Math Meth Oper Res (2000) 52:221–236

Mathematical Methods
of Operations Research

The position value for union stable systems

E. Algaba*, J. M. Bilbao*, P. Borm†, J. J. López*

* University of Seville, Escuela Superior de Ingenieros, Camino de los Descubrimientos s/n, 41092 Sevilla, Spain (E-mail: mbilbao@cica.es)
† Tilburg University, Department of Econometrics, P.O. Box 90153, 5000 LE Tilburg, The Netherlands (E-mail: P.E.M.Borm@kub.nl)

Manuscript received: October 1998

Abstract. In this paper, we study the position value for games in which partial cooperation exist, that is based on a *union stable* coalition system. The concept of *basis* is introduced for these systems, allowing for a definition of the *position value*. Moreover, an axiomatic characterization of the position value is provided for a specific class of union stable systems. Conditions under which convexity is inherited from the underlying game to the *conference game*, and the position value is a core vector of the *restricted game* are provided.

1991 Mathematics Subject Classification: 90D12

Key words: Union stable systems, conference game, position value

1 Introduction

In cooperative game theory, partial cooperation assumes that the formation of coalitions is restricted. Several models of partial cooperation have been proposed, among which are those derived from *communication situations* as introduced by Myerson [4]. This line of research was continued by Owen [8], van den Nouweland and Borm [6], Borm, Owen and Tijs [1], van den Nouweland, Borm and Tijs [7], and Potters and Reijnierse [9].

In Myerson's model, the bilateral relations among the players are represented by means of an undirected graph and the feasible coalitions are those that induce connected subgraphs. However, partial cooperation can not always be modelled by a graph, so the communication model has been generalized in several directions, for instance towards *conference structures* by Myerson [5] and *hypergraph communication situations* by van den Nouweland, Borm and Tijs [7].

In this article, we study partial cooperation structures, which satisfy the following property: Given any two feasible coalitions with a non-empty intersection, the union is another feasible coalition. This type of feasible coalition systems will be called *union stable systems*. In the partial cooperation context, this condition means that if two feasible coalitions have common elements, these ones will act as intermediaries between the two coalitions in order to establish meaningful cooperation in the whole group. A particular case of union stable systems are the communication situations.

Section 2 provides the formal definition of a union stable system, and the notions of *basis* and *supports*. In section 3, we first introduce the $\mathscr{F}$*-restricted games* and the *conference games*. The *position value* for games on union stable systems is defined, it is derived from the Shapley value of the conference game. This value was first introduced for communication situations by Meesen [3], and later, Borm, Owen and Tijs [1] gave an axiomatic characterization for the position value on the subclass of communication situations for which the communication graphs do not contain cycles. In this paper, this axiomatic characterization for the position value is generalized.

In section 4, the conditions under which the convexity is inherited from the underlying game to the conference game are investigated. For that, first a characterization of convexity in cooperative games due to Shapley [11] is extended. Moreover, we show that under these same conditions the position value is in the core of the restricted game.

2 On the basis of a union stable system

Definition 1. *Let $N = \{1, 2, \ldots, n\}$ be a finite set of players and $\mathscr{F} \subseteq 2^N$ a collection of feasible coalitions. The set system $\mathscr{F}$ is called union stable if for all $A, B \in \mathscr{F}$ with $A \cap B \neq \emptyset$ it is satisfied that $A \cup B \in \mathscr{F}$.*

A communication situation is a triple (N, v, E), where (N, v) is a game and (N, E) is a graph. It is easy to see that the collection $\mathscr{F}$, defined by

$$\mathscr{F} = \{S \subseteq N : (S, E(S)) \text{ is a connected subgraph of } (N, E)\},$$

is a union stable system. Notice that a union stable system can not always be modelled by a communication situation. For example, let $N = \{1, 2, 3, 4\}$ and $\mathscr{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2, 3\}, \{2, 3, 4\}, N\}$. This family is union stable, but does not coincide with the connected subgraph family of any graph.

Let $\mathscr{F}$ be a union stable family and $\mathscr{G} \subseteq \mathscr{F}$. We define inductively the families

$$\mathscr{G}^{(0)} = \mathscr{G}, \quad \mathscr{G}^{(n)} = \{S \cup T : S, T \in \mathscr{G}^{(n-1)},\ S \cap T \neq \emptyset\} \quad (n = 1, 2, \ldots)$$

Notice that $\mathscr{G}^{(0)} \subseteq \mathscr{G}^{(n-1)} \subseteq \mathscr{G}^{(n)} \subseteq \mathscr{F}$, since $\mathscr{G} \subseteq \mathscr{F}$ and $\mathscr{F}$ is union stable. The inductive process is finite because $\mathscr{F}$ is finite.

Definition 2. *Let $\mathscr{F}$ be a union stable system and let $\mathscr{G} \subseteq \mathscr{F}$. We define the closure $\bar{\mathscr{G}}$ by $\bar{\mathscr{G}} = \mathscr{G}^{(k)}$, where k is the smallest integer such that $\mathscr{G}^{(k+1)} = \mathscr{G}^{(k)}$.*

For any $S \in \mathscr{F}$, $\overline{\{S\}} = \{S\}$, the collection $\bar{\mathscr{G}}$ is union stable, and $\bar{\mathscr{F}} = \mathscr{F}$.

Example 1. Let $N = \{1, 2, 3, 4\}$ and consider the union stable family given by:

$$\mathscr{F} = \{\{1\}, \{3\}, \{1, 2\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, N\}.$$

For the collection $\mathscr{G} = \{\{1, 2\}, \{2, 3\}, \{2, 4\}\}$, note that

$$\mathscr{G}^{(1)} = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}\},$$

$$\mathscr{G}^{(2)} = \bar{\mathscr{G}} = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, N\}.$$

Let $\mathscr{F}$ be a union stable system and $\mathscr{G} \subseteq \mathscr{F}$. If $\mathscr{G}$ is union stable, there can be feasible coalitions which can be written as the union of two feasible coalitions with a non-empty intersection. So, we can consider the following set:

$$D(\mathscr{G}) = \{G \in \mathscr{G} : G = A \cup B,\ A \neq G,\ B \neq G,\ A, B \in \mathscr{G},\ A \cap B \neq \emptyset\}.$$

Note that $D(\mathscr{G})$ is composed of those feasible coalitions which can be written as the union of two distinct feasible coalitions with a non-empty intersection.

Definition 3. *Let $\mathscr{F}$ be a union stable system. The set $B(\mathscr{F}) = \mathscr{F} \setminus D(\mathscr{F})$, is called the basis of $\mathscr{F}$, and the elements of $B(\mathscr{F})$ are called supports of $\mathscr{F}$.*

Example 2. Let $N = \{1, 2, 3, 4\}$ and consider the union stable system

$$\mathscr{F} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{3, 4\},$$
$$\{1, 2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}.$$

Clearly, the set $D(\mathscr{F}) = \{\{1, 2, 3\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$ hence the basis is $B(\mathscr{F}) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{3, 4\}, \{2, 3, 4\}\}$.

By construction, the basis $B(\mathscr{F})$ of $\mathscr{F}$ is unique, non-empty if $\mathscr{F}$ is non-empty, and satisfies the following properties.

(a) If $\emptyset \in \mathscr{F}$, then $\emptyset \in B(\mathscr{F})$.
(b) If $\{j\} \in \mathscr{F}$, for some $j \in N$, then $\{j\} \in B(\mathscr{F})$.
(c) If $S \in \mathscr{F}$ is a minimal element in $(\mathscr{F}, \subseteq)$, then $S \in B(\mathscr{F})$.
(d) Let $S \in \mathscr{F}$, $|S| > 1$. If for all $T \in 2^S \cap \mathscr{F}$, $|T| \in \{0, 1, |S|\}$, then $S \in B(\mathscr{F})$.

In particular, if $S \in \mathscr{F}$ and $|S| \leq 2$, then $S \in B(\mathscr{F})$.

