SOLUTION CONCEPTS FOR GAMES ON CLOSURE SPACES

NIEVES JIMÉNEZ JIMÉNEZ
Ph. D. Thesis, University of Seville
Department of Applied Mathematics II

June 10, 1998

Contents

1 Introduction **4**
1.1 Cooperative games 4
1.2 Partial cooperation 10
1.3 Basic concepts 16
1.4 Synthesis of contents 19

2 Games on set families **23**
2.1 Games on feasible coalitions 24
2.2 Solution concepts 26
2.3 Games on closure spaces 37
2.4 Supermodular games 44

3 Games on convex geometries **48**
3.1 Convex geometries 49
3.2 The Weber set 54
3.3 Quasi-supermodular games 61
3.4 Selections and marginal worth vectors 69

4 Values on convex geometries **81**
4.1 Individual values and group values 82
4.2 Probabilistic values 82
4.3 Efficient values 89
4.4 Compatible-order values 91
4.5 The Shapley value 96

5 Simple games **101**
5.1 Preliminaries 101
5.2 Imputations and core 104

5.3 The Weber set 107
5.4 Stable sets 110
5.5 Bargaining sets 112
5.6 The Shapley value 120

6 Appendix 122
6.1 Selectope Algorithm 122
6.2 Weber Algorithm 125

Chapter 1

Introduction

1.1 Cooperative games

The first fundamentals of the theory of games were exposed by John von Neumann in 1928 [64] although it was later, in 1944, when the bases of this theory were established with the publication *Theory of Games and Economic Behaviour,* written by Oskar Morgenstern and John von Neumann. In this, it was shown how many social and economic situations could be described through strategic games and that those games are capable of a mathematical analysis.

In a general way, it could be said that the theory of games studies cooperation and conflict models, using mathematical methods; so, this could be divided into two fundamental lines: the theory of cooperative games and the theory of noncooperative games. This research corresponds to the first type of games and, within it, the so-called transferable utility games.

A *cooperative game* with *transferable utility* is a pair (N, v), where N is a finite set and $v : 2^N \longrightarrow \mathbb{R}$ is a function that assigns to each $S \subseteq N$ a real number satisfying that $v(\emptyset) = 0$.

The elements of $N = \{1, \ldots, n\}$ are called *players*, the subsets $S \in 2^N$ *coalitions* and $v(S)$ is the *worth* of the coalition S. For each S, $v(S)$ could be interpreted as the maximal gain or minimal cost that the players, which form the coalition S can achieve when they decide to cooperate and form a coalition. Throughout this work, we interpret the function v as the function that determines the maximal gain of each coalition. The function v is called *characteristic function* of the game and, generally, the cooperative game (N, v) is identified with the characteristic function v.

Many interesting situations, from the point of view of economic behaviour, could be conveniently modelled as a game in form of a characteristic function. This can be observed in the following example which the original formulation is due to Driessen [20].

Example 1.1.1 *We consider an economic situation in which there exist different companies that make two types of complementary commodities, A and B, which are usable in equal quantities only. We suppose that the set of companies is divided into two disjoint nonempty subsets P and Q, where each one of the companies of the set P makes only one unit of the commodity A daily. Similarly, each one of the companies of Q only makes α units of the commodity B daily. Furthermore, it is supposed that the output produced with one unit of both commodities can be sold at a net profit of one unit of money.*

The net profit function v which describes the largest possible monetary value of the output of the daily goods for any subset S of companies is given by

$$v(S) = min\{|S \cap P|\,,\ \alpha\,|S \cap Q|\}.$$

This economic situation can be modelled as a cooperative game (N, v) where its player set is $N = P \cup Q$, and its characteristic function v is precisely the daily net profit function.

As it has already been indicated, a game (N, v) is identified with its characteristic function and so, the different properties of the function v give rise to distinct types of games. An complete enough description of these can be seen in [20]. Here, only those which are used throughout this work are defined.

If $v\,(S) \leq v\,(T)\,$, for all $S \subseteq T \subseteq N$, then the game v is called *monotonic*. If also $v\,(S)$ only takes values in the set $\{0, 1\}$ for every coalition $S \subseteq N$, then the game is called *simple*.

A game v is *superadditive* if, for all $S, T \subseteq N$ such that $S \cap T = \emptyset$, it holds that

$$v\,(S \cup T) \geq v\,(S) + v\,(T)\,.$$

A special class of superadditive games are the so-called convex games. A game v is *convex* or *supermodular* if for any coalitions $S, T \subseteq N$, the following inequality is satisfied

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T).$$

Equivalently, a game v is convex if and only if for any coalitions $S, T \subseteq N$, such that $S \subseteq T$ and for all $i \in N \setminus T$, it holds that

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

Thus, for convex games, the marginal contribution of a player to a coalition is never less than the marginal contribution of the player to any coalition contained in that. The convex games were introduced by Shapley [57] and they are applied by modelling several situations that are studied in the economic sciences.

In general, we denote by Γ^N the set of all games (N, v). In this set, the following operations are introduced

$$\begin{array}{rll} + : & \Gamma^N \times \Gamma^N \longrightarrow \Gamma^N, & (v, w) \longmapsto v + w \\ \cdot : & \mathbb{R} \times \Gamma^N \longrightarrow \Gamma^N, & (\alpha, v) \longmapsto \alpha \cdot v \end{array}$$

define by

$$(v + w)(S) = v(S) + w(S), \quad (\alpha \cdot v)(S) = \alpha \cdot v(S),$$

for any $S \subseteq N$. With respect to these operations, Γ^N is a $2^n - 1$-dimensional vector space. One basis is formed by the set

$$\{u_T : T \subseteq N, T \neq \emptyset\},$$

where

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

These games u_T are called *unanimity games.*

From the introduction of the cooperative games, the most extensively studied problem has been how to divide the total savings among the players, since one of the principal rules, in a cooperative game, is to suppose that all players that participate in a game decide to cooperate among them and form the grand coalition N. This leads to the problem of distributing the amount $v(N)$ among them, and consequently, to define the concept of efficient payoff vector.

Each vector $x = (x_i)_{i\in N} \in \mathbb{R}^n$ is called *distribution* or *payoff vector,* since the coordinate x_i represents the payoff to player i. In a game v, a payoff vector x is called *efficient* if it distributes exactly the value of the coalition N among the players, i.e.,

$$\sum_{i=1}^{n} x_i = v(N).$$

The payoff vectors which satisfy this *efficiency principle* are called *preimputations* and a *solution* or *solution concept* on a nonempty collection of games is a function ψ which associates with any cooperative game v of this collection a subset $\psi(v)$ of its preimputation set.

Most of the proposed solution concepts for cooperative games require that the efficient payoff vectors meet the so-called *individual rationality principle* which demands that the payoff to each player i by the payoff vector x is at least the amount which the player can attain for himself in the game; that is, for all $i \in N$,

$$x_i \geq v(\{i\}).$$

The preimputations which satisfy the individual rationality principle are called *imputations* for the game v.

A satisfactory distribution criterium could be that not only every player, but also every coalition $S \in 2^N$ should receive at least the amount it can obtain by operating on its own, i.e., the payoff vector x verifies, for all $S \subseteq N$,

$$\sum_{i\in S} x_i \geq v(S).$$

The *core* of the game v consists of all efficient payoff vectors satisfying the above inequalities

$$Core(v) = \left\{x \in \mathbb{R}^n : x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \in 2^N\right\},$$

where $x(S) = \sum_{i\in S} x_i$ and $x(\emptyset) = 0$.

The core was introduced by Gillies [27] in 1953 and it is considered a very natural solution concept, although it has the trouble that, in many cases, it is empty. For the class of convex games, it can be affirmed that it is a nonempty set; nevertheless, even though the core is not empty, it could be small for obtaining reasonable solutions to certain games. This leads to consider other solution concepts.

From the foundation of game theory, many solution concepts have been proposed. In 1944, von Neumann and Morgenstern [65] introduced the concept of stable set (note that it is a previous concept to the core). The stable sets are described in terms of a relation between imputations called domination and therefore require that the imputation set is not empty.

If x and y are imputations for a game v, then x *dominates* y if there exists a nonempty coalition $S \subset N$ such that

$$x(S) \leq v(S) \qquad \text{and} \qquad x_i > y_i \quad \text{for all } i \in S.$$

In this case, the coalition S prefers the distribution x to y because every member of S obtains more, and S does not exceed its value with this imputation.

A subset E of the imputation set is a *stable set* if no imputation in E dominates another (*internal stability*) and any imputation outside the set E is dominated by some imputation in E (*external stability*).

In the most of the cases, the calculation of stable sets is very difficult since there are games with infinite stable sets. In contrast, there are games for which there does not exist stable sets (Lucas [41]).

Another solution concept is the so-called bargaining set, proposed by Aumann and Maschler [1]. This set is an extension of the core and it is united to the negotiation process since in this, the possible actions and answers made by the coalitions are taken into account. Actually, there are several versions of the bargaining set, but here the most classic is considered .

If x is an imputation for the game v, an *objection* of player $i \in N$ against another player $j \in N$ with respect to the imputation x is a pair (y, S) where $S \subset N$ is a coalition containing player i but not player j, and $y \in \mathbb{R}^{|S|}$ is a vector satisfying

$$y(S) = v(S) \quad \text{and} \quad y_k > x_k \quad \text{for every } k \in S.$$

A *counterobjection* to the objection (y, S) is a pair (z, T) where $T \subset N$ is a coalition containing player j but not player i, and $z \in \mathbb{R}^{|T|}$ is a vector satisfying

$$z(T) = v(T), \quad \text{and} \quad \begin{cases} z_k \geq x_k & \text{for} \quad k \in T \setminus S, \\ z_k \geq y_k & \text{for} \quad k \in S \cap T. \end{cases}$$

An objection is said *to be justified* if there does not exist counterobjection. The *Aumann-Maschler bargaining set* is the set of all imputations x for which there does not exist justified objection to x.

In 1978, Weber [66] proposed, as a solution concept, a set that contains the core and which is easier to compute. Furthermore, this set is always nonempty. The definition of the Weber set is based in the marginal worth vectors.

It is assumed that all players are ordered in a game (N, v) and it is taken into account all possible orders of the set of players; that is, the set Π_n of all permutation of N is considered. For every order $\pi \in \Pi_n$, we define the *marginal worth vector* $a^\pi(v) \in \mathbb{R}^n$ as the preimputation whose coordinates satisfy

$$a_i^\pi(v) = v\left(\pi^i \cup \{i\}\right) - v\left(\pi^i\right) \qquad \text{for all } i \in N,$$

where π^i is the set of the predecessors of player i in the order π. The vector $a^\pi(v)$ assigns to each player his marginal contribution in the order π.

The *Weber set* of game v is the convex hull of all marginal worth vectors, that is,

$$Weber(v) = conv\{a^\pi(v) : \pi \in \Pi_n\}.$$

In the most of the cases, the aforementioned solution concepts do not assign to the game a unique distribution, but a set of distributions. Also, solutions that assign to each game a unique payoff vector have been proposed and they are called values. Within theses values, the most popular is the Shapley value, introduced in 1953 by Shapley [56]. There are several interpretations of this concept. One of these allows to indicate that the *Shapley value* $\Phi(v) \in \mathbb{R}^n$ of game v is a weighted average of the marginal contributions of the players to the different coalitions, defined, for all $i \in N$, by

$$\Phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!\,(n-1-s)!}{n!} \left[v(S \cup \{i\}) - v(S)\right],$$

where $s = |S|$ and $n = |N|$.

Another form to introduce the Shapley value is based in the marginal worth vectors and corresponds to the following interpretation. Suppose the players enter a room one by one in a randomly chosen order. Each player gets the amount that he contributes to the coalition S already formed into the room when the player i enters the room; that is, i gets $v(S \cup \{i\}) - v(S)$. The Shapley value $\Phi(v)$ distributes to each player $i \in N$, the expected amount that he gets by this procedure; that is,

$$\Phi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi_n} \left[v\left(\pi^i \cup \{i\}\right) - v\left(\pi^i\right) \right].$$

Finally, it should be indicated that there exist other solution concepts as the *kernel* (Davis and Maschler [17]), the *prekernel* (Maschler, Peleg and Shapley [44]), the *nucleolus* (Schmeidler [60]) and the *τ-value* (Tijs [63]) among others, whose definitions are omitted as they are not studied in this research work.

1.2 Partial cooperation

In a general model of cooperative games it is assumed that there is no restrictions in the cooperation among the players and thus, each subgroup of player can join by forming a coalition. However, there are situations in which the cooperation among the players is not complete by certain reasons. For instance, there could be players that do not want to join because they are not related, have no common interests or simply because there exists some type of veto among them.

One of the first approximations to the partial cooperation, incorporating restrictive conditions to the formation of coalitions among the players, is the model of Aumann and Maschler [1] on *games with coalition structures.* In this model, the players are subdivided into classes forming a partition of the total set. That is, a coalition structure is a partition $\mathcal{B} = \{B_1, B_2, \ldots, B_k\}$ of the set N of players such that the cooperation is only possible among the players that belong to an element B_i of the coalition structure and there it is total. Later, in 1974, Aumann and Dréze [3] introduced in an axiomatic form the concept of value of a restricted game by a coalition structure.

