qquad \text{for all } j \in N \setminus \{i\}. \tag{7}$$

Note that $P_i(C_{T,i}) = V_i(C_{T,i}) = T$. From the property of $P-V$ bounds for sovereignty it follows that

$$\psi_i(C_{T,i}) = \frac{1}{|T|}. \tag{8}$$

From (7) and (8), $\psi(C_{T,i}) = \frac{1}{|T|}\mathbf{1}_{\{i\}}$. Since the sovereignty index also satisfies the properties used, it can be concluded that

$$\psi(C_{T,i}) = sov(C_{T,i}).$$

2. Inductive step. Take $T_1, \ldots, T_{m+1} \in 2^N \setminus \{\emptyset\}$ and $i_1, \ldots, i_{m+1} \in N$ with $i_k \in T_k$ for all $k = 1, \ldots, m+1$. Let

$$A = \bigcup_{k=1}^{m+1} C_{T_k,i_k}, \quad \hat{A} = \bigcup_{k=1}^{m} C_{T_k,i_k},$$

$$B = C_{T_{m+1},i_{m+1}}, \quad \hat{B} = \bigcup_{k=1}^{m} (C_{T_k,i_k} \cap C_{T_{m+1},i_{m+1}}).$$

Since ψ and sov satisfy the transfer property, it follows that

$$\psi(A) - \psi(\hat{A}) = \psi(B) - \psi(\hat{B}), \tag{9}$$

$$sov(A) - sov(\hat{A}) = sov(B) - sov(\hat{B}). \tag{10}$$

It is already known that

$$\psi(B) = sov(B). \tag{11}$$

By induction hypothesis, it is verified that

$$\psi(\hat{A}) = sov(\hat{A}). \tag{12}$$

Observe that if $i_k \neq i_{m+1}$, then $C_{T_k,i_k} \cap C_{T_{m+1},i_{m+1}}$ can be eliminated from the expression of $\hat{B}$, and if $i_k = i_{m+1}$ then $C_{T_k,i_k} \cap C_{T_{m+1},i_{m+1}} = C_{T_k \cup T_{m+1},i_k}$. With this in mind, and by induction hypothesis,

$$\psi(\hat{B}) = sov(\hat{B}). \tag{13}$$

From (9–13) it is concluded that

$$\psi\left(\bigcup_{k=1}^{m+1} C_{T_k,i_k}\right) = sov\left(\bigcup_{k=1}^{m+1} C_{T_k,i_k}\right).$$

□

4.3. A characterization of the influence index

In order to characterize the influence index, we will consider the following property, where ψ is a mapping from $\mathcal{A}^N$ into $\mathbb{R}^N$.

Let i and j be two different agents in N. If $i \notin P_j(A)$ then the influence of i over j in (N, A) should be zero. If $i \in P_j(A)$, then the influence of i over j would be expected to be positive and no greater than the sovereignty index of j. If $i \in V_j(A)$, then the influence of i over j would reasonably be greater or equal to the sovereignty index of j. These considerations along with the $P-V$ bounds of the sovereignty index lead us to consider the following property.

$P-V$ **bounds for influence.** If $A \in \mathcal{A}^N$ and $i \in N$, then

$$\sum_{\{j \in N \backslash \{i\}:\, i \in V_j(A)\}} \frac{1}{|P_j(A)|} \leqslant \psi_i(A) \leqslant \sum_{\{j \in N \backslash \{i\}:\, i \in P_j(A)\}} \frac{1}{|V_j(A)|}.$$

In the following theorem, this property together with the transfer property are shown to uniquely determine the influence index. The following lemma will be used in the proof of the theorem.

Lemma 4. *If $A \in \mathcal{A}^N$, $i \in N$ and $j \in A(N)$, then*

$$\Phi_{ij}(A) \leqslant sov_j(A),$$

and the equality holds if and only if i has veto power over j in (N, A).

Proof. Let $A \in \mathcal{A}^N$, $i \in N$ and $j \in A(N)$. Therefore,

$$\Phi_{ij}(A) = Sh_i(A_j) = \sum_{\{E\subseteq N:\, i\in E\}} p_E\big[A_j(E) - A_j(E \setminus \{i\})\big]$$
$$= \sum_{\{E\subseteq N:\, j\in A(E)\setminus A(E\setminus\{i\})\}} p_E. \tag{14}$$

and

$$sov_j(A) = \Phi_{jj}(A) = Sh_j(A_j) = \sum_{\{E\subseteq N:\, j\in E\}} p_E\big[A_j(E) - A_j(E \setminus \{j\})\big]$$
$$= \sum_{\{E\subseteq N:\, j\in A(E)\}} p_E. \tag{15}$$