Proposition 1. *Let $\mathscr{F}$ be a union stable system. The map $\varphi : 2^{\mathscr{F}} \to 2^{\mathscr{F}}$, defined by $\varphi(\mathscr{G}) = \bar{\mathscr{G}}$ is a closure operator, i.e.,*

(a) *For all $\mathscr{G} \in 2^{\mathscr{F}}$, $\mathscr{G} \subseteq \varphi(\mathscr{G})$.*
(b) *The relation $\mathscr{G} \subseteq \mathscr{R} \subseteq \mathscr{F}$ implies $\varphi(\mathscr{G}) \subseteq \varphi(\mathscr{R})$.*
(c) *For all $\mathscr{G} \in 2^{\mathscr{F}}$, $\varphi(\varphi(\mathscr{G})) = \varphi(\mathscr{G})$.*

This result assures that $(\mathscr{F}, \varphi(2^{\mathscr{F}}))$, where $\varphi(2^{\mathscr{F}}) = \{\varphi(\mathscr{G}) : \mathscr{G} \in 2^{\mathscr{F}}\}$, is a *closure space*. From now on the closure space $(\mathscr{F}, \varphi(2^{\mathscr{F}}))$ will be denoted by $(\mathscr{F}, -)$ and the elements of $\varphi(2^{\mathscr{F}})$ will be called *closed*. We can obtain:

Proposition 2. *Let* $(\mathscr{F}, -)$ *be the above closure space, and* $\mathscr{G} \in 2^{\mathscr{F}}$*. Then,* $\mathscr{G}$ *is closed if and only if* $\mathscr{G}$ *is union stable.*

Next, we provide two characterizations of the basis of a union stable system.

Proposition 3. *Let* $\mathscr{F}$ *be a union stable system and* $B(\mathscr{F})$ *the basis of* $\mathscr{F}$*. Then* $B(\mathscr{F})$ *is the minimal subset of* $\mathscr{F}$ *such that* $\overline{B(\mathscr{F})} = \mathscr{F}$*.*

Proof. We first prove that $\overline{B(\mathscr{F})} = \mathscr{F}$. We have that $\overline{B(\mathscr{F})} \subseteq \mathscr{F}$, since $B(\mathscr{F}) \subseteq \mathscr{F}$ and $\mathscr{F}$ is union stable. In order to prove the reverse inclusion, we use induction on the number of elements of feasible coalitions in $\mathscr{F}$. Clearly, the minimal elements in $(\mathscr{F}, \subseteq)$ belong to the basis and hence to $\overline{B(\mathscr{F})}$. Now, suppose $F \in \overline{B(\mathscr{F})}$ for all $F \in \mathscr{F}$ with $|F| < p$. Then, given $F \in \mathscr{F}$ with $|F| = p$, we have either $F \in B(\mathscr{F})$ or $F \notin B(\mathscr{F})$. In the first case $F \in \overline{B(\mathscr{F})}$. Otherwise, $F \in D(\mathscr{F})$ and hence, there are two feasible coalitions $S, T \in \mathscr{F}$, $S \neq F$, $T \neq F$, $S \cap T \neq \varnothing$ such that $S \cup T = F$. By using the induction hypothesis, since $|S| < p$ and $|T| < p$, we have that $S, T \in \overline{B(\mathscr{F})}$, and the union stability implies that $F = S \cup T \in \overline{B(\mathscr{F})}$. Finally, we note that $B(\mathscr{F})$ is a minimal subset of $\mathscr{F}$ such that $\overline{B(\mathscr{F})} = \mathscr{F}$ by construction. □

Proposition 4. *Let* $\mathscr{F}$ *be a union stable system and* $\mathscr{G} \subseteq \mathscr{F}$ *union stable. Then, with* $ex(\mathscr{G}) := \{G \in \mathscr{G} : \mathscr{G} \backslash \{G\}$ *is union stable*$\}$ *it holds that* $ex(\mathscr{G}) = B(\mathscr{G})$*.*

Proof. We first prove $ex(\mathscr{G}) \subseteq B(\mathscr{G})$. As $\overline{B(\mathscr{G})} = \mathscr{G}$, $\mathscr{G}$ is the smallest union stable system that contains $B(\mathscr{G})$. Let $G \in ex(\mathscr{G})$. Then $\mathscr{G} \backslash \{G\}$ is union stable. If $G \notin B(\mathscr{G})$, then $B(\mathscr{G}) \subseteq \mathscr{G} \backslash \{G\} \subset \mathscr{G}$ and hence $\mathscr{G}$ would not be the smallest union stable system that contains $B(\mathscr{G})$.

It remains to prove that $B(\mathscr{G}) \subseteq ex(\mathscr{G})$. For this, let $B \in B(\mathscr{G})$. We show that $\mathscr{G} \backslash \{B\}$ is union stable. Indeed, let $S, T \in \mathscr{G} \backslash \{B\}$, with $S \cap T \neq \varnothing$. Since $\mathscr{G}$ is union stable, $S \cup T \in \mathscr{G}$. On the other hand, $S \cup T \neq B$ since $S \cup T = B$ would imply $B \notin B(\mathscr{G})$. Hence, $S \cup T \in \mathscr{G} \backslash \{B\}$. □

The closure space $(\mathscr{F}, -)$ is a *convex geometry* (see Edelman and Jamison [2]) since, if $\mathscr{G} \subseteq \mathscr{F}$ is union stable, then $\overline{ex(\mathscr{G})} = \overline{B(\mathscr{G})} = \mathscr{G}$.

Definition 4. *Consider* $\mathscr{G} \subseteq 2^N$ *and let* $S \subseteq N$*. A set* $T \subseteq S$ *is called a* $\mathscr{G}$*-component of* S *if it is satisfied that* $T \in \mathscr{G}$ *and there exists no* $T' \in \mathscr{G}$ *such that* $T \subset T' \subseteq S$*.*

The $\mathscr{G}$-components of S are the maximal coalitions that belong to $\mathscr{G}$ and are contained in S. We denote by $C_{\mathscr{G}}(S)$ the set of the $\mathscr{G}$-components of S. Observe that the set $C_{\mathscr{G}}(S)$ may be the empty set.

Remark 1. Let (N, v, E) be a communication situation and

$$\mathscr{F} = \{S \subseteq N : (S, E(S)) \text{ is a connected subgraph of } (N, E)\}.$$

In this situation, the $\mathscr{F}$-components of any coalition $S \subseteq N$ are the connected components of the subgraph $(S, E(S))$ and form a partition of S. Moreover, the collection $\{B \in B(\mathscr{F}) : |B| \geq 2\}$ consists of the edges of graph (N, E).

Proposition 5. *The system $\mathcal{F} \subseteq 2^N$ is union stable if and only if for any $S \subseteq N$ such that $C_{\mathcal{F}}(S) \neq \emptyset$, the $\mathcal{F}$-components of S form a partition of a subset of S.*

Proof. Let $\mathcal{F}$ be union stable. Let S^1, S^2, $S^1 \neq S^2$, be maximal feasible coalitions of S. If $S^1 \cap S^2 \neq \emptyset$, then $S^1 \cup S^2 \in \mathcal{F}$ since $\mathcal{F}$ is union stable and $S^1 \cup S^2 \subseteq S$. This contradicts the fact that S^1 and S^2 are $\mathcal{F}$-components of S.

Conversely, assume for any S such that $C_{\mathcal{F}}(S) \neq \emptyset$, that its $\mathcal{F}$-components form a partition of a subset of S. Suppose that $\mathcal{F}$ is not union stable, then there are $A, B \in \mathcal{F}$, with $A \cap B \neq \emptyset$ and $A \cup B \notin \mathcal{F}$. Hence, there must be an $\mathcal{F}$-component $C_1 \in C_{\mathcal{F}}(A \cup B)$, with $A \subseteq C_1$ and an $\mathcal{F}$-component $C_2 \in C_{\mathcal{F}}(A \cup B)$, with $B \subseteq C_2$ such that $C_1 \neq C_2$. This contradicts the fact that the $\mathcal{F}$-components of $A \cup B$ are disjoint. □

Notice that in general the $\mathcal{F}$-components of S do not form a partition of S, but if $\mathcal{F}$ is a union stable system such that $\{i\} \in \mathcal{F}$, $\forall i \in N$, then they do. Now, some relations between $\mathcal{F}$-components, feasible coalitions, and supports of a union stable system are studied, which are used in the next sections.