Other authors that have developed this research line of cooperative games with coalition structures are Owen [52], Hart and Kurz [34], Levy and Mc Lean [39], Winter [68] [69] [70] and Mc Lean [46]. They consider the study of the partial cooperation when the coalition structure is given beforehand; that is, in exogenous form. In a parallel line, the endogenous formation of coalition structures, implicited in the theory of stable sets by John von Neumann and Oskar Morgenstern, is studied by researchers as Shenoy [59], Hart and Kurz [33] [38].

The fact that, in every element of the coalition structure, the cooperation is total, it demands that the relations among the players, if that happens, are transitive. This imposes a limitation to the application of this model. Because of this, Myerson [47], in 1977, in his seminal work *Graphs and Cooperation in Games,* proposed a different model by studying the partial cooperation.

In the model proposed by Myerson, the bilateral relations between the players are represented through a non-directed graph and it is understood that a coalition of players is feasible if the corresponding subgraph induced by this is connected. Thus, a game with partial cooperation among the players is denoted by a triple (N, v, G) where (N, v) is a cooperative game with transferable utility and $G = (N, E)$ is a cooperation graph where its set of vertices is the set N of players and its set of non-ordered edges E symbolizes the relation between pairs of players. Because of this, the characteristic function v associated to the game is modified since the cooperation is now partial and it has the *restricted game by the cooperation graph* which is represented by (N, v^G) where $v^G : 2^N \longrightarrow \mathbb{R}$ is given, for all $S \subseteq N$, by

$$v^G(S) = \sum_{S_i \in S/G} v(S_i).$$

That is, the worth of a coalition in the restricted game by the cooperation graph is a sum of values on the connected components of the subgraph induced by the coalition S.

In general, the triple (N, v, G) is called *communication situation* and, for this model of partial cooperation, Myerson studies, as solution concept, allocation rules of payoffs that are efficient in the connected components of the grand coalition N, fair and stable. That is, if we denote by SC^N the set of all communication situations defined on the set N,

$$SC^N = \{(N, v, G) : G \text{ is a graph } (N, E)\},$$

functions are studied

$$Y : SC^N \longrightarrow \mathbb{R}^n, \qquad (N, v, G) \longmapsto (Y_1(N, v, G), \ldots, Y_n(N, v, G)),$$

that satisfy the following properties:

1. *Efficient*: $\forall (N, v, G) \in SC^N$, $\forall S \in N/G$,

$$\sum_{i \in S} Y_i(N, v, G) = v(S).$$

2. *Fair*: $\forall (N, v, G) \in SC^N$, $\forall \{ij\} \in E$,

$$Y_i(N, v, G) - Y_i(N, v, G \setminus \{ij\}) = Y_j(N, v, G) - Y_j(N, v, G \setminus \{ij\}).$$

3. *Stable*: $\forall (N, v, G) \in SC^N$, $\forall \{ij\} \in E$,

$$Y_i(N, v, G) \geq Y_i(N, v, G \setminus \{ij\}), \quad Y_j(N, v, G) \geq Y_j(N, v, G \setminus \{ij\}).$$

Myerson proves that there exists a unique allocation rule of payoffs that satisfies the efficiency and fairness conditions: the Shapley value corresponding to the restricted game (N, v^G). This allocation rule of payoffs is known by the name of *Myerson value*:

$$\mu(N, v, G) = \Phi(N, v^G).$$

The model of partial cooperation introduced by Myerson has given rise to a research line with the work of Owen [53], Nouweland and Borm [49], Carreras [15], Nouweland, Borm and Tijs [50] among others. On the other hand, the endogenous formation of communication situations is analized by Aumann and Myerson [2] and Nouweland [51]. Finally, other models that keep up a close relation with the model initially created by Myerson, as the relations among the players is modelled by a non-directed graph, are studied by Bergantiños, Carreras and García Jurado [5], Calvo and Lasaga [14].

In 1980, in his work *Conference Structures and Fair Allocation Rules* [48], Myerson proposed the generalization of his model of partial cooperation with the use of communication hypergraphs since, for instance, if the cooperation should only be constituted by coalitions of three or more players, the model of representation through a cooperation graph could be not used. On the other hand, and motivated by problems of combinatorial optimization, other models of restricted cooperation like the models of Faigle and Kuipers arise, that will be commented on later.

The idea that underlies in the generalization of Myerson and that Nouweland picks up in her thesis is the issue to establish a broader setting where the relations among the players do not have to be represented by a non-oriented graph, but simply that the distinction between feasible and non feasible coalitions is made, although these have some type of predetermined structure. Following this research line, Bilbao [8] and López [40] begin to develop a model of cooperation based in the so-called *systems of feasible coalitions* and *partition systems,* which generalize the communication situations. The systems of feasible coalitions are collections $\mathcal{F}$ of subsets of N that contain the emptyset and the unitary coalitions. The partition systems are systems of feasible coalitions in which every coalition $S \subseteq N$ can be expresed as a finite disjoint union of feasible coalitions contained in S and maximal for the inclusion.

Then, if we consider the triple $(N, v, \mathcal{F})$ in which (N, v) is a game with transferable utility and $(N, \mathcal{F})$ is a system of feasible coalitions or a partition system, the restricted games corresponding by the system $\mathcal{F}$ are defined in the following way:

$$\tilde{v}^{\mathcal{F}} : 2^N \longrightarrow \mathbb{R}, \quad \tilde{v}^{\mathcal{F}}(S) = \max\left\{\sum v(S_i) : \{S_i\} \in P_{\mathcal{F}}(S)\right\}, \quad \forall S \subseteq N,$$

$$v^{\mathcal{F}} : 2^N \longrightarrow \mathbb{R}, \quad v^{\mathcal{F}}(S) = \sum_{S_i \in \Pi_S} v(S_i), \quad \forall S \subseteq N,$$

where $\mathcal{P}_{\mathcal{F}}(S)$ represents the set of all possible partitions of the coalition S in feasible coalitions and Π_S symbolizes the partition of the coalition S in maximal feasible coalitions contained in S. Obviously, they are two different models of characteristic functions associated to the partial cooperation although both are always closely relationed when it is worked in the context of a partition system and the original game (N, v) is superadditive. On the other hand, it is proved that a communication situation (N, v, G) is a particular case of a partition system.

The study of systems of feasible coalitions, partition systems and their respective restricted games have been continued with the analysis and characterization of the different solution concepts that exist for any cooperative game. In this study, the importance of the set of feasible coalitions $\mathcal{F}$ and its structure for the determination of them is made clear, and also for the transmision of properties of the characteristic function v to the functions $\tilde{v}^{\mathcal{F}}$ and $v^{\mathcal{F}}$. Furthermore, the set of unanimity games where its support is a feasible coalition is a basis for the set of restricted games by the partial cooperation, the *core* is characterized (under certain hypothesis) by the feasible coalitions, and the Shapley value of restricted games can be calculated by means of the dividends of Harsanyi for the feasible coalitions.

The previous observations imply that it is especially interesting to study the values of the game (N, v) only on the feasible coalitions. That is, define the game $(\mathcal{F}, v)$ with

$$v : \mathcal{F} \subseteq 2^N \longrightarrow \mathbb{R},$$

and later consider, if it proceeds and makes sense, the restricted game $(N, \tilde{v}^{\mathcal{F}})$ or $(N, v^{\mathcal{F}})$ as an extension of the characteristic function v of the game $(\mathcal{F}, v)$ to all subsets of N.

These last considerations connect with another model of partial cooperation, initiated fundamentally by Faigle [24] and Kuipers [37]. In this model, the cooperative game is defined on a family $\mathcal{F}$ of subsets of the set of players and the coalitions that belong to this family are called *feasible coalitions*, which have no determined structure. Later, the game is extended to those coalitions of N that can be expressed as union, not necessarily unique, of disjoint feasible coalitions.

On the other hand, it has been indicated before that the combinatory structure of $\mathcal{F}$ is important. Thus, the convexity of the game (N, v) is not transmitted to the games $(N, \tilde{v}^{\mathcal{F}})$, $(N, v^{\mathcal{F}})$ unless the family $\mathcal{F}$ is intersecting, i.e.,

$$A, B \in \mathcal{F}, \text{ with } A \cap B \neq \emptyset \implies A \cap B \in \mathcal{F}, \quad A \cup B \in \mathcal{F},$$

and the effective calculation of the Shapley value for the restricted game is computed more easily if $\mathcal{F}$ is a convex geometry, since this has the characteristic properties of the graphs denominated *trees* for which the calculation of the Myerson value is easier.

Connecting the reflections realised in the previous paragraphs, a research line has been opened in the past few years (see Bilbao [6] and Bilbao and Edelman [9]) in which the model proposed by Faigle is assumed and a cooperative game is studied defining it as a pair $(\mathcal{F}, v)$, where

$$v : \mathcal{F} \longrightarrow \mathbb{R}, \quad v(\emptyset) = 0,$$

and $\mathcal{F} \subseteq 2^N$ is a family of feasible coalitions, in the sense of Faigle, with a combinatory structure determined as lattice, closure space, intersecting family, convex geometry, matroid and greedoid, among others.

The central objective of this thesis is to introduce and analyze the aforementioned solution concepts in the above section for games defined on families of coalitions that, on some occasions, additional properties will be verified that will give it one structure or another.

This objective of generalization or study of solution concepts for cooperative games $(\mathcal{F}, v)$ is a first step of a wide research process which would try answer, in its more complete development, to the following questions:

- The new introduced solution concepts. Have they got the same characteristics and verify the same properties as their anologous in the context of a universal cooperation?
- Does it hold, among them, the same relations as those that already exist for the cooperative games with transferable utility?
- What is the relation between the introduced solution concepts and their homonymous, in a partial cooperation situation modelled by systems of feasible coalitions, partition systems or other model which requires an extension of the characteristic function $v : \mathcal{F} \longrightarrow \mathbb{R}$ to other function defined on any coalition?
- Knowing that one determined solution concept in cooperative games, can not always be the apropppriate for a problem or situation. Do they furnish the new introduced concepts more reasonable or aproppriate solutions to those situations in which the classical solutions of partial cooperation were empty or lack of logical sense in this context?

- How does the structure of the system of feasible coalitions affect to all questions before planed? And the type of characteristic function $v : \mathcal{F} \longrightarrow \mathbb{R}$ which gives rise to one or another class of cooperative game?

1.3 Basic concepts

In each cooperative game (N, v), the different coalitions that can be formed from the set of players N together with the inclusion relation form a partially ordered set. Because of this, it is necessary to present some concepts relating to these sets, using, in the following, the notations of Stanley [62] and Birkhoff [12]. In this section, the exposition of concepts and results relative to partially ordered sets will be limited to those that are used in the following chapters.

A *partially ordered set* (or *poset*) is a pair $(P, \leq)$ where P is a set, together with a binary relation denoted $\leq$ satisfying the reflexive, antisymmetric and transitive properties. We say that $\hat{1} \in P$ is *last element* of P if $x \leq \hat{1}$ for all $x \in P$. Similarly, we say that $\hat{0} \in P$ is *first element* of P if $\hat{0} \leq x$ for all $x \in P$.

As we have indicated before, an example of a poset is the set 2^N of all subsets of the set N, ordered by inclusion, i.e., if $A, B \in 2^N$, then $A \leq B$ in 2^N if and only if $A \subseteq B$. If N is finite, then $\left(2^N, \subseteq\right)$ is also finite.

If $Q \subseteq P$, we define a partial order in Q called *induced order* as follows: for $x, y \in Q$, we have $x \leq y$ in Q if and only if $x \leq y$ in P. The set Q is called *induced partially ordered subset* (or *induced subposet*) of P. Two special types of partially ordered subsets are the intervals and the chains.

If x and y are elements of the poset P and $x \leq y$, then the set

$$[x, y] = \{z \in P : x \leq z \leq y\}$$

is called *interval*. If every interval of $(P, \leq)$ is finite, then $(P, \leq)$ is called a *locally finite poset*.

We say two elements x and y of P are *comparable* if $x \leq y$ or $y \leq x$; otherwise, x and y are *incomparable*. A *chain* C of P is an induced subposet of P, in which any two elements are comparable; that is,

$$C \subseteq P \text{ is a chain if } x \leq y \text{ or } y \leq x \text{ for all } x, y \in C.$$

An *antichain* is a subset A of a poset P such that any two distinct elements of A are incomparable.

The *length* $l(C)$ of a finite chain is defined by $l(C) = |C| - 1$. The *length* or *rank* of a finite poset P is $l(P) = \max\{l(C) : C \text{ is a chain of } P\}$.

Another special type of induced subposet of P, are the order ideals of P. We say that $I \subseteq P$ is an *order ideal* of P when

$$\forall x \in I, \text{ if } y \leq x \Longrightarrow y \in I.$$

In particular, if $x \in P$, the set

$$\langle x \rangle = \{y \in P : y \leq x\},$$

is called *principal order ideal generated by* x.