It is clear that the sum (14) is less than or equal to the sum (15). Moreover, the equality holds if and only if $\{E \subseteq N : j \in A(E)\} = \{E \subseteq N : j \in A(E) \setminus A(E \setminus \{i\})\}$. However, by taking into consideration that $j \in A(N)$, this condition is equivalent to $i \in V_j(A)$. □

Theorem 5. *A mapping $\psi : \mathcal{A}^N \to \mathbb{R}^N$ is equal to the influence index if and only if it satisfies the properties of $P-V$ bounds for influence and transfer.*

Proof. It will first be proved that the influence index satisfies the properties mentioned in the theorem.

$P-V$ BOUNDS FOR INFLUENCE. Let $A \in \mathcal{A}^N$ and $i \in N$. Therefore,

$$infl_i(A) = \sum_{j\in N\setminus\{i\}} \Phi_{ij}(A),$$

which, taking into consideration that $\Phi_{ij}(A) > 0$ if and only if j depends partially on i in (N, A), is equal to

$$\sum_{\{j\in N\setminus\{i\}:\, i\in P_j(A)\}} \Phi_{ij}(A),$$

which, from Lemma 4, is less than or equal to

$$\sum_{\{j\in N\setminus\{i\}:\, i\in P_j(A)\}} sov_j(A),$$

which, from the $P-V$ bounds for sovereignty, is less than or equal to

$$\sum_{\{j\in N\setminus\{i\}:\, i\in P_j(A)\}} \frac{1}{|V_j(A)|}.$$

Hence the right-hand inequality has been proved. Let us prove the other inequality. It is known that

$$infl_i(A) = \sum_{\{j\in N\setminus\{i\}:\, i\in P_j(A)\}} \Phi_{ij}(A) \geqslant \sum_{\{j\in N\setminus\{i\}:\, i\in V_j(A)\}} \Phi_{ij}(A),$$

which, using the second statement of Lemma 4, is equal to

$$\sum_{\{j\in N\setminus\{i\}:\, i\in V_j(A)\}} sov_j(A),$$

which, from the $P-V$ bounds for sovereignty, is greater than or equal to

$$\sum_{\{j\in N\setminus\{i\}:\, i\in V_j(A)\}} \frac{1}{|P_j(A)|}.$$

TRANSFER PROPERTY. Let $A, \hat{A}, B, \hat{B} \in \mathcal{A}^N$ be such that $A(E) \setminus \hat{A}(E) = B(E) \setminus \hat{B}(E)$ and $\hat{A}(E) \setminus A(E) = \hat{B}(E) \setminus B(E)$ for every $E \subseteq N$. Let $i \in N$. It is clear that $A_j - \hat{A}_j = B_j - \hat{B}_j$ for every $j \in N$. Therefore,

$$\begin{aligned}\Phi_{ij}(A) - \Phi_{ij}(\hat{A}) &= Sh_i(A_j) - Sh_i(\hat{A}_j) = Sh_i(A_j - \hat{A}_j)\\ &= Sh_i(B_j - \hat{B}_j) = Sh_i(B_j) - Sh_i(\hat{B}_j)\\ &= \Phi_{ij}(B) - \Phi_{ij}(\hat{B}).\end{aligned}$$

Finally, we attain

$$infl_i(A) - infl_i(\hat{A}) = \sum_{j\in N\setminus\{i\}} \left(\Phi_{ij}(A) - \Phi_{ij}(\hat{A})\right)$$
$$= \sum_{j\in N\setminus\{i\}} \left(\Phi_{ij}(B) - \Phi_{ij}(\hat{B})\right) = infl_i(B) - infl_i(\hat{B}).$$

It will now be shown that these properties uniquely determine the influence index. The reasoning is similar to that followed in the case of the sovereignty index. Suppose that $\psi : \mathcal{A}^N \to \mathbb{R}^N$ satisfies the properties of $P - V$ bounds for influence and transfer. It must be proved that ψ is equal to the influence index.

Let $A \in \mathcal{A}^N$. If $A(N) = \emptyset$, then it is known from the $P - V$ bounds for influence, that $\psi(A) = infl(A) = 0$. If $A(N) \neq \emptyset$, then

$$A = \bigcup_{\{(T,i)\in 2^N\times N:\, i\in A(T)\}} C_{T,i}.$$

Therefore, if proof that $\psi = infl$ is required, it is sufficient to show, for every $m \in \mathbb{N}$, $T_1, \ldots, T_m \in 2^N \setminus \{\emptyset\}$ and $i_1, \ldots, i_m \in N$ with $i_k \in T_k$ for all $k = 1, \ldots, m$, that

$$\psi\left(\bigcup_{k=1}^{m} C_{T_k,i_k}\right) = infl\left(\bigcup_{k=1}^{m} C_{T_k,i_k}\right).$$

Let us prove this equality by strong induction on m.