Proposition 6. *Let $\mathcal{F}$ be a union stable system and $B(\mathcal{F})$ its basis. Then*

(a) *If $N \notin \mathcal{F}$, we define the partition $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p\}$ of the basis $B(\mathcal{F})$ by $\mathcal{B}_i = \{B \in B(\mathcal{F}) : B \subseteq N_i,\ N_i \in C_{\mathcal{F}}(N)\}$. Then, for all $B \in \mathcal{B}_i$, $B' \in \mathcal{B}_j$, with $i \neq j$, $1 \leq i, j \leq p$, we have $B \cap B' = \emptyset$.*
(b) *Let $\mathcal{I} \subseteq B(\mathcal{F})$, $\mathcal{J} \subseteq B(\mathcal{F})$ such that for all $B \in \mathcal{I}$ and for all $B' \in \mathcal{J}$, we have $B \cap B' = \emptyset$. Then*
   (i) *For all $S \in \overline{\mathcal{I}}$ and for all $S' \in \overline{\mathcal{J}}$, $S \cap S' = \emptyset$.*
   (ii) $\overline{\mathcal{I} \cup \mathcal{J}} = \overline{\mathcal{I}} \cup \overline{\mathcal{J}}$.
   (iii) $C_{\overline{\mathcal{I} \cup \mathcal{J}}}(N) = C_{\overline{\mathcal{I}}}(N) \cup C_{\overline{\mathcal{J}}}(N)$.

Proposition 7. *Let $\mathcal{F}$ be a union stable system and $B(\mathcal{F})$ its basis. Let $F \in \mathcal{F}$ with $|F| \geq 2$. Then, F can be written as a union of supports of size at least two.*

Proof. It is clear that F is a union of supports B_i, $i \in I$, with non-empty intersections. If $|B_j| = 1$, there is B_k, $k \in I$, $k \neq j$ such that $B_k \cap B_j \neq \emptyset$, and hence $B_j \subseteq B_k$ and $|B_k| \geq 2$. Therefore F can be written as a union of supports of size at least two with a non-empty intersection. □

3 The position value: properties and axiomatization

We now consider the *$\mathcal{F}$-restricted game* and the *conference game*, derived from a cooperative game and a union stable system $\mathcal{F}$.

Definition 5. *Let (N, v) be a cooperative game in coalitional form and $\mathcal{F} \subseteq 2^N$ a union stable system. The $\mathcal{F}$-restricted game $v^{\mathcal{F}} : 2^N \to \mathbb{R}$, is defined by $v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T)$.*

The $\mathcal{F}$-restricted game measures the economic values of the coalitions assuming that the entire cooperation structure given by $\mathcal{F}$ can be considered. Thus, the $\mathcal{F}$-restricted game focuses on the role of a player in creating economic possibilities and establishing meaningful communication among the players.

Definition 6. *Let* (N, v) *be a cooperative game and* $\mathscr{F} \subseteq 2^N$ *a union stable system. Let* $\mathscr{B}$ *be the basis of* $\mathscr{F}$ *and* $\mathscr{C} = \{B \in \mathscr{B} : |B| \geq 2\}$. *The conference game is the game* $(\mathscr{C}, v^{\mathscr{C}})$ *where* $v^{\mathscr{C}} : 2^{\mathscr{C}} \to \mathbb{R}$, *is defined by* $v^{\mathscr{C}}(\mathscr{A}) = v^{\bar{\mathscr{A}}}(N)$.

The game $(\mathscr{C}, v^{\mathscr{C}})$ is well defined since for each $\mathscr{A} \subseteq \mathscr{C}$, its closure $\bar{\mathscr{A}}$ is a union stable system. The conference game measures the economic value of the grand coalition when specific parts of the cooperation structure are considered. Note that if the game (N, v) is zero-normalized, i.e., $v(\{i\}) = 0$ for all $i \in N$, then $v^{\mathscr{C}}(\mathscr{C}) = v^{\bar{\mathscr{C}}}(N) = v^{\mathscr{F}}(N)$.

Remark 2. Let (N, v, E) be a communication situation. The family $\mathscr{F}$, defined by $\mathscr{F} = \{S \subseteq N : (S, E(S))$ is a connected subgraph of $(N, E)\}$, is a union stable system. The two above definitions extend the *point game* and the *arc game* respectively. The arc game was introduced by Borm, Owen, and Tijs [1], and for this situation we have that $\mathscr{C} = \{\{i, j\} : \{i, j\} \in E\}$.

A *union stable cooperation structure* is a triple $(N, v, \mathscr{F})$ where $N = \{1, \ldots, n\}$ is the set of players, (N, v) is a game $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$, and $\mathscr{F}$ a union stable system. For convenience, we assume from now on that the underlying game (N, v) is zero-normalized.

The position value for graph communication situations was first introduced in Meesen [3] and studied in Borm, Owen and Tijs [1]. This value was extended to hypergraphs communication situations in van den Nouweland, Borm and Tijs [7]. Now, this value will be defined in a union stable cooperation structure, by assigning to each support its Shapley value in the conference game and then dividing the value of each support equally among its participants.

Let $(N, v, \mathscr{F})$ be a union stable cooperation structure. On $\mathscr{C} \subseteq \mathscr{B}$, we have defined the game $(\mathscr{C}, v^{\mathscr{C}})$ that is called the conference game. Therefore, we can consider the Shapley value associated to the game $(\mathscr{C}, v^{\mathscr{C}})$, $\Phi(\mathscr{C}, v^{\mathscr{C}}) \in \mathbb{R}^{\mathscr{C}}$.

Definition 7. *Let* $(N, v, \mathscr{F})$ *be a union stable cooperation structure. For* $i \in N$ *the position value* $\pi_i(N, v, \mathscr{F})$ *is given by*

$$\pi_i(N, v, \mathscr{F}) = \sum_{C \in \mathscr{C}_i} \frac{1}{|C|} \Phi_C(\mathscr{C}, v^{\mathscr{C}}),$$

where $\mathscr{C}_i = \{C \in \mathscr{C} : i \in C\}$, *denotes the set of all supports with at least two players in which i participates.*

Example 3. Let $N = \{1, 2, 3, 4\}$ and $\mathscr{F} = \{\{1\}, \{1, 2, 3\}, \{2, 3, 4\}, N\}$. Then $\mathscr{B} = \{\{1\}, \{1, 2, 3\}, \{2, 3, 4\}\}$, and $\mathscr{C} = \{\{1, 2, 3\}, \{2, 3, 4\}\}$. Let $v : 2^N \to \mathbb{R}$ be the game defined by $v(\emptyset) = 0$, $v(S) = |S| - 1$, for all non-empty $S \subseteq N$. Thus, $v^{\mathscr{F}}(S) = |S| - 1$, if $S \in \mathscr{F}$, and $v^{\mathscr{F}}(S) = 0$, otherwise. The conference game is

$\mathscr{A} \subseteq \mathscr{C}$	$\bar{\mathscr{A}}$	$C_{\bar{\mathscr{A}}}(N)$	$v^{\mathscr{C}}(\mathscr{A})$
$\{\{1,2,3\}\}$	$\{\{1,2,3\}\}$	$\{\{1,2,3\}\}$	2
$\{\{2,3,4\}\}$	$\{\{2,3,4\}\}$	$\{\{2,3,4\}\}$	2
$\{\{1,2,3\},\{2,3,4\}\}$	$\{\{1,2,3\},\{2,3,4\},\{1,2,3,4\}\}$	$\{\{1,2,3,4\}\}$	3

The Shapley values are $\Phi_{\{1,2,3\}}(\mathscr{C}, v^{\mathscr{C}}) = \Phi_{\{2,3,4\}}(\mathscr{C}, v^{\mathscr{C}}) = 3/2$, hence the position value is $\pi(N, v, \mathscr{F}) = (1/2, 1, 1, 1/2)$.

The set of all union stable cooperation structures on N will be denoted by $US^N = \{(N, v, \mathscr{F}) : \mathscr{F}$ is union stable$\}$. An *allocation rule* on US^N is a map γ that assigns to each union stable cooperation structure $(N, v, \mathscr{F})$ a payoff vector, $\gamma(N, v, \mathscr{F}) \in \mathbb{R}^N$ which is *component-efficient* and *component-dummy*:

(1) For all $(N, v, \mathscr{F}) \in US^N$, $M \in C_{\mathscr{F}}(N)$, we have $\sum_{i \in M} \gamma_i(N, v, \mathscr{F}) = v(M)$.
(2) For all $i \notin \bigcup_{M \in C_{\mathscr{F}}(N)} M$, we have $\gamma_i(N, v, \mathscr{F}) = 0$.

In order to prove that the position value is an allocation rule, we need the following result.