If $x, y \in P$, then we say y *covers* x if $x < y$ and if no element $z \in P$ satisfies $x < z < y$. Thus, y covers x if and only if

$$x < y \quad \text{and} \quad [x, y] = \{x, y\}.$$

An element $x \in P$ is called *atom* if x covers the first element of P. A *maximal* element of a subset X of P is an element a such that $a < x$ for no $x \in X$.

The *Hasse diagram* of a finite poset P is a graph whose vertices are the elements of P and whose edges are determined by the cover relations.

An *upper bound* of a subset X of a poset P is an element $a \in P$ such that $x \leq a$ for all $x \in X$. If in addition, for every upper bound y of X, then $a \leq y$, the element $a \in P$ is the *least upper bound* of X; it is denoted by $\sup X$. Dually one can define the concepts of *lower bound* and *greatest lower bound* of X or $\inf X$. A *lattice* is a poset P for which every pair of elements of P has least upper bound and greatest lower bound. Usually, we denote:

$$a \wedge b = \inf\{a, b\}, \qquad a \vee b = \sup\{a, b\}.$$

A lattice P is *complete* when each of its subsets X has a greatest lower bound and a least upper bound in P.

An important class of lattices from the combinatorial point of view are the distributive lattices. A lattice is *distributive* if it verifies

$$\begin{aligned} x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \\ x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) . \end{aligned}$$

A special type of distributive lattice is the boolean; that is, the lattice 2^N of all subsets of an arbitrary set N. Any chain is also a distributive lattice.

In the following, we expose some concepts regarding the incidence algebra of a locally finite poset and the Möbius inversion formula.

Let P be a locally finite poset and let $\mathbb{K}$ be a field (usually $\mathbb{R}$ or $\mathbb{C}$). We say that

$$f : P \times P \longrightarrow \mathbb{K}$$

is an *incidence function* of P over $\mathbb{K}$ if $f(x, y) = 0$ when $x \not\leq y$. This definition implies that an incidence function is zero when it is evaluated over pairs that are not intervals of P. We denote by $I(P, \mathbb{K})$ the set formed by the incidence functions of P over $\mathbb{K}$. This set has the structure of a vector space with the addition and multiplication by scalar operations. Moreover, we can define a second internal operation called *convolution* as follows:

$$(f * g)(x, y) = \sum_{\{z : x \leq z \leq y\}} f(x, z)\, g(z, y)$$

for every $(x, y) \in P \times P$. This internal operation $*$ has the Kronecker function δ as neutral element, defined by

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

The set $I(P, \mathbb{K})$ together with the addition, multiplication by scalar and convolution is an algebra over $\mathbb{K}$ designed, normally, by *incidence algebra* $I(P, \mathbb{K})$.

A function of $I(P, \mathbb{K})$ is the *zeta function* ζ, defined by

$$\zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

If the locally finite poset P is the set of all subsets of a finite set N, then the zeta function, fix any $S \in 2^N, S \neq \emptyset$, is the function

$$\zeta_S : 2^N \longrightarrow \mathbb{R}, \quad \zeta_S(T) = \zeta(S,T) = \begin{cases} 1 & \text{if } S \subseteq T, \\ 0 & \text{otherwise,} \end{cases}$$

that can be recognized as a *unanimity game*. This form of interpreting the unanimity game is used by Faigle and Kern [25].

If $f \in I(P, \mathbb{K})$, then the inverse function for the convolution exists if and only if $f(x,x) \neq 0$, for all $x \in P$. This implies that the zeta function ζ has inverse, called *Möbius function* of P and it is symbolized by μ. Its calculation can be realized through the following recurrent formulas

$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{\{z: x \leq z < y\}} \mu(x,z) & \text{if } x < y \text{ in } P. \end{cases}$$

As a fundamental result related with the Möbius function, to throw *Möbius inversion formula* into relief, which establishes that if P is a poset where every principal order ideal is finite and $f, g : P \longrightarrow \mathbb{C}$, then for all $x \in P$:

$$g(x) = \sum_{y \leq x} f(y) \Longleftrightarrow f(x) = \sum_{y \leq x} \mu(y,x)\, g(y).$$

If P is the boolean algebra 2^N, the Möbius function of P is given by

$$\mu(T,S) = (-1)^{|S|-|T|}.$$

Hence, the Möbius inversion formula for 2^N establishes: if $f, g : P \longrightarrow \mathbb{C}$, then

$$g(S) = \sum_{T \subseteq S} f(T) \Longleftrightarrow f(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} g(T).$$

1.4 Synthesis of contents

In the second chapter, following the line of Faigle already commented on, the partial cooperation is modelled defining the characteristic function of the game only on the feasible coalitions. The family of feasible coalitions will not have, at first, any determined structure, we only demand that the coalition formed by all players and the empty coalition are feasible. We consider the vector space constituted for all games defined on a family of feasible coalitions and we prove that the collections of unanimity games and identity games, defined in this model of cooperation, form bases of this vector space.

In the second section, different solution concepts are defined for these games as the imputation set, the core and the stable sets. We establish different relations between them generalizing results for cooperative games with transferable utility. Furthermore, when the family of feasible coalitions is atomic, we introduce another solution concept called selectope where its definition is based on the dividends of Harsanyi; some relations between this set and the core are determined noting that, unlike what happens for cooperative games (N, v), the core is not contained to the selectope. On the other hand, the games for which the selectope is a subset of the core are characterized that will be called almost positive games.

In the third section, we give the family of feasible coalitions as structure of closure space and, in particular, intersecting family. In the case that the family of feasible coalitions is an atomic intersecting family, we prove that the core is always a subset of the selectope.

In the last section, with the idea of generalizing results related to the stable sets and the core, the supermodular games are introduced. Thus, we prove that, if the family of feasible coalitions is an atomic intersecting family, the game is supermodular on this family and satisfies an additional condition, then the core is the unique stable set.

In the third chapter, we introduce a special class of closure spaces, the so-called convex geometries and, in the first section, some of its properties are established and the most relevant concepts are defined. The family 2^N of the subsets of the set of all players is a convex geometry and thus, the cooperative games are a particular case of games defined on these families of sets.

In the second section, we include a new solution concept for games defined on convex geometries: the Weber set. We study this concept with the idea of establishing its relation with the core. In a cooperative game, the Weber set always contains the core. However it is not true if the convex geometry is distinct to 2^N.

In the relation between the Weber set and the core, the quasi-supermodular games play an important part, which are defined in the third section. We prove that the quasi-supermodular games are the only ones for which their Weber set is a subset of the core, establishing a characterization of these games.

Finally, in the last part of this chapter, we prove that, for every game on an atomic convex geometry, the Weber set is a subset of the selectope, establishing the relations between their extreme points.

In the fourth chapter, we extend the work of Weber [67] on probabilistic values to games defined on convex geometries (see Bilbao, Lebrón and Jiménez [10]). The first part of this chapter centres on the study of individual values, in which each player evaluates his possibilities in the participation in the different games. We introduce the probabilistic values and we observe in detail the axioms that characterize such values. These axioms are sequentially introduced observing how they have repercussions on the probabilistic value expression.

In the second part of the chapter we study the group values, that is, functions which associate to each game a vector where its components represent the value that every player has in the participation in the game. We prove the relation between the efficient group values and those vectors where their components are probabilistic values. We introduced the compatible-order values, we study their relation with the previous ones and we conclude that the preimputation set associated to a game by the family of compatible-order values is its Weber set. Finally, we obtain the Shapley value using the axioms introduced throughout this chapter.

In the fifth chapter, we study a special class of games which are called simple games. These games were introduced by von Neumann and Morgenstern (1944) and permit to model distinct economic, social and political situations. In this chapter, results already known in games with total cooperation regarding the different solution concepts are extended for this class of games. In concrete, we give characterizations of the imputation set, the core and the Weber set. We prove that every simple game on an atomic family with nonempty imputation set, has at least one stable set and we determine which simple games on atomic convex geometries have the core as unique stable set.

In the last part of the chapter, we study several notions of bargaining sets for simple games and the relations between them, the core and the Weber set are established.

In the appendix, we present two algorithms implemented with the program of symbolic calculus MATHEMATICA [71], in which the vectors of the Weber set and the selectope are computed for games defined on families of feasible coalitions.

Chapter 2

Games on set families

In the first chapter the necessity of studying more general models of cooperative games in which there are restrictions in the cooperation among the players has been justified, as the reality of some applications need to establish intermediate possibilities between the total cooperation and non cooperation.

In the model that we present here, we study situations in which the communication among the players is represented by a family $\mathcal{L}$ of subsets of the set of players N, whose elements will be called *feasible coalitions.* One form of modelling the partial cooperation, already commented on in the introduction, has been by the modification of the characteristic function of the game (N, v), giving rise to a new game called, in Bilbao [7] and López [40], *restricted game by the feasible coalitions system.* This form of modelling the partial cooperation is a possibility facing others. An alternative suggested by Faigle in *Cores of Games with Restricted Cooperation* (1989) is to consider that the characteristic function of a game with partial cooperation must only be defined for the feasible coalitions. In this chapter and in the following chapters, we will follow the model of Faigle and we will observe how the total cooperation then arises as a particular case.

Although it should be desirable that the set of feasible coalitions have no determinant, here we require that the set of all players N and the emptyset are feasible coalitions. Futhermore, in some cases, the family of feasible coalitions will have a determined structure, depending on the cooperation rules that are established, but in all cases known results for a cooperative game (N, v) are generalized.

2.1 Games on feasible coalitions

Taking into account all the above considerations and understanding then that there exist situations in which there are restrictions in the cooperation among the players and therefore, the fact that all coalitions are not possible makes it logical to think that only those coalitions that can be formed have a worth in the game. Thus, we define the games by functions that only assign values to the feasible coalitions.

Definition 2.1.1 *A game is a triple* $(N, v, \mathcal{L})$*, where* N *is a finite set,* $\mathcal{L}$ *is a family of subsets of* N *such that* $\emptyset, N \in \mathcal{L}$*, and* v *is a function* $v : \mathcal{L} \longrightarrow \mathbb{R}$ *such that* $v(\emptyset) = 0$*.*

During this work, we consider the set $N = \{1, 2, \ldots, n\}$. Its elements are called *players* and the elements of the family $\mathcal{L}$ *feasible coalitions*. When $\{i\} \in \mathcal{L}$ for all $i \in N$, the family $\mathcal{L}$ is called *atomic*. Just like the cooperative games with transferable utility (N, v), the game $(N, v, \mathcal{L})$ is identified with the function v and we say that it is a game defined on the family $\mathcal{L}$. Note that the definition of cooperative game is a particular case of this, in which $\mathcal{L}$ is considered as the set 2^N of all subsets of N.

We denote by $\Gamma(\mathcal{L})$ the set of all games defined on the family $\mathcal{L}$ of subsets of N, which always includes among its elements the sets $\emptyset$ and N. In the set $\Gamma(\mathcal{L})$ we introduce two operations

$$\begin{array}{rrl} + : & \Gamma(\mathcal{L}) \times \Gamma(\mathcal{L}), & (v, w) \longmapsto v + w \\ \cdot : & \mathbb{R} \times \Gamma(\mathcal{L}), & (\alpha, v) \longmapsto \alpha \cdot v \end{array}$$

defined, for every $S \in \mathcal{L}$, by

$$(v + w)(S) = v(S) + w(S), \qquad (\alpha \cdot v)(S) = \alpha \cdot v(S).$$

With respect to this addition and multiplication by scalar, the set $\Gamma(\mathcal{L})$ is a vector space. There are two special collections of games in $\Gamma(\mathcal{L})$ taking values in $\{0, 1\}$, the *unanimity games* and the *identity games* which are defined as follows.

For any nonempty coalition $T \in \mathcal{L}$, the corresponding *unanimity game*, that is represented by ζ_T, is defined as $\zeta_T : \mathcal{L} \longrightarrow \mathbb{R}$

$$\zeta_T(S) = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise,} \end{cases}$$

for every coalition $S \in \mathcal{L}$.

The *identity game* $\delta_T : \mathcal{L} \longrightarrow \mathbb{R}$ is defined by

$$\delta_T(S) = \begin{cases} 1 & \text{if } S = T, \\ 0 & \text{otherwise,} \end{cases}$$

for every coalition $S \in \mathcal{L}$.

The relevance of these collections of games is made clear in the following result.

Proposition 2.1.1 *The dimension of the vector space* $\Gamma(\mathcal{L})$ *is* $|\mathcal{L}| - 1$, *and the collections*

$$\{\zeta_T : T \in \mathcal{L}, T \neq \emptyset\} \quad \textit{and} \quad \{\delta_T : T \in \mathcal{L}, T \neq \emptyset\}$$

are two bases of $\Gamma(\mathcal{L})$.

Proof. It is evident that the collection $\{\delta_T : T \in \mathcal{L}, T \neq \emptyset\}$ is a basis of $\Gamma(\mathcal{L})$ since every $v \in \Gamma(\mathcal{L})$ can be written as

$$v = \sum_{\{T \in \mathcal{L}, T \neq \emptyset\}} v(T)\, \delta_T,$$

and also, if $\sum_{\{T \in \mathcal{L}, T \neq \emptyset\}} \alpha_T \delta_T = \theta$, where θ denoted to the null game ($\theta(S) = 0$, for all $S \in \mathcal{L}$), it is immediate that $\alpha_T = 0$, for every nonempty coalition $T \in \mathcal{L}$.