1. Base case. Let $T \in 2^N\setminus\{\emptyset\}$ and $i \in T$. We can derive, from the $P - V$ bounds for influence, that

$$\psi_j(C_{T,i}) = \begin{cases} \dfrac{1}{|T|}, & \text{if } j \in T \setminus \{i\}, \\ 0, & \text{if } j \in (N \setminus T) \cup \{i\}. \end{cases}$$

   Since the influence index also satisfies the property used, it follows that

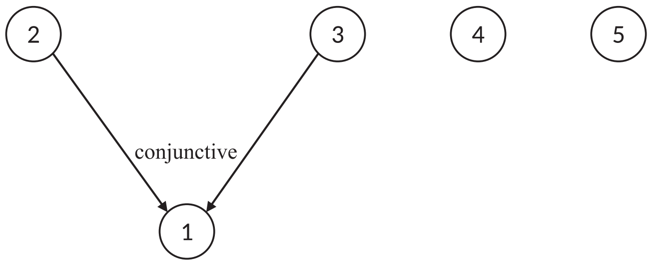
$$\psi(C_{T,i}) = infl(C_{T,i}).$$

2. Inductive step. The reasoning here is equal to that followed in the case of the sovereignty index. □

5. Example

Let $N = \{1, 2, 3, 4, 5\}$. Three authorization operators $A, B, C \in \mathcal{A}^N$ are now described and $\Phi(A)$, $\Phi(B)$ and $\Phi(C)$ are calculated.

(a) Suppose that agents 2 and 3 have veto power over 1 and agents 2, 3, 4 and 5 are autonomous, that is, they need no authorization from any other agent within any coalition. The diagram below provides an illustration:



This situation can be described by the authorization operator $A \in \mathcal{A}^N$ given by

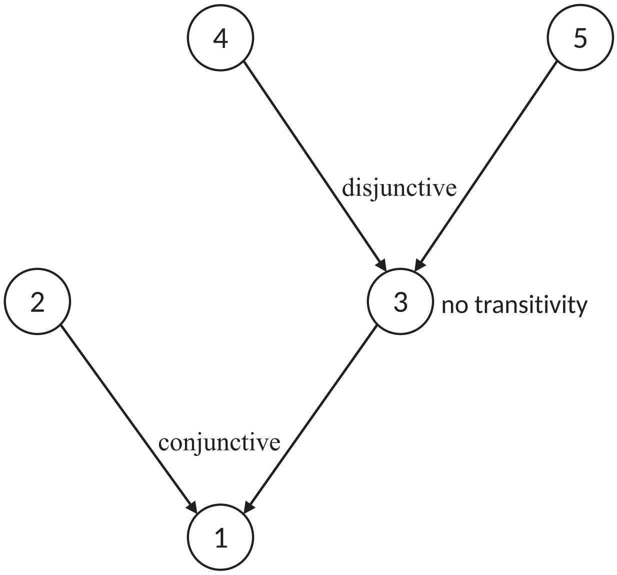
$$A(E) = \begin{cases} E \setminus \{1\}, & \text{if } \{2,3\} \not\subseteq E, \\ E, & \text{otherwise.} \end{cases}$$

If $\Phi(A)$ is calculated, then

$$\Phi(A) = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since 2 and 3 have veto power over 1, then the influence of 2 (or 3) over 1 must be equal to the sovereignty index of 1. In addition, agent 1 does not depend partially on either 4 or 5, and hence the influence of 4 (or 5) over 1 is equal to zero. Finally, agents 2, 3, 4 and 5 are autonomous, therefore they have the maximum possible sovereignty.

(b) Starting from the situation described above, suppose that now agent 3 always needs authorization from either 4 or 5. Let us illustrate the new situation:



The expression "no transitivity" means that an agent does not need to be authorized to cooperate within a coalition in order to give permission to other agents. For instance, in the situation described, agent 3 is not authorized to cooperate within coalition {1, 2, 3}, but agent 3 allows agent 1 to cooperate within that coalition. This situation can be described by the authorization operator $B \in \mathcal{A}^N$, given by