Lemma 1. *Let $(N, v, \mathscr{F})$ be a union stable cooperation structure with $N \notin \mathscr{F}$ and $(\mathscr{C}, v^{\mathscr{C}})$ the associated conference game. Define the partition $\{\mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_p\}$ of $\mathscr{C}$ by $\mathscr{D}_i = \{D \in \mathscr{C} : D \subseteq N_i,\ N_i \in C_{\mathscr{F}}(N)\}$. Then $v^{\mathscr{C}} = \sum_{i=1}^{p} v^{\mathscr{D}_i}$, where the games $(\mathscr{C}, v^{\mathscr{D}_i})$ are defined by $v^{\mathscr{D}_i}(\mathscr{A}) = v^{\mathscr{C}}(\mathscr{A} \cap \mathscr{D}_i)$, for all $\mathscr{A} \subseteq \mathscr{C}$, $1 \le i \le p$.*

Proof. If $\mathscr{A} \subseteq \mathscr{C}$, then

$$v^{\mathscr{C}}(\mathscr{A}) = v^{\mathscr{C}}\left(\bigcup_{i=1}^{p} (\mathscr{A} \cap \mathscr{D}_i)\right) = v^{\overline{\bigcup_{i=1}^{p}(\mathscr{A} \cap \mathscr{D}_i)}}(N)$$

$$= \sum_{M \in C_{\overline{\bigcup_{i=1}^{p}(\mathscr{A} \cap \mathscr{D}_i)}}(N)} v(M) = \sum_{M \in \bigcup_{i=1}^{p} C_{\overline{\mathscr{A} \cap \mathscr{D}_i}}(N)} v(M)$$

$$= \sum_{i=1}^{p} \left[\sum_{M \in C_{\overline{\mathscr{A} \cap \mathscr{D}_i}}(N)} v(M) \right] = \sum_{i=1}^{p} v^{\mathscr{C}}(\mathscr{A} \cap \mathscr{D}_i) = \sum_{i=1}^{p} v^{\mathscr{D}_i}(\mathscr{A}).$$

The equality in the second line follows directly from proposition 6 (b) (iii). □

Theorem 1. *The position value $\pi : US^N \to \mathbb{R}^N$, is an allocation rule.*

Proof. We first prove that the position value satisfies component-dummy. If $i \notin \bigcup_{M \in C_{\mathscr{F}}(N)} M$, this player is not in any feasible coalition. Therefore $\mathscr{C}_i = \emptyset$ and $\pi_i(N, v, \mathscr{F}) = 0$, by definition.

To prove component-efficiency, let $M \in C_{\mathscr{F}}(N)$. Then either $|M| = 1$ or $|M| > 1$. If $|M| = 1$, then $M = \{i\}$ and $\mathscr{C}_i = \emptyset$. Hence, $\sum_{i \in M} \pi_i(N, v, \mathscr{F}) = 0 = v(\{i\})$, since v is zero-normalized. If $|M| > 1$, then $\mathscr{C}_i \neq \emptyset$, for all $i \in M$. Moreover as $M \in C_{\mathscr{F}}(N)$, denoting all basis elements which are contained in M by B_k, $k \in K$, we have $M = \bigcup_{k \in K} B_k$. Hence, if $i \in M$, then $\mathscr{C}_i \subseteq \{B_k\}_{k \in K}$. Suppose on the contrary that $C \in \mathscr{C}_i$ and $C \neq B_k$ for all $k \in K$. Then, we have $C \cap (N \backslash M) \neq \emptyset$. Moreover, since $i \in M \cap C$, it follows that $M \cup C \in \mathscr{F}$, but this contradicts that $M \in C_{\mathscr{F}}(N)$. Therefore, applying lemma 1 and using the properties of the Shapley value:

$$
\begin{aligned}
\sum_{i \in M} \pi_i(N, v, \mathscr{F}) &= \sum_{i \in M} \left[\sum_{C \in \mathscr{C}_i} \frac{1}{|C|} \Phi_C(\mathscr{C}, v^{\mathscr{C}}) \right] \\
&= \sum_{\{\{B_k\}_{k \in K} : |B_k| \geq 2\}} \left[|B_k| \frac{1}{|B_k|} \right] \Phi_{B_k}(\mathscr{C}, v^{\mathscr{C}}) \\
&= \sum_{\{\{B_k\}_{k \in K} : |B_k| \geq 2\}} \Phi_{B_k}(\mathscr{C}, v^{\mathscr{C}}) \\
&= \sum_{\{\{B_k\}_{k \in K} : |B_k| \geq 2\}} \Phi_{B_k}(\mathscr{C}, v^{\mathscr{D}_k}) = v^{\mathscr{D}_k}(\{B_k\}_{k \in K}) \\
&= v^{\mathscr{C}}(\{B_k\}_{k \in K} \cap \mathscr{D}_k) = v^{\mathscr{C}}(\{B_k\}_{k \in K}) = v(M). \qquad \square
\end{aligned}
$$

Definition 8. *An allocation rule $\gamma : US^N \to \mathbb{R}^N$ is additive if $\gamma(N, v + w, \mathscr{F}) = \gamma(N, v, \mathscr{F}) + \gamma(N, w, \mathscr{F})$, for all $(N, v, \mathscr{F}), (N, w, \mathscr{F}) \in US^N$.*

Definition 9. *The support $H \in \mathscr{C}$ is called superfluous for $(N, v, \mathscr{F}) \in US^N$ if $v^{\mathscr{C}}(\mathscr{A}) = v^{\mathscr{C}}(\mathscr{A} \backslash \{H\})$, for all $\mathscr{A} \subseteq \mathscr{C}$, i.e., if H is a null player in the conference game. An allocation rule $\gamma : US^N \to \mathbb{R}^N$ has the superfluous support property if $\gamma(N, v, \mathscr{F}) = \gamma(N, v, \overline{\mathscr{B} \backslash \{H\}})$, for all $(N, v, \mathscr{F}) \in US^N$ and for every superfluous support $H \in \mathscr{C}$ for $(N, v, \mathscr{F})$.*

Theorem 2. *The position value π satisfies additivity and the superfluous support property.*

Proof. Additivity follows directly from the additivity of the Shapley value. Let $(N, v, \mathscr{F}) \in US^N$. Let $H \in \mathscr{C}$ be a superfluous support. We have to prove $\pi(N, v, \mathscr{F}) = \pi(N, v, \overline{\mathscr{B} \backslash \{H\}})$. For each $i \in N$, by definition

$$
\pi_i(N, v, \mathscr{F}) = \sum_{C \in \mathscr{C}_i} \frac{1}{|C|} \Phi_C(\mathscr{C}, v^{\mathscr{C}}),
$$

$$
\pi_i(N, v, \overline{\mathscr{B} \backslash \{H\}}) = \sum_{C \in \mathscr{C}_i \backslash \{H\}} \frac{1}{|C|} \Phi_C(\mathscr{C} \backslash \{H\}, v^{\mathscr{C} \backslash \{H\}}).
$$

If $i \in H$, we can deduce

$$\pi_i(N, v, \mathscr{F}) = \sum_{C \in \mathscr{C}_i \setminus \{H\}} \frac{1}{|C|} \Phi_C(\mathscr{C}, v^{\mathscr{C}}) + \frac{1}{|H|} \Phi_H(\mathscr{C}, v^{\mathscr{C}}),$$

where $\Phi_H(\mathscr{C}, v^{\mathscr{C}}) = 0$, since H is a superfluous support and hence a null player in the conference game $(\mathscr{C}, v^{\mathscr{C}})$. Therefore, for all $i \in N$,

$$\pi_i(N, v, \mathscr{F}) = \sum_{C \in \mathscr{C}_i \setminus \{H\}} \frac{1}{|C|} \Phi_C(\mathscr{C}, v^{\mathscr{C}}).$$