In order to prove that the collection $\{\zeta_T : T \in \mathcal{L}, T \neq \emptyset\}$ is a basis of $\Gamma(\mathcal{L})$ it is sufficient to show that it is an independent linear system. If we consider $\sum_{\{T \in \mathcal{L}, T \neq \emptyset\}} \alpha_T \zeta_T = \theta$ and this null linear combination is applied to the different coalitions of $\mathcal{L}$, we obtain a homogenous linear system with the trivial solution as unique solution. □

As a consequence of this proposition, every game $v \in \Gamma(\mathcal{L})$ is a linear combination of the unanimity games. The coefficients can be calculated using the Möbius function. Indeed, assume $v \in \Gamma(\mathcal{L})$ such that

$$v = \sum_{\{T \in \mathcal{L}, T \neq \emptyset\}} \alpha_T \zeta_T.$$

For every nonempty coalition $S \in \mathcal{L}$, we have

$$v(S) = \sum_{\{T \in \mathcal{L}, T \neq \emptyset, T \subseteq S\}} \alpha_T.$$

As $(\mathcal{L} \setminus \{\emptyset\}, \subseteq)$ is a finite partially ordered set, if we consider the functions $v : \mathcal{L} \setminus \{\emptyset\} \longrightarrow \mathbb{R}$ and $f : \mathcal{L} \setminus \{\emptyset\} \longrightarrow \mathbb{R}$, given by $f(T) = \alpha_T$, we can apply the Möbius Inversion Formula (see p. 20), and we obtain

$$\alpha_S = \sum_{\{T \in \mathcal{L}, T \neq \emptyset, T \subseteq S\}} \mu(T, S)\, v(T).$$

□

2.2 Solution concepts

As it has already been indicated in the previous chapter, one of the most extensively studied problems in the theory of cooperative games has been how divide the amount $v(N)$ among the players who participate in the game v when they decide to cooperate and form the grand coalition N.

In this section, some solution concepts for games defined on a family $\mathcal{L}$ are generalized, understanding as a solution concept any subset of efficient vectors in $\mathbb{R}^n$, that is, any subset of the set of *preimputations*. The *preimputation set* of a game $v : \mathcal{L} \longrightarrow \mathbb{R}$, which is denoted by $I^*(\mathcal{L}, v)$, is the set

$$I^*(\mathcal{L}, v) = \left\{ x \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N) \right\}.$$

The *imputations* are the preimputations that satisfy the individual rationality principle for those players who can participate in the game v forming a unitary coalition, that is,

$$I(\mathcal{L}, v) = \{x \in I^*(\mathcal{L}, v) \ : \ x_i \geq v(\{i\}) \ \text{ if } \ \{i\} \in \mathcal{L}\}.$$

By definition, we have that $I(\mathcal{L}, v) \subseteq I^*(\mathcal{L}, v)$, but although $I^*(\mathcal{L}, v) \neq \emptyset$, it could be $I(\mathcal{L}, v) = \emptyset$. However, we can observe that if the family $\mathcal{L}$ does not contain any unitary coalition, then $I(\mathcal{L}, v) = I^*(\mathcal{L}, v)$, and hence $I(\mathcal{L}, v) \neq \emptyset$. On the other hand, if the family $\mathcal{L}$ contains at least one unitary coalition, the following cases may happen:

1. If $\mathcal{L}$ is not atomic (i.e., there exists $j \in N$ such that $\{j\} \notin \mathcal{L}$), then $I(\mathcal{L}, v) \neq \emptyset$, since we can define the vector $x = (x_i)_{i \in N}$ in the following way
$$x_i = \begin{cases} v(\{i\}) & \text{if } \{i\} \in \mathcal{L}, \\ v(N) - \sum\limits_{\{k\} \in \mathcal{L}} v(\{k\}) & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$
that satisfies the conditions to affirm that $x \in I(\mathcal{L}, v)$.

2. If, for all $i \in N$, we have that $\{i\} \in \mathcal{L}$, that is, $\mathcal{L}$ is atomic, then
$$I(\mathcal{L}, v) \neq \emptyset \quad \text{if and only if} \quad v(N) \geq \sum_{i \in N} v(\{i\}).$$

Unlike the imputation set of a cooperative game v, the imputation set of a game on a family $\mathcal{L}$ may or may not be bounded, depending on the family $\mathcal{L}$. Note that the set $I(\mathcal{L}, v)$ is determined by a finite number of inequalities and thus, it constitutes a polyhedron. The theory of polyhedrons (Schrijver [61]) provides some useful definitions and results for establishing a necessary and sufficient condition on the family $\mathcal{L}$ so that the imputation set is bounded. A set $P \subseteq \mathbb{R}^n$ is called a *polyhedron* if there exist a matrix A and a column vector b such that

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}.$$

The polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is bounded if and only if

$$\{x \in \mathbb{R}^n : Ax \leq 0\} = \{0\}.$$

Furthermore, a nonempty polyhedron P is bounded if and only if it is the convex hull of a finite number of vectors, that is, if there exist $x_1, \ldots, x_t$ such that $P = \operatorname{conv}\{x_1, \ldots, x_t\}$.

Proposition 2.2.1 *Let $v \in \Gamma(\mathcal{L})$. The following statements are equivalent:*

(a) The family $\mathcal{L}$ is atomic.

(b) The set $I(\mathcal{L}, v)$ is bounded.

Proof. As $I(\mathcal{L}, v)$ is a bounded set if and only if

$$\left\{x \in \mathbb{R}^n : \sum_{i \in N} x_i = 0,\ x_i \geq 0, \text{ for all } \{i\} \in \mathcal{L}\right\} = \{0\},$$

it is evident that, for this, it is a necessary and sufficient condition that the family $\mathcal{L}$ is atomic. □

From the imputation definition it is immediate to deduce that when the family $\mathcal{L}$ is atomic and $v(N) = \sum_{i \in N} v(\{i\})$, the imputation set is reduced to a unique distribution.

Proposition 2.2.2 *Let $v \in \Gamma(\mathcal{L})$ be a game on an atomic family $\mathcal{L}$. Then $I(\mathcal{L}, v) = \{(v(\{1\}), \ldots, v(\{n\}))\}$ if and only if $v(N) = \sum_{i \in N} v(\{i\})$.*

Given a cooperative game (N, v), one of the most studied solution concepts has been the *core.* In this context of games with partial cooperation, it is natural to consider in the definition of the core only the feasible coalitions in $\mathcal{L}$.

Definition 2.2.1 *Let $v \in \Gamma(\mathcal{L})$. The core of the game v is the set*

$$Core(\mathcal{L}, v) = \{x \in \mathbb{R}^n \ : \ x(N) = v(N),\ x(S) \geq v(S),\ \text{ for all } S \in \mathcal{L}\},$$

where $x(S) = \sum_{i \in S} x_i$ for every $S \in \mathcal{L}$, $x(\emptyset) = 0$.

Obviously, all vectors of the core are imputations and thus, the core can be an empty set. If $\mathcal{L}$ is atomic, the imputation set $I(\mathcal{L}, v)$ is bounded and hence the following proposition is immediate.

Proposition 2.2.3 *Let $v \in \Gamma(\mathcal{L})$. If the family $\mathcal{L}$ is atomic, then the polyhedron $Core(\mathcal{L}, v)$ is bounded.*

The reverse is not true in general. Upon taking $N = \{1, 2, 3\}$, the family $\mathcal{L} = \{\emptyset, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$, and the game v on $\mathcal{L}$ defined by $v(S) = |S| - 1$ for every nonempty coalition $S \in \mathcal{L}$. In this case,

$$Core(\mathcal{L}, v) = \left\{x \in \mathbb{R}^n_+ : x_1 + x_2 + x_3 = 2,\ x_1 \leq 1,\ x_2 \leq 1,\ x_3 \leq 2\right\}$$

is a bounded set but however, the family $\mathcal{L}$ is not atomic.

In order to secure the reverse, it is necessary to establish an additional condition on the family $\mathcal{L}$, which will be introduced in the next section.

When the imputation set is nonempty, we can define a new solution concept, the so-called *stable sets*. For this, we introduce the notion of *dominance* between imputations.

Definition 2.2.2 *Let $v \in \Gamma(\mathcal{L})$ and $x, y \in I(\mathcal{L}, v)$. The imputation y dominates the imputation x, $y \operatorname{dom} x$, if there exists a nonempty coalition $S \in \mathcal{L}$ such that*

$$y(S) \leq v(S) \quad \text{and} \quad y_i > x_i \text{ for all } i \in S.$$

Note that these conditions exclude the dominance through the grand coalition N and those unitary coalitions in $\mathcal{L}$. Similarly, we define the relation of dominance between preimputations.

Definition 2.2.3 *Let $v \in \Gamma(\mathcal{L})$. A nonempty set $E \subseteq I(\mathcal{L}, v)$ is a stable set for the game v if it satisfies the following conditions:*

(1) Internal stability: if $x \in E$, $y \in E$, then x no dominates y.

(2) External stability: if $x \in I(\mathcal{L}, v) \setminus E$, then there exists $y \in E$ such that $y \operatorname{dom} x$.

The following result characterizes the core as the set of all preimputations which are not dominated by other preimputations.

Proposition 2.2.4 *Let $v \in \Gamma(\mathcal{L})$. Then,*

$$Core(\mathcal{L}, v) = \{x \in I^*(\mathcal{L}, v) : \textit{for all } y \in I^*(\mathcal{L}, v), \ \ y \textit{ no dominates } x\}.$$

Proof. Let $x \in Core(\mathcal{L}, v)$ and $y \in I^*(\mathcal{L}, v)$ such that $y \operatorname{dom} x$. Then, there exists a nonempty coalition $S \in \mathcal{L}$ satisfying $x(S) < y(S) \leq v(S)$, but this is a contradiction because $x \in Core(\mathcal{L}, v)$.

In order to prove the other inclusion, assume $x \in I^*(\mathcal{L}, v) \setminus Core(\mathcal{L}, v)$. Then, there exists a nonempty coalition $S \in \mathcal{L}$, $S \neq N$ such that $x(S) < v(S)$. If we define the vector $y \in \mathbb{R}^n$ by

$$y_i = \begin{cases} x_i + \dfrac{1}{|S|}\left[v(S) - x(S)\right] & \text{if } i \in S, \\ \dfrac{1}{|N \setminus S|}\left[v(N) - v(S)\right] & \text{if } i \notin S, \end{cases}$$

it follows that this vector satisfies $y(N) = v(N)$, $y(S) = v(S)$ and $y_i > x_i$ for all $i \in S$. Hence, $y \operatorname{dom} x$. □

In the following, we analyze the dominance between imputations and study its relation with the core. Moreover, we prove that if the core of a game $v \in \Gamma(\mathcal{L})$ is stable, then it is a unique stable set for the game. In the case $\mathcal{L} = 2^N$, the results are known [20]. The following proposition is a direct extension and therefore we will omit the proof.

Proposition 2.2.5 *Let $v \in \Gamma(\mathcal{L})$. Then,*

$$Core(\mathcal{L}, v) \subseteq \{x \in I(\mathcal{L}, v) : \textit{for all } y \in I(\mathcal{L}, v), \ y \textit{ no dominates } x\}.$$

Theorem 2.2.6 *Let $v \in \Gamma(\mathcal{L})$ and let E be a stable set for v. Then*

(1) $Core(\mathcal{L}, v) \subseteq E$.

(2) If $Core(\mathcal{L}, v)$ is a stable set, then $E = Core(\mathcal{L}, v)$.

Proof. Let E be a stable set for the game v.

(1) Assume that there exists $x \in Core(\mathcal{L}, v) \setminus E$. Because E is a stable set, there exists $y \in E$ such that $y \operatorname{dom} x$, but it is a contradiction with the Proposition 2.10.

(2) Taking into account part (1), it is sufficient to prove the reverse inclusion. Assume that there exists $x \in E \setminus Core(\mathcal{L}, v)$. As $Core(\mathcal{L}, v)$ is a stable set, there exists $y \in Core(\mathcal{L}, v)$ such that $y \operatorname{dom} x$. Therefore, $\{x, y\} \subseteq E$ and satisfies that $y \operatorname{dom} x$, but this is a contradiction with the stability of E. □

As it has already been indicated, the above solution concepts are defined on the imputation set and therefore demand that the imputation set is nonempty. We now introduce a new solution concept, the *selectope*, for which is not necessary that $I(\mathcal{L}, v)$ is nonempty. This solution concept was introduced by Hammer, Peled and Sorensen [31] and investigated by Derks, Haller and Peters [19] in the case of cooperative games. Now, we extend this concept for games $v \in \Gamma(\mathcal{L})$ and we consider that the family of feasible coalitions $\mathcal{L}$ is an atomic family. The definition of the selectope is based in the dividends of Harsanyi [32].