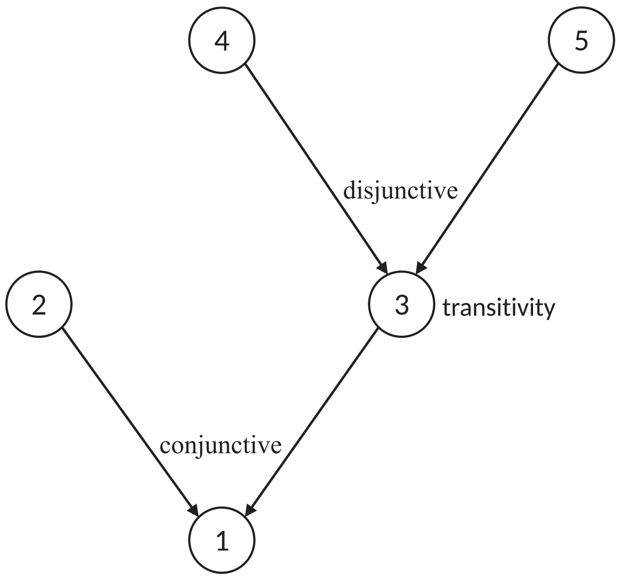
$$B(E) = \begin{cases} E \setminus \{1,3\}, & \text{if } \{2,3\} \not\subseteq E \text{ and } \{4,5\} \cap E = \emptyset, \\ E \setminus \{1\}, & \text{if } \{2,3\} \not\subseteq E \text{ and } \{4,5\} \cap E \neq \emptyset, \\ E \setminus \{3\}, & \text{if } \{2,3\} \subseteq E \text{ and } \{4,5\} \cap E = \emptyset, \\ E, & \text{if } \{2,3\} \subseteq E \text{ and } \{4,5\} \cap E \neq \emptyset. \end{cases}$$

If $\Phi(B)$ is calculated, then

$$\Phi(B) = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 1 & 0 \\ 0 & 0 & \frac{1}{6} & 0 & 1 \end{pmatrix}.$$

Agents 4 and 5 have gained influence over 3. This influence is less than the sovereignty index of 3, since none of them have veto power over 3.

(c) Based on the above scenario, suppose now that there is transitivity in the dependence relationships, that is, an agent needs to be authorized to cooperate within a coalition in order to give permission to other agents in the coalition. In our situation this means that agent 3 cannot authorize 1 to cooperate within a coalition that does not contain either 4 or 5.



This situation can be described by the authorization operator $C \in \mathcal{A}^N$ given by

$$C(E) = \begin{cases} E \setminus \{1,3\}, & \text{if } \{4,5\} \cap E = \emptyset, \\ E \setminus \{1\}, & \text{if } \{2,3\} \not\subseteq E \text{ and } \{4,5\} \cap E \neq \emptyset, \\ E, & \text{if } \{2,3\} \subseteq E \text{ and } \{4,5\} \cap E \neq \emptyset. \end{cases}$$

If $\Phi(C)$ is calculated, then

$$\Phi(C) = \begin{pmatrix} \frac{3}{10} & 0 & 0 & 0 & 0 \\ \frac{3}{10} & 1 & 0 & 0 & 0 \\ \frac{3}{10} & 0 & \frac{2}{3} & 0 & 0 \\ \frac{1}{20} & 0 & \frac{1}{6} & 1 & 0 \\ \frac{1}{20} & 0 & \frac{1}{6} & 0 & 1 \end{pmatrix}.$$

Since agent 3 cannot authorize 1 without being authorized by 4 or 5, these agents have gained influence over 1.

6. Conclusions and remarks

We have introduced authorization structures, which can be used to model hierarchies. A value for authorization structures has been defined and characterized. This value describes the strength of the dependency relationships induced by the structure. This has enabled the concepts of sovereignty and influence to be introduced.

Although the concept of authorization structure was inspired by the set functions which, given a permission structure, assign to each coalition the disjunctive or the conjunctive sovereign part of the coalition, authorization structures can enjoy a much wider application. In fact, authorization structures can be used to model hierarchies which are neither disjunctive nor conjunctive, or hierarchies in which the dependency relationships are not transitive. Moreover, they can also describe non-hierarchical structures. Therefore, the value that we have studied can be applied to a great range of situations.

Other approaches could be considered for further research. Within our framework, the dependency relationships mean that certain agents can block the actions of other agents. However, other kinds of dependence could be considered. For instance, we could study situations in which certain agents have coercive power over others, or situations in which the dependency relationships are fuzzy, in the sense that the agents enjoy a certain degree of freedom. Another possibility for future research would be to apply our value to the study of centrality in digraphs. Each digraph can be assigned an authorization structure. The Shapley authorization correspondence applied to that authorization structure could describe how much each node of the digraph depends on other nodes for it to be connected.

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