This implies that it suffices to prove that $\Phi_C(\mathscr{C}, v^{\mathscr{C}}) = \Phi_C(\mathscr{C} \setminus \{H\}, v^{\mathscr{C} \setminus \{H\}})$, for all $C \in \mathscr{C} \setminus \{H\}$. Applying the Shapley formula,

$$\begin{aligned}
\Phi_C(\mathscr{C}, v^{\mathscr{C}}) &= \sum_{\{\mathscr{S} \subseteq \mathscr{C} : C \in \mathscr{S}\}} \gamma(\mathscr{S})[v^{\mathscr{C}}(\mathscr{S}) - v^{\mathscr{C}}(\mathscr{S} \setminus \{C\})] \\
&= \sum_{\{\mathscr{S} \subseteq \mathscr{C} : C \in \mathscr{S}, H \in \mathscr{S}\}} \gamma(\mathscr{S})[v^{\mathscr{C}}(\mathscr{S}) - v^{\mathscr{C}}(\mathscr{S} \setminus \{C\})] \\
&\quad + \sum_{\{\mathscr{S} \subseteq \mathscr{C} : C \in \mathscr{S}, H \notin \mathscr{S}\}} \gamma(\mathscr{S})[v^{\mathscr{C}}(\mathscr{S}) - v^{\mathscr{C}}(\mathscr{S} \setminus \{C\})],
\end{aligned}$$

where $\gamma(\mathscr{S}) = (s-1)!(c-s)!/c!,\ s = |\mathscr{S}|,\ c = |\mathscr{C}|$. To each coalition $\mathscr{S} \subseteq \mathscr{C}$ which contains the superfluous support H corresponds – when H is deleted – a coalition which does not contain it, and this relation is bijective. Therefore,

$$\begin{aligned}
\Phi_C(\mathscr{C}, v^{\mathscr{C}}) &= \sum_{\{\mathscr{S} \subseteq \mathscr{C} : C \in \mathscr{S}, H \in \mathscr{S}\}} \gamma(\mathscr{S})[v^{\mathscr{C}}(\mathscr{S} \setminus \{H\}) - v^{\mathscr{C}}((\mathscr{S} \setminus \{C\}) \setminus \{H\})] \\
&\quad + \sum_{\{\mathscr{S} \subseteq \mathscr{C} : C \in \mathscr{S}, H \notin \mathscr{S}\}} \gamma(\mathscr{S})[v^{\mathscr{C}}(\mathscr{S}) - v^{\mathscr{C}}(\mathscr{S} \setminus \{C\})] \\
&= \sum_{\{\mathscr{S} \subseteq \mathscr{C} : C \in \mathscr{S}, H \notin \mathscr{S}\}} (\gamma(\mathscr{S} \cup \{H\}) + \gamma(\mathscr{S}))[v^{\mathscr{C}}(\mathscr{S}) - v^{\mathscr{C}}(\mathscr{S} \setminus \{C\})] \\
&= \sum_{\{\mathscr{S} \subseteq \mathscr{C} \setminus \{H\} : C \in \mathscr{S}\}} \frac{(s-1)!(c-s-1)!}{(c-1)!} [v^{\mathscr{C}}(\mathscr{S}) - v^{\mathscr{C}}(\mathscr{S} \setminus \{C\})] \\
&= \sum_{\{\mathscr{S} \subseteq \mathscr{C} \setminus \{H\} : C \in \mathscr{S}\}} \frac{(s-1)!(c-s-1)!}{(c-1)!} \\
&\quad \times [v^{\mathscr{C} \setminus \{H\}}(\mathscr{S}) - v^{\mathscr{C} \setminus \{H\}}(\mathscr{S} \setminus \{C\})] \\
&= \Phi_C(\mathscr{C} \setminus \{H\}, v^{\mathscr{C} \setminus \{H\}}).
\end{aligned}$$

Note that $v^{\mathscr{C}}(\mathscr{S}) = v^{\mathscr{C} \setminus \{H\}}(\mathscr{S})$ because $\mathscr{S} \subseteq \mathscr{C} \setminus \{H\}$. □

Let $(N, v, \mathscr{F})$ be a union stable cooperation structure. The *influence* of a player i is given by $I_i(N, v, \mathscr{F}) = \sum_{C \in \mathscr{C}_i} 1/|C|$.

The triple $(N, v, \mathscr{F}) \in US^N$ is called *support anonymous* if there exists a function $f : \{0, 1, \ldots, |\mathscr{C}|\} \to \mathbb{R}$ such that $v^{\mathscr{C}}(\mathscr{A}) = f(|\mathscr{A}|)$, for all $\mathscr{A} \subseteq \mathscr{C}$, i.e., if the conference game $(\mathscr{C}, v^{\mathscr{C}})$ is anonymous.

An allocation rule $\gamma : US^N \to \mathbb{R}^N$ has the *influence property* if for each $(N, v, \mathscr{F}) \in US^N$ that is support anonymous there exists an $\alpha \in \mathbb{R}$ such that $\gamma_i(N, v, \mathscr{F}) = \alpha I_i(N, v, \mathscr{F})$, for all $i \in N$.

So, if the value of the grand coalition only depends on the number of supports that are present, the payoffs to the players are proportional to its influence.

Theorem 3. *The position value π satisfies the influence property.*

Proof. Consider $(N, v, \mathscr{F}) \in US^N$ that is support anonymous. This means that the game $(\mathscr{C}, v^{\mathscr{C}})$ is anonymous, hence $\Phi_C(\mathscr{C}, v^{\mathscr{C}}) = v^{\mathscr{C}}(\mathscr{C})/|\mathscr{C}|$, for all $C \in \mathscr{C}$. Thus, for all $i \in N$, we have

$$\pi_i(N, v, \mathscr{F}) = \sum_{C \in \mathscr{C}_i} \frac{1}{|C|} \Phi_C(\mathscr{C}, v^{\mathscr{C}}) = \sum_{C \in \mathscr{C}_i} \frac{1}{|C|} \left(\frac{1}{|\mathscr{C}|} v^{\mathscr{C}}(\mathscr{C}) \right)$$

$$= \frac{v^{\mathscr{C}}(\mathscr{C})}{|\mathscr{C}|} \sum_{C \in \mathscr{C}_i} \frac{1}{|C|} = \frac{v^{\mathscr{C}}(\mathscr{C})}{|\mathscr{C}|} I_i(N, v, \mathscr{F}).$$

We may conclude with the choice of $\alpha = v^{\mathscr{C}}(\mathscr{C})/|\mathscr{C}|$. □

The following example illustrates the influence property of the position value in a union stable cooperation structure.

Example 4. Let $N = \{1, 2, 3, 4\}$ and let $(N, v, \mathscr{F})$ be the triple as considered in Example 3. Then, the influence $I_1 = 1/3$, since $\mathscr{C}_1 = \{\{1, 2, 3\}\}$, $I_2 = I_3 = 2/3$, since $\mathscr{C}_2 = \mathscr{C}_3 = \{\{1, 2, 3\}, \{2, 3, 4\}\}$, and $I_4 = 1/3$, since $\mathscr{C}_4 = \{\{2, 3, 4\}\}$. So, $I(N, v, \mathscr{F}) = (1/3, 2/3, 2/3, 1/3)$. In this case, $(N, v, \mathscr{F})$ is a support anonymous system. Indeed, let $f : \{0, 1, 2\} \to \mathbb{R}$ be such that $f(0) = 0, f(1) = 2$, and $f(2) = 3$, then $v^{\mathscr{C}}(\mathscr{A}) = f(|\mathscr{A}|)$, $\mathscr{A} \subseteq \mathscr{C}$. The position value is $\pi(N, v, \mathscr{F}) = 3/2(1/3, 2/3, 2/3, 1/3)$, where the number $\alpha = 3/2$, as it was shown in the proof of theorem 3.

Lemma 2. *Let $\mathscr{F}$ be a union stable system, such that the expression of each non-unitary feasible coalition as a union of non-unitary supports is unique. If $S = \bigcup_{i \in I} S_i$, and $T = \bigcup_{j \in J} T_j$ with $S_i, T_j \in \mathscr{C}$, for all i,j, then $S \subseteq T$ if and only if $\{S_i\}_{i \in I} \subseteq \{T_j\}_{j \in J}$.*

Proof. Let $S \subseteq T$. If $\{S_i\}_{i \in I}$ is not contained in $\{T_j\}_{j \in J}$, then there is S_k, $k \in I$, such that $S_k \neq T_j$, for all $j \in J$. Hence, $T = \bigcup_{j \in J} T_j = (\bigcup_{j \in J} T_j) \cup S_k$ since $S_k \subseteq S \subseteq T$, and consequently, the expression of T as a union of supports is not unique in contradiction with the hypothesis. The converse is obvious. □

Next, we obtain an axiomatic characterization of the position value on the class of the union stable cooperation structures $(N, v, \mathscr{F})$ such that:

(1) For all $S, T \in \mathscr{F}$, with $|S \cap T| \geq 2$ we have $S \cap T \in \mathscr{F}$.
(2) All non-unitary feasible coalitions can be written in a unique way as a union of non-unitary supports.