We define, recursively, the *dividends for the game* $v \in \Gamma(\mathcal{L})$, for every coalition $S \in \mathcal{L}$, by

$$\triangle_v(S) = \begin{cases} 0 & \text{if } S = \emptyset, \\ v(S) - \sum_{\{T \in \mathcal{L}, T \subset S\}} \triangle_v(T) & \text{otherwise.} \end{cases}$$

It is clear, that for every $v \in \Gamma(\mathcal{L})$ and $S \in \mathcal{L}$,

$$v(S) = \sum_{\{T \in \mathcal{L}, T \subseteq S\}} \triangle_v(T).$$

and hence every $v \in \Gamma(\mathcal{L})$ can be uniquely represented by

$$v = \sum_{\{T \in \mathcal{L}, T \neq \emptyset\}} \triangle_v(T)\, \zeta_T$$

where $\zeta_T \in \Gamma(\mathcal{L})$ is the unanimity game for the nonempty coalition $T \in \mathcal{L}$.

Definition 2.2.4 *A selector on an atomic family $\mathcal{L}$ is a function $\alpha : \mathcal{L} \setminus \{\emptyset\} \longrightarrow N$ with $\alpha(S) \in S$ for every nonempty coalition $S \in \mathcal{L}$.*

We denote by $\mathcal{A}(\mathcal{L})$ the set of all selectors on $\mathcal{L}$. Note that the number of selectors on $\mathcal{L}$ is equal to $\prod_{\{T\in\mathcal{L},T\neq\emptyset\}} |T|$.

Definition 2.2.5 *The corresponding selection to the selector $\alpha \in \mathcal{A}(\mathcal{L})$ is the vector $m^{\alpha}(v) \in \mathbb{R}^n$ defined by*

$$m_i^{\alpha}(v) = \sum_{\{S\in\mathcal{L}:\alpha(S)=i\}} \triangle_v(S)$$

for every $i \in N$ and $v \in \Gamma(\mathcal{L})$. The selectope for a game $v \in \Gamma(\mathcal{L})$ is defined by

$$Sel(\mathcal{L}, v) = conv\{m^{\alpha}(v) \in \mathbb{R}^n : \alpha \in \mathcal{A}(\mathcal{L})\}.$$

Obviously, for every game $v \in \Gamma(\mathcal{L})$, and $\alpha \in \mathcal{A}(\mathcal{L})$, we have that $m^{\alpha}(v) \in I^*(\mathcal{L}, v)$, and hence, by the convexity of the preimputation set,

$$Sel(\mathcal{L}, v) \subseteq I^*(\mathcal{L}, v).$$

At the end of this work, we include an appendix in which an algorithm implemented with the program of symbolic calculus MATHEMATICA is presented, where the possible selectors on a family of feasible coalitions and the corresponding selections for a game $v \in \Gamma(\mathcal{L})$ are computed.

Example 2.2.1 *Let* $N = \{1, 2, 3\}$ *and consider the family*

$$\mathcal{L} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 2, 3\}\}.$$

Let $v \in \Gamma(\mathcal{L})$ *be the game defined by*

$$v(S) = \begin{cases} |S| & \textit{if } |S| \geq 2, \\ 0 & \textit{otherwise.} \end{cases}$$

There are 6 selectors on $\mathcal{L}$ *which are listed below along with the corresponding selections. The unitary coalitions do not appear in the table because the* $\triangle_v(\{i\}) = 0$ *for all* $i \in N$.

α	$\{1,2\}$	$\{1,2,3\}$	$m_1^\alpha(v)$	$m_2^\alpha(v)$	$m_3^\alpha(v)$
1	1	1	3	0	0
2	1	2	2	1	0
3	1	3	2	0	1
4	2	1	1	2	0
5	2	2	0	3	0
6	2	3	0	2	1

TABLE 2.1

In consequence, we have

$$Sel(\mathcal{L}, v) = conv\{(3,0,0), (2,1,0), (2,0,1), (1,2,0), (0,3,0), (0,2,1)\}$$

Furthermore, note that, in relation with the core, we have that

$$Core(\mathcal{L}, v) = Sel(\mathcal{L}, v)$$

as we illustrate in the following figure.

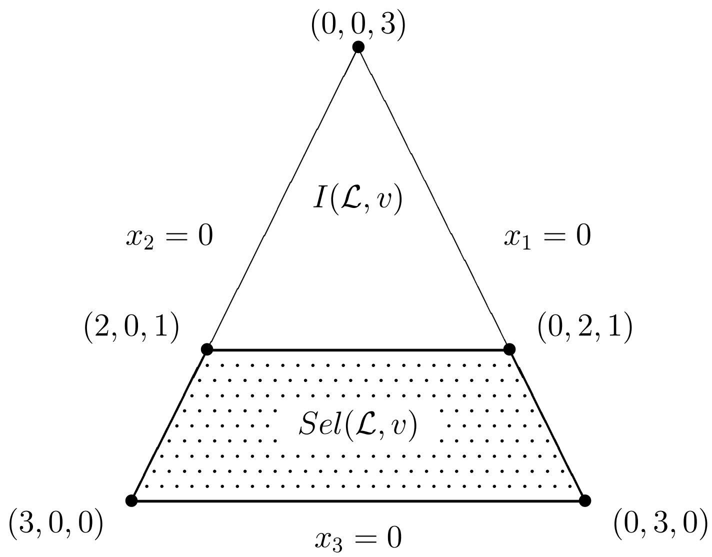

FIGURE 2.1

The inclusion $Core\,(\mathcal{L}, v) \subseteq Sel\,(\mathcal{L}, v)$ is always verified if $\mathcal{L} = 2^N$ (Hammer, Peled and Sorensen, [31]). However, this result is not true, in general, if the family of feasible coalitions $\mathcal{L}$ is different to 2^N. The other inclusion, $Sel\,(\mathcal{L}, v) \subseteq Core\,(\mathcal{L}, v)$, is due to a property of the characteristic function of game v. To check that these inclusions are not true in general, take the following example.

Example 2.2.2 *Let $N = \{1, 2, 3, 4\}$ and consider the family*

$$\mathcal{L} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{2,3\}, \{1,2,3,4\}\}.$$

We define the game $v \in \Gamma(\mathcal{L})$, for every nonempty coalition $S \in \mathcal{L}$, by

$$v(S) = \begin{cases} -1 & \text{if } |S| = 1, \\ 2 & \text{otherwise.} \end{cases}$$

If we calculate the dividends of the game, we get

$$\triangle_v(S) = \begin{cases} -1 & \text{if } |S| = 1, \\ 4 & \text{if } |S| = 2, \\ -2 & \text{if } S = N. \end{cases}$$

There are 16 selectors on $\mathcal{L}$ and we show the selections in the following table:

α	$\{1,2\}$	$\{2,3\}$	$\{1,2,3,4\}$	$m_1^\alpha(v)$	$m_2^\alpha(v)$	$m_3^\alpha(v)$	$m_4^\alpha(v)$
1	1	2	1	1	3	−1	−1
2	1	2	2	3	1	−1	−1
3	1	2	3	3	3	−3	−1
4	1	2	4	3	3	−1	−3
5	1	3	1	1	−1	3	−1
6	1	3	2	3	−3	3	−1
7	1	3	3	3	−1	1	−1
8	1	3	4	3	−1	3	−3
9	2	2	1	−3	7	−1	−1
10	2	2	2	−1	5	−1	−1
11	2	2	3	−1	7	−3	−1
12	2	2	4	−1	7	−1	−3
13	2	3	1	−3	3	3	−1
14	2	3	2	−1	1	3	−1
15	2	3	3	−1	3	1	−1
16	2	3	4	−1	−3	3	−3

TABLE 2.2

On the other hand, $Core(\mathcal{L}, v)$ is given by the vectors of $\mathbb{R}^4$ such that

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 2 \\ x_1 & + & x_2 & & & & & \geq & 2 \\ & & x_2 & + & x_3 & & & \geq & 2 \end{array}$$

$$x_1,\ x_2,\ x_3,\ x_4 \geq -1.$$

In this example, it is verified that

$$Sel(\mathcal{L}, v) \not\subseteq Core(\mathcal{L}, v) \quad y \quad Core(\mathcal{L}, v) \not\subseteq Sel(\mathcal{L}, v),$$

because the vector $(3, 3, -3, -1) \in Sel(\mathcal{L}, v) \setminus Core(\mathcal{L}, v)$ since $x_3 < -1$ and the vector $(0, 2, 0, 0) \in Core(\mathcal{L}, v) \setminus Sel(\mathcal{L}, v)$ as it can not be written as a convex combination of $m^\alpha(v)$, with $\alpha \in \mathcal{A}(\mathcal{L})$.

As it has already been indicated, the inclusion $Sel(\mathcal{L}, v) \subseteq Core(\mathcal{L}, v)$ characterizes a class of games called *almost positive*.

Definition 2.2.6 *A game $v \in \Gamma(\mathcal{L})$ is called almost positive if the dividends of all non-unitary coalitions are nonnegative, that is*

$$\triangle_v(S) \geq 0, \quad \text{for all } S \in \mathcal{L} \text{ with } |S| \geq 2.$$

Theorem 2.2.7 *Let $v \in \Gamma(\mathcal{L})$ be a game on an atomic family. The following statements are equivalent:*

(a) $Sel(\mathcal{L}, v) \subseteq Core(\mathcal{L}, v)$.

(b) $Sel(\mathcal{L}, v) \subseteq I(\mathcal{L}, v)$.

(c) The game v is almost positive.

Proof. It is evident that (a) implies (b), since $Core(\mathcal{L}, v) \subseteq I(\mathcal{L}, v)$ by definition.

The relation (b) forces that the game v is almost positive since, if there exists a coalition $S \in \mathcal{L}$ with $|S| \geq 2$ and $\triangle_v(S) < 0$, we could consider $\alpha \in \mathcal{A}(\mathcal{L})$ such that $\alpha(S) = i$ and $\alpha(T) \neq i$ for all $T \in \mathcal{L}$, $T \neq S, \{i\}$ and then $m_i^{\alpha}(v) = v(\{i\}) + \triangle_v(S) < v(\{i\})$ and it is a contradiction with (b).

Finally, if v is an almost positive game, for every selector α, we have that $m^{\alpha}(v) \in Core(\mathcal{L}, v)$. Indeed, for every nonempty coalition $S \in \mathcal{L}$, we have

$$\begin{aligned} \sum_{i \in S} m_i^{\alpha}(v) &= \sum_{i \in S} \sum_{\{T \in \mathcal{L} : \alpha(T) = i\}} \triangle_v(T) \\ &\geq \sum_{\{T \in \mathcal{L}, T \subseteq S\}} \triangle_v(T) \\ &= v(S), \end{aligned}$$

where the inequality follows because all dividends of game v are nonnegative for the non-unitary coalitions. □

Note that, from this proposition, we deduce that the almost positive games defined on atomic families have a nonempty core.

As it has been seen in the example 2.15, the inclusion $Core(\mathcal{L}, v) \subseteq Sel(\mathcal{L}, v)$ does not hold in general if $\mathcal{L}$ is an atomic family distinct to 2^N. This result is true if we consider the family $\mathcal{L}$ with some additional conditions.

2.3 Games on closure spaces

In the following, we assume that a fundamental rule of cooperation is established on the family of feasible coalitions $\mathcal{L}$: when two coalitions are feasible, their intersection is also feasible. This requirement which is imposed on the cooperation among players has an interpretation that makes the study that it is going to be realized in many real cases useful. It is supposed that the players find some benefit when all are joined, but it is also logical to think that the possible coalitions among them are made attending common interests (they share certain ideas, are the same nationality, belong to a certain company) and so those common players from two coalitions also form one coalition jointly defend a set, presumably, of wider interests. This rule of cooperation gives the family $\mathcal{L}$ a determined structure which is introduced in the following.

Definition 2.3.1 *A closure space on N is a family $\mathcal{L} \subseteq 2^N$ which satisfies the following properties:*

(1) $\emptyset \in \mathcal{L}$ and $N \in \mathcal{L}$,

(2) If $A \in \mathcal{L}$ and $B \in \mathcal{L}$, then $A \cap B \in \mathcal{L}$.

The elements of a closure space are called *closed sets.*

In the following, we will assume that every closure space is a closure space on the set of players N.

When a family $\mathcal{L}$ is a closure space, we can associate to each set $A \in 2^N$ the set $\overline{A}$, defined by

$$\overline{A} = \bigcap \{S \in \mathcal{L} : A \subseteq S\},$$

such that it satisfies

(C1) $A \subseteq \overline{A}$,

(C2) $\overline{A} = A$ if and only if $A \in \mathcal{L}$, (C3) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$,

Note that it allows to indicate that the operator $- : 2^N \longrightarrow 2^N$ is a closure operator [21] with the additional property $\overline{\emptyset} = \emptyset$. Conversely, departing from any closure operator as above, we can consider the closure space $\mathcal{L}$ determined by the closed sets of the operator.

Taking into account this reasoning , it is immediate to check that $(\mathcal{L}, \subseteq)$ is a complete lattice. Indeed, for all $A, B \in \mathcal{L}$, we have that

$$A \wedge B = \inf \{A, B\} = A \cap B,$$

$$A \vee B = \sup \{A, B\} = \overline{A \cup B}.$$

Some examples of closure spaces on N are the following:

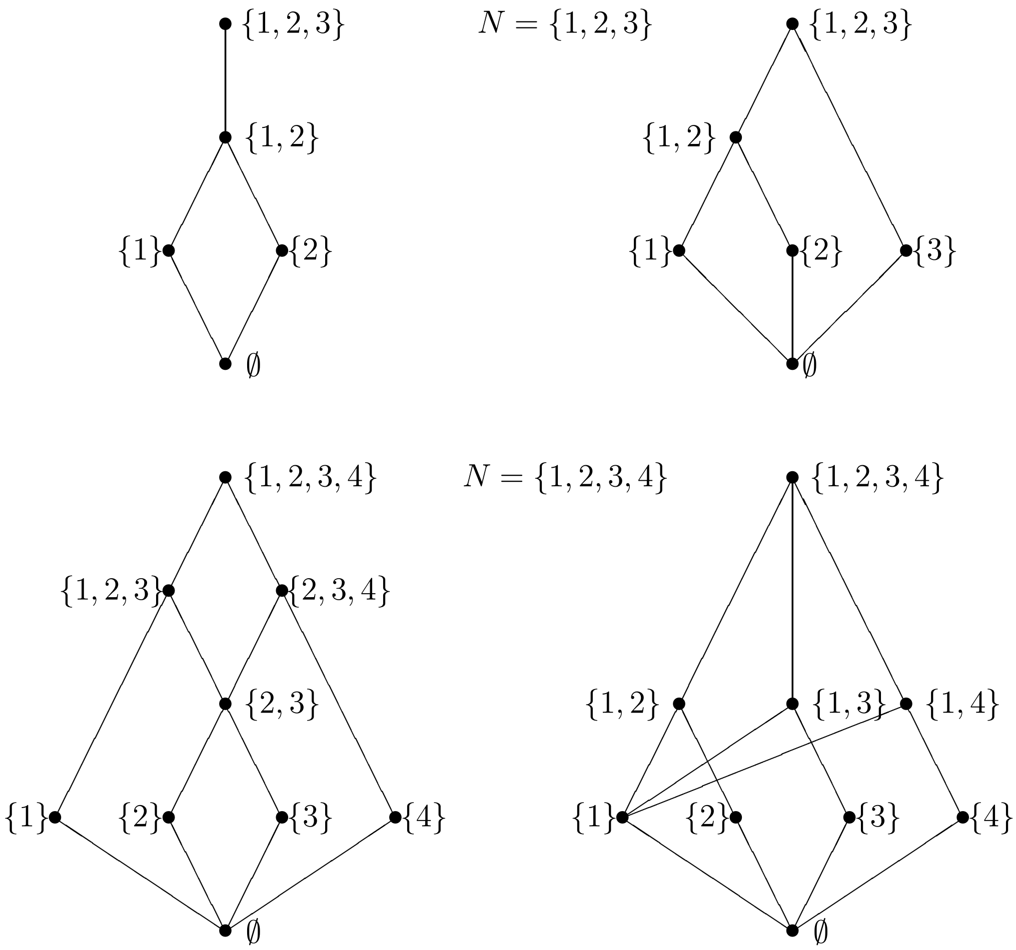

FIGURE 2.2

One of the objectives of introducing additional conditions on the family of feasible coalitions is to establish sufficient hypothesis on $\mathcal{L}$ so that the inclusion $Core\,(\mathcal{L}, v) \subseteq Sel\,(\mathcal{L}, v)$ is true. Note that the selectope of a game $v \in \Gamma(\mathcal{L})$ is defined on atomic families and is always a bounded set. So that the above inclusion makes sense, $Core\,(\mathcal{L}, v)$ has to also be a bounded polyhedron and, as it has already been indicated, the core of a game $v \in \Gamma(\mathcal{L})$ is a bounded set if the family $\mathcal{L}$ is atomic. Although it is not relevant in the relation between core and selectope, we can observe that, if $\mathcal{L}$ is a closure space, this sufficient condition is also made necessary.

Proposition 2.3.1 *Let $\mathcal{L}$ be a closure space and let $v \in \Gamma(\mathcal{L})$ be a game such that $Core(\mathcal{L}, v) \neq \emptyset$. The following statements are equivalent:*

(a) The core of the game v is a bounded polyhedron.

(b) The family $\mathcal{L}$ is atomic.

Proof. The Proposition 2.6 establishes that if the family $\mathcal{L}$ is atomic, but not necessarily a closure space, the core is a bounded polyhedron. Thus, it is sufficient to prove the other implication. Assume that there exists $j \in N$ such that $\{j\} \notin \mathcal{L}$, and consider $\overline{\{j\}} \in \mathcal{L}$. Let $k \in \overline{\{j\}}$ with $k \neq j$. We define the vector $x \in \mathbb{R}^n$ in the following way

$$x_i = \begin{cases} -1 & \text{if } i = j, \\ 1 & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

We have that $x(N) = 0$ and also $x(S) \geq 0$, for all $S \in \mathcal{L}$, because if $j \in S$ then $\overline{\{j\}} \subseteq S$ and hence $k \in S$. This proves that

$$\{x \in \mathbb{R}^n : x(N) = 0,\ x(S) \geq 0 \text{ para toda } S \in \mathcal{L}\} \neq \{0\},$$

and it leads to a contradiction because the polyhedron $Core(\mathcal{L}, v)$ would not be bounded. □

Two special cases of closure spaces are the *intersecting families* [30] and the *convex geometries* [21]. In this section, only the intersecting families are defined as the convex geometries will be discussed in detail in the next chapter.

Definition 2.3.2 *An intersecting family is a closure space $\mathcal{L}$ which satisfies that*

$$S, T \in \mathcal{L} \text{ with } S \cap T \neq \emptyset \Longrightarrow S \cup T \in \mathcal{L}.$$

Note that all closure spaces on a set of three players are intersecting families.

Before establishing the next theorem, it is necessary to present some concepts regarding the construction of the so-called *maximal closed sets* of a coalition $S \subseteq N$. For this, we consider that the family of feasible coalitions $\mathcal{L}$ is an atomic intersecting family.

Let $S \subseteq N$, we denote by $\mathcal{L}_S$ the set of all closed sets of $\mathcal{L}$ contained in S, that is

$$\mathcal{L}_S = \{T \in \mathcal{L} : T \subseteq S\}.$$

Obviously, $\mathcal{L}_S \neq \emptyset$ since $\emptyset, \{i\} \in \mathcal{L}_S$ for all $i \in S$. Moreover, $\mathcal{L}_S$ is an atomic intersecting family on S.

If we consider $\mathcal{L}_S$ ordered by inclusion, then $(\mathcal{L}_S, \subseteq)$ is a partially ordered set where its first element is $\emptyset$. Let $\Pi_{\mathcal{L}}(S)$ be the set of all maximal elements of $(\mathcal{L}_S, \subseteq)$. Each one of the elements of $\Pi_{\mathcal{L}}(S)$ is called *maximal closed set* of S.

Proposition 2.3.2 *Let $\mathcal{L}$ be an atomic intersecting family. For every $S \subseteq N$, the coalitions of $\Pi_{\mathcal{L}}(S)$ form a partition of S, and this partition is unique.*

Proof. Note that if $S \in \mathcal{L}$, then $\Pi_{\mathcal{L}}(S) = \{S\}$ and therefore, the proposition is trivially true. If $S \notin \mathcal{L}$ and we suppose that $\Pi_{\mathcal{L}}(S) = \{S_1, \ldots, S_k\}$, then $k \geq 2$ and $\cup_{i=1}^{k} S_i \subseteq S$. Also, $S \subseteq \cup_{i=1}^{k} S_i$ because $\mathcal{L}$ is an atomic family.

In order to prove that the coalitions in $\Pi_{\mathcal{L}}(S)$ form a partition of S, we show that $S_i \cap S_j = \emptyset$ for all $1 \leq i, j \leq k, i \neq j$. Indeed, if $S_i \cap S_j \neq \emptyset$ then $S_i \cup S_j \in \mathcal{L}$ since $\mathcal{L}$ is an intersecting family and also $S_i \cup S_j \subseteq S$, hence $S_i \cup S_j \in \mathcal{L}_S$, but this is a contradiction because S_i and S_j are maximal closed sets of S. Obviously, by construction $\Pi_{\mathcal{L}}(S)$ is unique. □

In the preceding section, the selectope was defined and necessary and sufficient conditions so that $Sel(\mathcal{L}, v) \subseteq Core(\mathcal{L}, v)$ were established. The other inclusion is true for every $v \in \Gamma(\mathcal{L})$, if the atomic family of feasible coalitions $\mathcal{L}$ is an intersecting family.

Theorem 2.3.3 *Let $\mathcal{L}$ be an atomic intersecting family and let $v \in \Gamma(\mathcal{L})$. Then,*

$$Core(\mathcal{L}, v) \subseteq Sel(\mathcal{L}, v).$$

Proof. Assume that there exists $x \in Core(\mathcal{L}, v)$ such that $x \notin Sel(\mathcal{L}, v)$. According to the convexity and closedness of $Sel(\mathcal{L}, v)$ and applying the

Separation Theorem (see Rockafellar [55]), there exists $y \in \mathbb{R}^n$ such that

$$z \cdot y > x \cdot y \qquad \text{for all } z \in Sel\left(\mathcal{L}, v\right).$$

In particular for $z = m^{\alpha}\left(v\right)$ with $\alpha \in \mathcal{A}\left(\mathcal{L}\right)$. If the components of vector y are ordered in decreasing order,

$$y_{i_1} \geq y_{i_2} \geq \cdots \geq y_{i_{n-1}} \geq y_{i_n},$$

then

$$\begin{aligned}
x \cdot y &= \sum_{j=1}^{n} x_{i_j} y_{i_j} = y_{i_n} \sum_{j=1}^{n} x_{i_j} + \sum_{k=1}^{n-1} \left(y_{i_k} - y_{i_{k+1}}\right) \sum_{j=1}^{k} x_{i_j} \\
&\geq y_{i_n} v\left(N\right) + \sum_{k=1}^{n-1} \left(y_{i_k} - y_{i_{k+1}}\right) \sum_{S \in \Pi_{\mathcal{L}}\left(\left\{i_1, i_2, \ldots, i_k\right\}\right)} v\left(S\right) \\
&= y_{i_1} v\left(\left\{i_1\right\}\right) + \sum_{j=2}^{n} y_{i_j} \left(\sum_{S \in \Pi_{\mathcal{L}}\left(\left\{i_1, i_2, \ldots, i_j\right\}\right)} v\left(S\right) - \sum_{S \in \Pi_{\mathcal{L}}\left(\left\{i_1, i_2, \ldots, i_{j-1}\right\}\right)} v\left(S\right) \right) \\
&= y_{i_1} \triangle_v\left(\left\{i_1\right\}\right) \\
&\quad + \sum_{j=2}^{n} y_{i_j} \left(\sum_{S \in \Pi_{\mathcal{L}}\left(\left\{i_1, i_2, \ldots, i_j\right\}\right)} \sum_{T \subseteq S} \triangle_v\left(T\right) - \sum_{S \in \Pi_{\mathcal{L}}\left(\left\{i_1, i_2, \ldots, i_{j-1}\right\}\right)} \sum_{T \subseteq S} \triangle_v\left(T\right) \right) \\
&= y_{i_1} \triangle_v\left(\left\{i_1\right\}\right) \\
&\quad + \sum_{j=2}^{n} y_{i_j} \left(\sum_{\left\{T \in \mathcal{L}, T \subseteq \left\{i_1, i_2, \ldots, i_j\right\}\right\}} \triangle_v\left(T\right) - \sum_{\left\{T \in \mathcal{L}, T \subseteq \left\{i_1, i_2, \ldots, i_{j-1}\right\}\right\}} \triangle_v\left(T\right) \right) \\
&= \sum_{j=1}^{n} y_{i_j} \left(\sum_{\left\{T \in \mathcal{L}, T \subseteq \left\{i_1, i_2, \ldots, i_j\right\}, i_j \in T\right\}} \triangle_v\left(T\right) \right),
\end{aligned}$$

where the last but one equality follows by Proposition 2.21. Obviously, it is sufficient to take the selector $\alpha \in \mathcal{A}\left(\mathcal{L}\right)$ such that

$$m_{i_j}^{\alpha}\left(v\right) = \sum_{\left\{T \in \mathcal{L}, T \subseteq \left\{i_1, i_2, \ldots, i_j\right\}, i_j \in T\right\}} \triangle_v\left(T\right)$$

for all $1 \leq j \leq n$ in order to have a contradiction. This selector $\alpha \in \mathcal{A}\left(\mathcal{L}\right)$ is defined by

$$\alpha(S) = i_k, \quad \text{where } k = \max\{p : i_p \in S\}$$

for all $S \in \mathcal{L}$. □

As it was made clear by defining the dividends of Harsanyi, every game $v \in \Gamma(\mathcal{L})$ can be written as a linear combination of the unanimity games where the coefficients are the dividends of the coalitions in $\mathcal{L}$. So, if we consider

$$v^+ = \sum_{\{S\in\mathcal{L}:\triangle_v(S)>0\}} \triangle_v(S)\,\zeta_S \quad \text{and} \quad v^- = \sum_{\{S\in\mathcal{L}:\triangle_v(S)<0\}} -\triangle_v(S)\,\zeta_S,$$

then $v = v^+ - v^-$.