We will denote by USI^N the subclass of US^N where the above two conditions are true. The following result is a generalization of the characterization for the position value given by Borm, Owen and Tijs [1], since in a communication situation the non-unitary supports are the edges of the graph and the feasible coalitions are the connected ones and the subclass of communication situations for which the communication graphs do not contain cycles satisfy the two above conditions.

Theorem 4. *The position value is the unique allocation rule on USI^N that satisfies additivity, the superfluous support property and the influence property.*

Proof. To prove uniqueness, let $\pi, \gamma : USI^N \to \mathbb{R}^N$ be two allocation rules that satisfy the three properties. We prove that $\pi(N, v, \mathscr{F}) = \gamma(N, v, \mathscr{F})$. The game (N, v) is zero-normalized and $v = \sum_{\{T \subseteq N : |T| \geq 2\}} \alpha_T u_T$, where u_T are the unanimity games. Moreover, since π and γ are additive allocation rules, it suffices to show $\pi(N, \alpha u_T, \mathscr{F}) = \gamma(N, \alpha u_T, \mathscr{F})$ for all $T \subseteq N$, $|T| \geq 2$, $\alpha \in \mathbb{R}$. Two cases will be distinguished:

(a) There exists a coalition $S \in \mathscr{F}$ such that $T \subseteq S$.
(b) There is no coalition $S \in \mathscr{F}$ such that $T \subseteq S$.

We first consider (b). As there is no coalition $S \in \mathscr{F}$ such that $T \subseteq S$

$$(\alpha u_T)^{\mathscr{C}}(\mathscr{A}) = \alpha \sum_{M \in C_{\overline{\mathscr{A}}}(N)} u_T(M) = 0, \quad \text{for all } \mathscr{A} \subseteq \mathscr{C},$$

which implies that the conference game associated to αu_T is the null game, and consequently, $\Phi_C(\mathscr{C}, (\alpha u_T)^{\mathscr{C}}) = 0$, for all $C \in \mathscr{C}$. So $\pi_i(N, \alpha u_T, \mathscr{F}) = 0$, for all $i \in N$. On the other hand, if $(\alpha u_T)^{\mathscr{C}} = 0$, it means that each support of $\mathscr{C}$ is superfluous and hence $\gamma(N, \alpha u_T, \mathscr{F}) = \gamma(N, \alpha u_T, \overline{\mathscr{B} \backslash \mathscr{C}})$. Moreover, the triple $(N, \alpha u_T, \overline{\mathscr{B} \backslash \mathscr{C}})$ is support anonymous and then the influence property implies that there is a $\beta \in \mathbb{R}$ such that $\gamma(N, \alpha u_T, \overline{\mathscr{B} \backslash \mathscr{C}}) = \beta I(N, \alpha u_T, \overline{\mathscr{B} \backslash \mathscr{C}}) = \mathbf{0}$. Therefore $\pi(N, \alpha u_T, \mathscr{F}) = \gamma(N, \alpha u_T, \mathscr{F}) = \mathbf{0}$.

Now consider (a). By assumption, the set $\{F \in \mathscr{F} : T \subseteq F\}$ is non-empty and define $\overline{T} = \bigcap \{F \in \mathscr{F} : T \subseteq F\}$. We have that $\overline{T} \neq \emptyset$ and by condition (1) of USI^N it follows that $\overline{T} \in \mathscr{F}$. It is also immediate that $\overline{T}$ is the minimal feasible set that contains the set T. Proposition 7 implies there are $B_i \in \mathscr{C}$ such that $\overline{T} = \bigcup_{i \in I} B_i$. Thus, the conference game associated to αu_T is

$$(\alpha u_T)^{\mathscr{C}} : 2^{\mathscr{C}} \to \mathbb{R}, \quad (\alpha u_T)^{\mathscr{C}}(\mathscr{A}) = \begin{cases} \alpha & \text{if } \overline{T} \in \bar{\mathscr{A}}, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $\bar{T} \in \bar{\mathscr{A}} \Leftrightarrow \{B_i\}_{i\in I} \subseteq \mathscr{A}$, since each non-unitary feasible coalition can be written in a unique way as a union of non-unitary supports. Hence

$$(\alpha u_T)^{\mathscr{C}}(\mathscr{A}) = \begin{cases} \alpha & \text{if } \{B_i\}_{i\in I} \subseteq \mathscr{A}, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that all supports $B \in \mathscr{C}$ such that $B \notin \{B_i\}_{i\in I}$, are superfluous in the conference game, and so the superfluous support property implies that

$$\pi(N, \alpha u_T, \mathscr{F}) = \pi(N, \alpha u_T, \mathscr{F}'), \quad \text{and} \quad \gamma(N, \alpha u_T, \mathscr{F}) = \gamma(N, \alpha u_T, \mathscr{F}'),$$

where $\mathscr{F}' = \bigcup\{\{j\} : \{j\} \in \mathscr{F}\} \cup (\overline{\{B_i\}}_{i\in I})$.

The conference game associated to αu_T in $\mathscr{F}'$ is support anonymous since,

$$(\alpha u_T)^{\{B_i\}_{i\in I}}(\mathscr{A}) = \begin{cases} \alpha & \text{if } \mathscr{A} = \{B_i\}_{i\in I}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, applying the influence property for the allocation rules π and γ, there is a $\beta, \delta \in \mathbb{R}$ such that $\pi(N, \alpha u_T, \mathscr{F}') = \delta I(N, \alpha u_T, \mathscr{F}')$ and $\gamma(N, \alpha u_T, \mathscr{F}') = \beta I(N, \alpha u_T, \mathscr{F}')$. So, if $i \in N\backslash\bar{T}$ then $I_i(N, \alpha u_T, \mathscr{F}') = 0$, and therefore $\pi_i(N, \alpha u_T, \mathscr{F}') = \gamma_i(N, \alpha u_T, \mathscr{F}') = 0$. Let $i \in \bar{T}$. Since $\bar{T} \in C_{\mathscr{F}'}(N)$, component-efficiency implies

$$\sum_{i\in\bar{T}} \gamma_i(N, \alpha u_T, \mathscr{F}') = \alpha u_T(\bar{T}) = \sum_{i\in\bar{T}} \pi_i(N, \alpha u_T, \mathscr{F}'),$$

and thus,

$$\sum_{i\in\bar{T}} \beta I_i(N, \alpha u_T, \mathscr{F}') = \sum_{i\in\bar{T}} \delta I_i(N, \alpha u_T, \mathscr{F}').$$

Subtracting, $\sum_{i\in\bar{T}}(\beta - \delta)I_i(N, \alpha u_T, \mathscr{F}') = 0$, and as for $\bar{T} \in C_{\mathscr{F}'}(N)$ it is satisfied that $\sum_{i\in\bar{T}} I_i(N, \alpha u_T, \mathscr{F}') \neq 0$, it is deduced that $\beta = \delta$. $\square$

4 Convexity

We study now the conditions under which the convexity is inherited from the underlying game to the conference game and we show that under the same conditions the position value is in the core of the restricted game. Shapley [11] showed that a game (N, v) is convex if and only if $v(T_1 \cup T_2) - v(T_1) - v(T_2) \geq v(S_1 \cup S_2) - v(S_1) - v(S_2)$, where $T_1 \cap T_2 = \emptyset$, $S_1 \subseteq T_1$, $S_2 \subseteq T_2$. In the following result we will extend this condition. For this we will use that a game (N, v) is convex if and only if $v(T \cup R) - v(T) \geq v(S \cup R) - v(S)$, for all $S \subseteq T \subseteq N\backslash R$.

Lemma 3. *If the game (N, v) is convex, then*

$$v\left(\bigcup_{i=1}^{k} T_i\right) - \sum_{i=1}^{k} v(T_i) \geq v\left(\bigcup_{i=1}^{k} S_i\right) - \sum_{i=1}^{k} v(S_i), \quad \textit{for all } k = 2, 3, \ldots,$$

where $T_i \cap T_j = \emptyset$, $i \neq j$, and $S_i \subseteq T_i$, for all i.