Using this descomposition of the game v, we can establish the next result.

Theorem 2.3.4 *Let $\mathcal{L}$ be an atomic intersecting family and let $v \in \Gamma(\mathcal{L})$. Then*

(a) $Sel(\mathcal{L}, v^+) = Core(\mathcal{L}, v^+)$ and $Sel(\mathcal{L}, v^-) = Core(\mathcal{L}, v^-)$.

(b) $$Sel(\mathcal{L}, v) = Core(\mathcal{L}, v^+) - Core(\mathcal{L}, v^-) = \{x \in \mathbb{R}^n : x = y - z,\ y \in Core(\mathcal{L}, v^+),\ z \in Core(\mathcal{L}, v^-)\}.$$

Proof. (a) It is a direct consequence of Theorem 2.17 and Theorem 2.22.

(b) For every $\alpha \in \mathcal{A}(\mathcal{L})$, we have that $m^\alpha(v) = m^\alpha(v^+) - m^\alpha(v^-)$, and by part (a), $m^\alpha(v) \in Core(\mathcal{L}, v^+) - Core(\mathcal{L}, v^-)$, and hence

$$Sel(\mathcal{L}, v) \subseteq Core\left(\mathcal{L}, v^+\right) - Core\left(\mathcal{L}, v^-\right).$$

In order to prove the reverse inclusion, in view of part (a), it is sufficient to prove that for any $\alpha, \beta \in \mathcal{A}(\mathcal{L})$, we have that $m^\alpha(v^+) - m^\beta(v^-) \in Sel(\mathcal{L}, v)$. Note that if we define $\gamma \in \mathcal{A}(\mathcal{L})$, for every nonempty coalition $S \in \mathcal{L}$, by

$$\gamma(S) = \begin{cases} \alpha(S) & \text{if } \triangle_v(S) \geq 0, \\ \beta(S) & \text{if } \triangle_v(S) < 0, \end{cases}$$

then $m^\gamma(v) = m^\alpha(v^+) - m^\beta(v^-)$ and so $m^\gamma(v) \in Sel(\mathcal{L}, v)$. □

Note that if the family of feasible coalitions $\mathcal{L}$ is atomic, but not necessarily an intersecting family, then we can only assert one of the inclusions in part (a) and (b).

2.4 Supermodular games

In this section, we study the aforementioned solution concepts: core and stable sets for a game defined on an intersecting family, demanding some additional conditions to the game. One of these conditions will be that the game is supermodular on the family of feasible coalitions.

Faigle and Kern [26] introduce the supermodularity property for a game on a distributive lattice. In the model introduced in the preceding section, a closure space $\mathcal{L}$ is a lattice with

$$S \vee T = \overline{S \cup T} \qquad \text{and} \qquad S \wedge T = S \cap T,$$

for all $S, T \in \mathcal{L}$. Then, we propose the following definition of supermodular game on $\mathcal{L}$.

Definition 2.4.1 *Let $\mathcal{L}$ be a closure space. A game $v \in \Gamma(\mathcal{L})$ is supermodular if, for any $S, T \in \mathcal{L}$, it holds*

$$v(\overline{S \cup T}) + v(S \cap T) \geq v(S) + v(T).$$

Note that in the case $\mathcal{L} = 2^N$, we obtain the classic definition of convex or supermodular game. If the family $\mathcal{L}$ is an intersecting family, Faigle [24] define a supermodularity property for a game $v \in \Gamma(\mathcal{L})$ weakening the preceding requirement on the characteristic function, giving rise to the following definition.

Definition 2.4.2 *Let $\mathcal{L}$ be an intersecting family. A game $v \in \Gamma(\mathcal{L})$ is called intersecting supermodular if for every pair $S, T \in \mathcal{L}$ with $S \cap T \neq \emptyset$, it holds*

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T).$$

Faigle [24] considers a family $\mathcal{L}$ of subsets of N (nothing is required of this family) and denominates $\tilde{\mathcal{L}}$ the set of coalitions of N that can be expressed as disjoint unions of elements of $\mathcal{L}$; that is, if $A \in \tilde{\mathcal{L}}$ then

$$A = A_1 \cup A_2 \cup \ldots \cup A_p,$$

where $A_i \in \mathcal{L}$, for all $i = 1, \ldots, p$ and also $A_i \cap A_j = \emptyset$ for $i \neq j$. Note that, in general, $\tilde{\mathcal{L}} \neq 2^N$. On the other hand, he defines the function

$$\tilde{v} : \tilde{\mathcal{L}} \longrightarrow \mathbb{R}, \quad \tilde{v}(A) = \max \left\{ \sum_i v(A_i) \right\}$$

where the maximum is taken among all partitions of A in elements of $\mathcal{L}$; that is, among all the so-called *$\mathcal{L}$-partitions* of A.

With this idea, Faigle shows that if $v : \mathcal{L} \longrightarrow \mathbb{R}$ is an intersecting supermodular game, then the game $\tilde{v} : \tilde{\mathcal{L}} \longrightarrow \mathbb{R}$ is supermodular. The next proposition is a corollary of this result for an atomic family $\mathcal{L}$ since, in this case, $\tilde{\mathcal{L}} = 2^N$.

Proposition 2.4.1 *Let $\mathcal{L}$ be an atomic intersecting family. If $v \in \Gamma(\mathcal{L})$ is an intersecting supermodular game, then $\tilde{v} : 2^N \longrightarrow \mathbb{R}$ is a supermodular game.*

The following proposition establishes one relation between the core of $v \in \Gamma(\mathcal{L})$ and the core of the extension $\tilde{v} \in \Gamma\left(2^N\right)$ and is used in the proof of the next theorem.

Proposition 2.4.2 *Let $\mathcal{L}$ be an atomic family. If $v \in \Gamma(\mathcal{L})$ satisfies that $v(N) = \tilde{v}(N)$, then $Core(\mathcal{L}, v) = Core\left(2^N, \tilde{v}\right)$.*

Proof. If $x \in Core\left(2^N, \tilde{v}\right)$, then $x(N) = \tilde{v}(N) = v(N)$, and also $x(S) \geq \tilde{v}(S)$, for all $S \subseteq N$. If $S \in \mathcal{L}$, then $\{S\}$ is a partition of S, and hence $x(S) \geq \tilde{v}(S) \geq v(S)$.

In order to prove the reverse inclusion, let $x \in Core(\mathcal{L}, v)$ and $\{S_k\}$ be an $\mathcal{L}$-partition of S. We have that

$$x(S) = \sum_{i \in S} x_i = \sum_k \left[\sum_{i \in S_k} x_i \right] = \sum_k x(S_k) \geq \sum_k v(S_k),$$

and hence

$$x(S) \geq \max \left\{ \sum_i v(T_i) : \{T_i\} \text{ is an } \mathcal{L}\text{-partition of } S \right\} = \tilde{v}(S).$$

□

Note that if $v(N) \neq \tilde{v}(N)$, then $Core(\mathcal{L}, v) \bigcap Core\left(2^N, \tilde{v}\right) = \emptyset$.

It is immediate to check that the condition $v(N) = \tilde{v}(N)$ is equivalent to that for every $\mathcal{L}$-partition of N, $\{P_1, \ldots, P_r\} \subseteq \mathcal{L}$, the following inequality is satisfied

$$v(P_1) + \cdots + v(P_r) \leq v(N).$$

In view of this observation, and using the above proposition, we can prove the next result, which establishes the existing relation between the core and the stable sets for intersecting supermodular games.

Theorem 2.4.3 *Let $\mathcal{L}$ be an atomic intersecting family and let $v \in \Gamma(\mathcal{L})$ be an intersecting supermodular game satisfying that, for every $\mathcal{L}$-partition of N, $\{P_1, \ldots, P_r\} \subseteq \mathcal{L}$, $v(P_1) + \cdots + v(P_r) \leq v(N)$ holds. Then, $Core(\mathcal{L}, v)$ is a stable set (the unique stable set).*

Proof. First of all, note that, with the hypothesis of the theorem, and applying the Propositions 2.26 and 2.27 for the game $\tilde{v} : 2^N \longrightarrow \mathbb{R}$ we have that $I(\mathcal{L}, v) = I\left(2^N, \tilde{v}\right)$, $Core(\mathcal{L}, v) = Core\left(2^N, \tilde{v}\right)$ and also the game $\tilde{v} : 2^N \longrightarrow \mathbb{R}$ is supermodular and, therefore, $Core\left(2^N, \tilde{v}\right) \neq \emptyset$.

As, by Proposition 2.10, the core is always internally stable, we only prove the external stability, that is, every imputation that is not in the core is dominated by one imputation of the core. If $x \in I(\mathcal{L}, v) \setminus Core(\mathcal{L}, v)$, then $x \in I\left(2^N, \tilde{v}\right) \setminus Core\left(2^N, \tilde{v}\right)$ and as the game $\tilde{v}$ is supermodular, $Core\left(2^N, \tilde{v}\right)$ is stable. Thus, there exists $y \in Core\left(2^N, \tilde{v}\right)$ such that $y \operatorname{dom} x$ in the game $\tilde{v}$. Then, for some nonempty coalition $S \in 2^N$, we have that $y(S) \leq \tilde{v}(S)$ and $y_i > x_i$ for all $i \in S$.

Let $\{S_1, \ldots, S_k\}$ be an $\mathcal{L}$-partition of S for which the maximum $\tilde{v}(S)$ is obtained, i.e.

$$\tilde{v}(S) = \sum_{j=1}^{k} v(S_j).$$

For this $\mathcal{L}$-partition of S, we have that $y_i > x_i$, for all $i \in S_j$, with

$1 \leq j \leq k$, and also,

$$\sum_{i \in S} y_i = \sum_{j=1}^{k} \left(\sum_{i \in S_j} y_i \right) \leq \sum_{j=1}^{k} v\left(S_j\right).$$

From this, we deduce that there exists at least one coalition $S_j \in \mathcal{L}$ verifying that $y\left(S_j\right) \leq v\left(S_j\right)$ and therefore, $y \operatorname{dom} x$ using the coalition S_j in the game v. $\square$

Chapter 3

Games on convex geometries

This chapter is dedicated to the study of games defined on a special class of closure spaces, the so-called *convex geometries.* For this, we present, in the first section, the general model which was established by Edelman and Jamison in *The theory of convex geometries* [21], defining some concepts and relevant properties.

The interest in the particular study of games defined on families of feasible coalitions which have the structure of a convex geometry is derived, initially, from the observations indicated by G. Owen [53] when he analyzes the simplifications which are produced in the computation of the Myerson value if, in the communication situation (N, v, G), the graph which models the partial cooperation among the players is a tree. Later, the special characteristics of these singular communication situations (N, v, G), in which $G = (N, E)$ is a tree have been made clear in the work, among others, of Borm, Nouweland, Owen and Tijs [13] when the problems of allocation of costs are analyzed and integral formulas are found for the calculation of the Myerson value; of Grafe, Mauleon and Iñarra [28] in the search of a procedure for computing the nucleolus; and Potters and Reijnierse [54] in the study of an equilibrium character of the restricted games by communication graphs, its nucleolus and the relations between the bargaining set and the core.

In general, the specific properties of the trees, in relation with other types of graphs, lie in its connected character, in that the intersection of connected subgraphs leads to a connected subgraph and in any connected graph is the graph originated by the convex hull of its extreme points. These characteristics and the results obtained in the work indicated before makes it interesting to plan, in this model of partial cooperation, the study of games defined on families of feasible coalitions which have a combinatorial structure with analogous properties to the mentioned before. This makes the convex geometries be considered.

3.1 Convex geometries

In this section, we define the concept of convex geometry, describe its fundamental properties and present some examples.

Definition 3.1.1 *A convex geometry on N is a family $\mathcal{L} \subseteq 2^N$ which satisfies the following properties:*

(G1) $\emptyset \in \mathcal{L}$,

(G2) If $A, B \in \mathcal{L}$, *then* $A \cap B \in \mathcal{L}$,

(G3) If $A \in \mathcal{L}$ *and* $A \neq N$, *there exists* $j \in N \setminus A$ *such that* $A \cup \{j\} \in \mathcal{L}$.

The elements of a convex geometry is called *convex sets*. Note that the properties $(G1)$ and $(G3)$ imply that $N \in \mathcal{L}$. On the other hand, $\mathcal{L} = 2^N$ is a convex geometry on N.

As it has already been indicated, the property $(G2)$, which characterizes the closure spaces, it is logical not only from the mathematical point of view, but also from the real point of view. The property $(G3)$ could be interpreted in the following way: the coalition formed by all players is reached by sequential processes of one by one incorporation of the participants in the game.