Proof. Let $S_i \subseteq T_i$ for all i and $T_i \cap T_j = \emptyset$. It is satisfied that

$$v(T_1 \cup T_2 \cup T_3 \cup \cdots \cup T_k) - v(T_1) \geq v(S_1 \cup T_2 \cup T_3 \cup \cdots \cup T_k) - v(S_1),$$

since $S_1 \subseteq T_1 \subseteq N\backslash\{T_2 \cup \cdots \cup T_k\}$. Similarly,

$$v(S_1 \cup T_2 \cup T_3 \cup \cdots \cup T_k) - v(T_2) \geq v(S_1 \cup S_2 \cup T_3 \cup \cdots \cup T_k) - v(S_2),$$

since $S_2 \subseteq T_2 \subseteq N\backslash\{S_1 \cup T_3 \cup \cdots \cup T_k\}$. Following in the same way:

$$v(S_1 \cup S_2 \cup \cdots \cup S_{k-1} \cup T_k) - v(T_k) \geq v(S_1 \cup S_2 \cup \cdots \cup S_{k-1} \cup S_k) - v(S_k).$$

since $S_k \subseteq T_k \subseteq N\backslash\{S_1 \cup S_2 \cup \cdots \cup S_{k-1}\}$. Now, by adding all k inequalities we find $v(T_1 \cup T_2 \cup \cdots \cup T_k) - \sum_{i=1}^{k} v(T_i) \geq v(S_1 \cup S_2 \cup \cdots \cup S_k) - \sum_{i=1}^{k} v(S_i)$. □

Theorem 5. *Let $(N, v, \mathscr{F}) \in USI^N$. If (N, v) is convex, then the conference game $(\mathscr{C}, v^{\mathscr{C}})$ is convex.*

Proof. We have to prove that given $C \in \mathscr{C}$ and $\mathscr{S} \subseteq \mathscr{T} \subseteq \mathscr{C}\backslash\{C\}$ it holds that $v^{\mathscr{C}}(\mathscr{T} \cup \{C\}) - v^{\mathscr{C}}(\mathscr{T}) \geq v^{\mathscr{C}}(\mathscr{S} \cup \{C\}) - v^{\mathscr{C}}(\mathscr{S})$. Taking into account the definition of the conference game, this boils down to

$$\sum_{T \in C_{\overline{\mathscr{T} \cup \{C\}}}(N)} v(T) - \sum_{T \in C_{\overline{\mathscr{T}}}(N)} v(T) \geq \sum_{S \in C_{\overline{\mathscr{S} \cup \{C\}}}(N)} v(S) - \sum_{S \in C_{\overline{\mathscr{S}}}(N)} v(S).$$

Let $C_{\overline{\mathscr{T}}}(N) = \{T_1, T_2, \ldots, T_k\} \cup \{T_{k+1}, T_{k+2}, \ldots, T_h\}$ such that $T_i \cap C \neq \emptyset$ for all $i = 1, \ldots, k$ and $T_i \cap C = \emptyset$ for all $i = k+1, \ldots, h$. Let $i = 1, \ldots, k$, we first show that $|T_i \cap C| = 1$. Suppose $|T_i \cap C| \geq 2$, then $T_i \cap C \in \mathscr{F}$, since $(N, v, \mathscr{F}) \in USI^N$ and $T_i \cap C \neq C$ because otherwise $C \subseteq T_i$ and then $C \in \overline{\mathscr{T}}$ but this contradicts the fact that $\mathscr{T} \subseteq \mathscr{C}\backslash\{C\}$. As $T_i \cap C \subseteq T_i \in \overline{\mathscr{T}}$ and $T_i \cap C \in \mathscr{F}$, then $T_i \cap C$ is either a support or a union of non-unitary supports of $\mathscr{T}$. Hence, since $C = C \cup (T_i \cap C)$ and $T_i \cap C \neq C$, we deduce that C could be expressed in two different ways as a union of supports contradicting that $(N, v, \mathscr{F}) \in USI^N$. Clearly, $T_1 \cap C \neq \emptyset, \ldots, T_k \cap C \neq \emptyset$, and $T_1 \cup C \in \mathscr{F}, \ldots, T_k \cup C \in \mathscr{F}$, hence $T_1 \cup \cdots \cup T_k \cup C \in \mathscr{F}$. Moreover, as we showed above, every $T_1, \ldots, T_k$ contains a unique element of C. For this reason, the maximal feasible coalitions of N in $\overline{\mathscr{T} \cup \{C\}}$ will be $\{T_1 \cup \cdots \cup T_k \cup C, T_{k+1}, T_{k+2}, \ldots, T_h\}$. Therefore, we obtain

$$\sum_{T \in C_{\overline{\mathscr{T} \cup \{C\}}}(N)} v(T) - \sum_{T \in C_{\overline{\mathscr{T}}}(N)} v(T) = v(T_1 \cup \cdots \cup T_k \cup C) - \sum_{i=1}^{k} v(T_i).$$

An analogous reasoning leads to

$$\sum_{S \in C_{\overline{\mathscr{S} \cup \{C\}}}(N)} v(S) - \sum_{S \in C_{\overline{\mathscr{S}}}(N)} v(S) = v(S_1 \cup \cdots \cup S_p \cup C) - \sum_{i=1}^{p} v(S_i),$$

with $S_i \in C_{\overline{\mathscr{S}}}(N)$, $S_i \cap C \neq \emptyset$, $i = 1, \ldots, p$ and each S_i contains a unique element of C. Consequently, the following inequality remains to be proved:

$$v(T_1 \cup \cdots \cup T_k \cup C) - \sum_{i=1}^{k} v(T_i) \geq v(S_1 \cup \cdots \cup S_p \cup C) - \sum_{i=1}^{p} v(S_i). \tag{1}$$

First, $T_1 \cup \cdots \cup T_k \cup C = T_1 \cup \cdots \cup T_k \cup C'$, where $C' \subseteq C$ is defined by $C' = \{i \in C : i \notin T_1 \cup \cdots \cup T_k\}$. Moreover, $S_1 \cup \cdots \cup S_p \cup C = S_1 \cup \cdots \cup S_p \cup C''$, where $C'' \subseteq C$ is defined by $C'' = \{i \in C : i \notin S_1 \cup \cdots \cup S_p\}$.

On the other hand, since $\mathscr{S} \subseteq \mathscr{T}$ we have $\overline{\mathscr{S}} \subseteq \overline{\mathscr{T}}$ and if S_i is a maximal feasible coalition of N in $\overline{\mathscr{S}}$, then $S_i \subseteq T_i$ for a unique maximal feasible coalition T_i of N in $\overline{\mathscr{T}}$. So, $p \leq k$ and without loss of generality $S_1 \subseteq T_1, \ldots, S_p \subseteq T_p$.

Since $S_i \subseteq T_i$ they must contain the same element of C, hence $S_i \cap C = T_i \cap C$. Notice that $C' \subseteq C'' \subseteq C$, therefore we have that $C'' = C' \cup (C'' \setminus C') \subseteq C$, and for each T_j, where $j = p+1, \ldots, k$, there is an element $s(j) \in (C'' \setminus C')$ such that $s(j) \in T_j$ and consequently $s(j) \neq s(i)$ for all $i, j \in \{p+1, \ldots, k\}$, $i \neq j$. By lemma 3, convexity of (N, v) implies

$$\begin{aligned} & v(T_1 \cup \cdots \cup T_p \cup T_{p+1} \cup \cdots \cup T_k \cup C') - v(T_1) - \cdots - v(T_k) - v(C') \\ & \quad \geq v(S_1 \cup \cdots \cup S_p \cup \{s(p+1)\} \cup \cdots \cup \{s(k)\} \cup C') - v(S_1) \\ & \qquad - \cdots - v(S_p) - v(\{s(p+1)\}) - \cdots - v(\{s(k)\}) - v(C'). \end{aligned}$$

By simplifying the above expression and taking into account that the game (N, v) is zero-normalized, we find

$$\begin{aligned} & v(T_1 \cup \cdots \cup T_p \cup T_{p+1} \cup \cdots \cup T_k \cup C') - \sum_{i=1}^{k} v(T_i) \\ & \quad \geq v(S_1 \cup \cdots \cup S_p \cup (C'' \setminus C') \cup C') - \sum_{i=1}^{p} v(S_i), \end{aligned}$$

and as $T_1 \cup \cdots \cup T_k \cup C' = T_1 \cup \cdots \cup T_k \cup C$, and moreover

$$S_1 \cup \cdots \cup S_p \cup (C'' \setminus C') \cup C' = S_1 \cup \cdots \cup S_p \cup C,$$

we may conclude that the inequality (1) is satisfied. □

We now describe the core of the $\mathscr{F}$-restricted game in terms of the feasible coalitions.

Proposition 8. *Let* $(N, v, \mathscr{F})$ *a union stable cooperation structure, then*

$$C(v^{\mathscr{F}}) = \{x \in \mathbb{R}_+^N : x(N) = v^{\mathscr{F}}(N),\ x(S) \geq v(S) \text{ for all } S \in \mathscr{F}\}.$$

Proof. Let $x \in C(v^{\mathscr{F}})$, then $x(N) = v^{\mathscr{F}}(N)$, and $x(S) \geq v^{\mathscr{F}}(S)$, for all $S \subseteq N$. Therefore, $x(N) = v^{\mathscr{F}}(N)$, $x(S) \geq v^{\mathscr{F}}(S) = v(S)$, for all $S \in \mathscr{F}$ and furthermore $x(\{i\}) \geq v^{\mathscr{F}}(\{i\}) = 0$. On the other hand, let $x \in \mathbb{R}^N_+$ such that $x(N) = v^{\mathscr{F}}(N)$, and $x(S) \geq v(S)$, for all $S \in \mathscr{F}$. Then, for all $S \subset N$, if $C_{\mathscr{F}}(S) = \varnothing$ then $v^{\mathscr{F}}(S) = 0$ and as $x \in \mathbb{R}^N_+$, we have $x(S) \geq v^{\mathscr{F}}(S) = 0$. If $C_{\mathscr{F}}(S) \neq \varnothing$, set $C_{\mathscr{F}}(S) = \{S_1, S_2, \ldots, S_k\}$ with $\bigcup_{p=1}^k S_p \subseteq S$ and hence,

$$x(S) = \sum_{i \in S} x_i \geq \sum_{p=1}^{k} \left[\sum_{i \in S_p} x_i \right] \geq \sum_{p=1}^{k} v(S_p) = v^{\mathscr{F}}(S). \qquad \square$$

Theorem 6. *Let $(N, v, \mathscr{F}) \in USI^N$ such that (N, v) is a convex game. Then, $\pi(N, v, \mathscr{F}) \in C(v^{\mathscr{F}})$.*

Proof. By theorem 5, $(\mathscr{C}, v^{\mathscr{C}})$ is convex and, therefore $\Phi(\mathscr{C}, v^{\mathscr{C}}) \in C(v^{\mathscr{C}})$. We first show that $\Phi(\mathscr{C}, v^{\mathscr{C}}) \geq 0$. By definition

$$\Phi_C(\mathscr{C}, v^{\mathscr{C}}) = \sum_{\{\mathscr{S} \subseteq \mathscr{C} : C \notin \mathscr{S}\}} \gamma(\mathscr{S})[v^{\mathscr{C}}(\mathscr{S} \cup \{C\}) - v^{\mathscr{C}}(\mathscr{S})], \quad C \in \mathscr{C}.$$

So, we have to prove $v^{\mathscr{C}}(\mathscr{S} \cup \{C\}) - v^{\mathscr{C}}(\mathscr{S}) \geq 0$. By convexity of $v^{\mathscr{C}}$ we have $v^{\mathscr{C}}(\mathscr{S} \cup \{C\}) - v^{\mathscr{C}}(\mathscr{S}) \geq v^{\mathscr{C}}(\{C\})$, therefore it suffices to show that $v^{\mathscr{C}}(\{C\}) \geq 0$. Indeed, $v^{\mathscr{C}}(\{C\}) = \sum_{T \in C_{\overline{\{C\}}}(N)} v(T) = v(C) \geq 0$ since v is superadditive and zero-normalized. As $\Phi_C(\mathscr{C}, v^{\mathscr{C}}) \geq 0$ then $\pi(N, v, \mathscr{F}) \in \mathbb{R}^N_+$. We now prove that $\sum_{i \in N} \pi_i(N, v, \mathscr{F}) = v^{\mathscr{F}}(N)$. Putting $C_{\mathscr{F}}(N) = \{N_1, \ldots, N_k\}$, we have

$$\begin{aligned}
\sum_{i \in N} \pi_i(N, v, \mathscr{F}) &= \sum_{i \notin \bigcup_{j=1}^k N_j} \pi_i(N, v, \mathscr{F}) + \sum_{i \in \bigcup_{j=1}^k N_j} \pi_i(N, v, \mathscr{F}) \\
&= \sum_{i \in \bigcup_{j=1}^k N_j} \pi_i(N, v, \mathscr{F}) = \sum_{j=1}^{k} \left[\sum_{i \in N_j} \pi_i(N, v, \mathscr{F}) \right] \\
&= \sum_{j=1}^{k} v(N_j) = v^{\mathscr{F}}(N),
\end{aligned}$$

since the position value is component-efficient and satisfies component-dummy because it is an allocation rule.

Finally, we prove that $\sum_{i \in S} \pi_i(N, v, \mathscr{F}) \geq v(S)$, for all $S \in \mathscr{F}$, $S \neq \varnothing$. If S is a unitary coalition $\{i\}$, $\pi_i(N, v, \mathscr{F}) \geq v(\{i\}) = 0$. If S is not a unitary coalition, as S is feasible, it is a union of non-unitary supports. Moreover, this expression is unique. Thus, let $S = \bigcup_{k \in K} S_k$, $S_k \in \mathscr{C}$. Then

$$\sum_{i \in S} \pi_i(N, v, \mathscr{F}) = \sum_{i \in S} \left[\sum_{C \in \mathscr{C}_i} \frac{1}{|C|} \Phi_C(\mathscr{C}, v^{\mathscr{C}}) \right].$$

As $\mathscr{C}_i \supseteq \{S_k : i \in S_k\}$ and $\Phi_C(\mathscr{C}, v^{\mathscr{C}}) \geq 0$, we obtain that

$$\sum_{i \in S} \pi_i(N, v, \mathscr{F}) \geq \sum_{i \in S} \left[\sum_{\{S_k : i \in S_k\}} \frac{1}{|S_k|} \Phi_{S_k}(\mathscr{C}, v^{\mathscr{C}}) \right]$$

$$= \sum_{k \in K} \left[\frac{1}{|S_k|} |S_k| \right] \Phi_{S_k}(\mathscr{C}, v^{\mathscr{C}})$$

$$\geq v^{\mathscr{C}}(\{S_k\}_{k \in K}) = \sum_{M \in C_{\overline{\{S_k\}}_{k \in K}}(N)} v(M) = v(S),$$

since $S \in \overline{\{S_k\}}_{k \in K}$ and $S = \bigcup_{k \in K} S_k$. The last inequality is due to the fact that $\Phi_C(\mathscr{C}, v^{\mathscr{C}}) \in C(v^{\mathscr{C}})$. □

References

[1] Borm P, Owen G, Tijs S (1992) On the position value for communication situations. SIAM J Disc Math 5:305–320

[2] Edelman PH, Jamison RE (1985) The theory of convex geometries. Geom Dedicata 19:247–270

[3] Meesen R (1988) Communication games. Ph D thesis, University of Nijmegen, The Netherlands (in Dutch)

[4] Myerson RB (1977) Graphs and cooperation in games. Math of Oper Res 2:225–229

[5] Myerson R (1980) Conference structures and fair allocation rules. Int J of Game Theory 9:169–182

[6] Nouweland, A van den, and Borm, P (1991) On the convexity of communication games. Int J of Game Theory 19:421–430

[7] Nouweland A, Borm P, Tijs S (1992) Allocation rules for hypergraph communication situations. Int J of Game Theory 20:255–268

[8] Owen G (1986) Values of graph-restricted games. SIAM J Alg Disc Meth 7:210–220

[9] Potters J, Reijnierse H (1995) Γ-Component additive games. Int J of Game Theory 24:49–56

[10] Shapley LS (1953) A value for *n*-person games. In: Kuhn HW, Tucker AW (eds) Contributions to the theory of games II, Princeton University Press, New Jersey, pp. 307–317

[11] Shapley LS (1971) Cores of convex games. Int J of Game Theory 1:11–26