In relation with the interpretation of the property $(G3)$, Edelman and Jamison [21] define a *compatible ordering* with a convex geometry $\mathcal{L}$ as a total order of the elements of N, $i_1 < i_2 < \cdots < i_n$, such that

$$\{i_1, i_2, \ldots, i_k\} \in \mathcal{L}, \quad \text{for all } \ 1 \leq k \leq n.$$

A compatible ordering with $\mathcal{L}$ corresponds exactly to a *maximal chain* in $\mathcal{L}$. A maximal chain C of $\mathcal{L}$ is an ordered collection of convex sets

$$\emptyset = C_0 \subset C_1 \subset \cdots \subset C_{n-1} \subset C_n = N,$$

such that there does not exist a convex set M nor an index $0 \leq j \leq n-1$ such that $C_j \subset M \subset C_{j+1}$. That is, C is a chain which is not contained in any larger chain. We denote by $\mathcal{C}(\mathcal{L})$ the set of all maximal chains of $\mathcal{L}$.

Another relevant concept in a convex geometry is the so-called extreme point. Given a convex set $A \in \mathcal{L}$, an element $a \in A$ is an *extreme point* of A when $A \setminus \{a\} \in \mathcal{L}$. By simplifying the notation, we will write $A \setminus a$ instead of $A \setminus \{a\}$. We denote by $ex(A)$ the set of all extreme points of A.

We exemplify the previous notions for a determined convex geometry.

Example 3.1.1 *Consider* $N = \{1, 2, 3\}$ *and the convex geometry*

$$\mathcal{L} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, N\},$$

where its Hasse diagram is

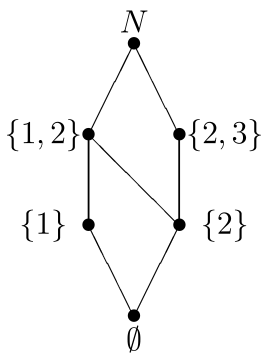

FIGURE 3.1

There are three maximal chains in $\mathcal{L}$*,*

$$C_1 : \emptyset \subset \{1\} \subset \{1,2\} \subset \{1,2,3\}\,,$$
$$C_2 : \emptyset \subset \{2\} \subset \{1,2\} \subset \{1,2,3\}\,,$$
$$C_3 : \emptyset \subset \{2\} \subset \{2,3\} \subset \{1,2,3\}\,.$$

On the other hand, $ex\,(\{1,2\}) = \{1,2\}$, $ex\,(\{2,3\}) = \{3\}$ *and* $ex\,(\{1,2,3\}) = \{1,3\}$.

Note that when $\mathcal{L} = 2^N$ there are $n!$ maximal chains corresponding to all orderings of players. Furthermore, for any $S \in 2^N$, all its points are extreme.

The extreme points permit the identification of the convex geometries with the closure spaces which satisfy the finite property of Minkowski-Krein-Milman: *every closed set is the closure of its extreme points.*

The convex geometries can also be defined as the closure spaces that satisfy the so-called *anti-exchange* property: given a closed set A and two elements x and y, $x \neq y$, belong to $N \setminus A$, then $y \in \overline{A \cup \{x\}}$ implies $x \notin \overline{A \cup \{y\}}$.

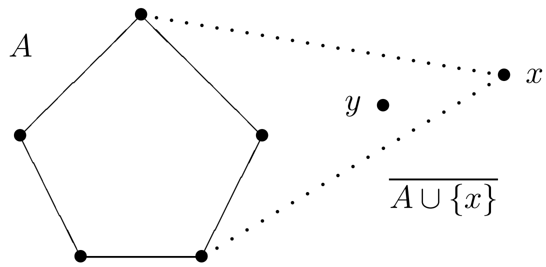

FIGURE 3.2

These equivalences in the identification of a convex geometry are picked up in the next theorem where its proof is due to Edelman [21, Theorem 2.1].

Theorem 3.1.1 *Let $\mathcal{L}$ be a closure space. The following statements are equivalent:*

(a) The family $\mathcal{L}$ is a convex geometry.

(b) The family $\mathcal{L}$ satisfies the anti-exchange property.

(c) For every $C \in \mathcal{L}$, it holds that $C = \overline{ex(C)}$.

As we have been indicated in the introduction to this chapter, the interest in the study of convex geometries comes from the search of combinatorial structures that generalize, if it is possible, the obtained results with other structures of cooperation used in the theory of games. So, we show some examples of set families with structure of convex geometry which have appeared in the literature relationed with the partial cooperation.

Example 3.1.2 *A communication situation is a triple (N, v, G), where (N, v) is a game and $G = (N, E)$ is a graph. If G is a connected and cycle-complete graph, then the family of all coalitions of N that induce connected subgraphs, that is*

$$\mathcal{L} = \{S \subseteq N \ : \ (S, E(S)) \text{ is connected}\},$$

is a convex geometry [21, Theorem 3.7].

Example 3.1.3 *The family of convex subsets of a finite partially ordered set* $(P, \leq)$ *is a convex geometry (Birkhoff and Bennett [11, Theorem 3]). In this context, a set* S *of* P *is a convex set if* $a \in S$, $b \in S$ *and* $a \leq b$ *implies* $[a, b] \subseteq S$. *We denote by* $Co(N)$ *the corresponding convex geometry considering* $N = \{1, 2, \ldots, n\}$ *with the natural order of its elements. Thus, the elements of* $Co(N)$ *are*

$$[i, j] = \{i, i+1, \ldots, j-1, j\}, \quad \textit{for } 1 \leq i \leq j \leq n,$$

and this model which has an antecedent in the policy order of Axelrod [4], have been used by Edelman [22] for the study of power indices in voting games.

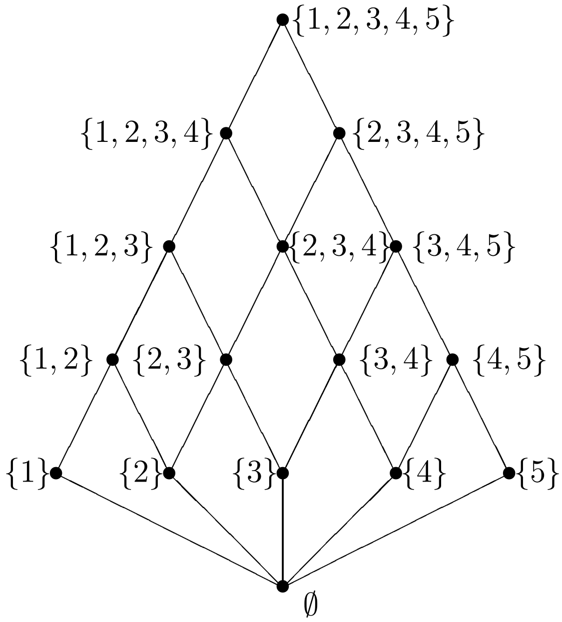

FIGURE 3.3. *The convex geometry* $Co(\{1, 2, 3, 4, 5\})$.

Example 3.1.4 *Let* $(P, \leq)$ *be a partially ordered set. The operator defined, for any* $X \subseteq P$, *by*

$$X \longrightarrow \overline{X} = \{y \in P : y \leq x \textit{ for some } x \in X\},$$

is a closure operator on P *and its closed sets are the order ideals of* P. *This lattice is denoted by* $J(P)$. *As the union and the intersection of order ideals* *is an order ideal, it follows that* $J(P)$ *is a distributive sublattice of* 2^P. *Also,* $J(P)$ *is a convex geometry closed under the union and the set of extreme points of* $S \in J(P)$ *is the set of all maximal points* $Max(S)$. *When the set* P *is finite, there is a one to one correspondence between the antichains of* P *and the order ideals. Faigle and Kern [25][26], study games defined on distributive lattices:* $(\mathcal{C}, v)$ *and* $(\mathcal{A}, v)$, *where* $\mathcal{C}$ *is* $J(P)$ *and* $\mathcal{A}$ *is the set of antichains of a hierarchy (partially ordered set in which each element has at most one element that covers it).*

3.2 The Weber set

In 1978, Weber [66] proposed as a solution concept for a cooperative game, a subset of the preimputation set where its definition is based in the marginal worth vectors. If we consider all possible permutations of the set of players N, each permutation $i_1, i_2, \ldots, i_n$ can be interpreted as a sequential process of formation of the grand coalition. Beginning from the emptyset, first the player i_1 is incorporated, next the player i_2 and so sucessively until the incorporation of the player i_n give rise to the coalition N. In each one of these processes, every player can evaluate his contribution to the coalition where he has been incorporated, and is then reflected in a vector which is called *marginal worth vector*. The j-th component of this vector represents the contribution of player j to the coalition of his predecessors.

In order to extend the idea of Weber to a game v defined on a family $\mathcal{L} \subseteq 2^N$, we consider that the family of feasible coalitions is a convex geometry.

When $\mathcal{L}$ is a convex geometry, each one of maximal chains determines a permutation of the set of players and so, a sequential order in the formation of the grand coalition (a compatible order according to the definition of Edelman). Given $i \in N$, and a maximal chain C, the set

$$C(i) = \{j \in N : j \leq i \text{ in the chain } C\},$$

represents the coalition of $\mathcal{L}$ formed by the player i and his predecessors in the chain C. Obviously, $i \in ex(C(i))$ since $C(i) \setminus i \in \mathcal{L}$.

Definition 3.2.1 *Let $C \in \mathcal{C}(\mathcal{L})$ be a maximal chain. If $v \in \Gamma(\mathcal{L})$, the marginal worth vector with respect to the chain C in game v is the vector $a^C(v) \in \mathbb{R}^n$, where its components are*

$$a_i^C(v) = v(C(i)) - v(C(i) \setminus i).$$

In the appendix, we present an algorithm for computing the marginal worth vectors.

From the following proposition, we deduce that the marginal worth vectors associated to the maximal chains $C \in \mathcal{C}(\mathcal{L})$ are preimputations for the game v.

Proposition 3.2.1 *Let $v \in \Gamma(\mathcal{L})$, where $\mathcal{L}$ is a convex geometry. Then,*

$$\sum_{j \in S} a_j^C(v) = v(S),$$

for each $C \in \mathcal{C}(\mathcal{L})$ and for every convex set S in the chain C.

Proof. Given $C \in \mathcal{C}(\mathcal{L})$, for each $k \in N$, we denote by S_k the coalition with cardinal k of the maximal chain C. If $S_0 = \emptyset$ and $S_k = \{i_1, i_2, \ldots, i_k\}$, for all $1 \leq k \leq n$, we have that

$$\begin{aligned}
\sum_{j \in S_k} a_j^C(v) &= \sum_{j=1}^{n} a_{i_j}^C(v) \\
&= \sum_{j=1}^{n} \left[v(C(i_j)) - v(C(i_j) \setminus i_j)\right] \\
&= \sum_{j=1}^{n} \left[v(S_j) - v(S_{j-1})\right] \\
&= v(S_j).
\end{aligned}$$

Note that for $S_n = N$, we have $\sum_{j \in N} a_j^C(v) = v(N)$. □

According to this result, every marginal worth vector is an efficient vector that satisfies at least n equalities among the inequalities that define the core of the game. Hence, we can affirm that any marginal worth vector is an exterior point of the core or a vertex of the core, as a point of a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is a vertex of this if and only if verifies n independent equalities of $Ax = b$.

Corollary 3.2.2 *Let $v \in \Gamma(\mathcal{L})$, where $\mathcal{L}$ is a convex geometry. If the vector $a^C(v) \in Core(\mathcal{L}, v)$, then $a^C(v)$ is a vertex of the polyhedron $Core(\mathcal{L}, v)$.*

Definition 3.2.2 *Let $\mathcal{L}$ be a convex geometry. The Weber set of $v \in \Gamma(\mathcal{L})$ is the convex hull of the marginal worth vectors, that is*

$$Weber(\mathcal{L}, v) = \text{conv}\left\{a^C(v) : C \in \mathcal{C}(\mathcal{L})\right\}.$$

As the preimputation set is a convex set and the marginal worth vectors are preimputations, we have that

$$Weber(\mathcal{L}, v) \subseteq I^*(\mathcal{L}, v).$$

However, in general, the vectors of the Weber set are not imputations. It is easy to see, taking into account that $I(\mathcal{L}, v)$ is a convex set, that $Weber(\mathcal{L}, v) \subseteq I(\mathcal{L}, v)$ if all marginal worth vectors are imputations. For this, a sufficient condition is that the game v is *zero-monotonic*, a concept that is defined as follows.

Definition 3.2.3 *A game $v \in \Gamma(\mathcal{L})$ is monotonic when for all $S, T \in \mathcal{L}$ with $S \subseteq T$ it holds that $v(S) \leq v(T)$.*

Definition 3.2.4 *The zero-normalization of a game $v \in \Gamma(\mathcal{L})$ is the game $v_0 \in \Gamma(\mathcal{L})$ defined by*

$$v_0(S) = v(S) - \sum_{\{j \in S : \{j\} \in \mathcal{L}\}} v(\{j\}), \quad \textit{for all } S \in \mathcal{L}.$$

Definition 3.2.5 *A game $v \in \Gamma(\mathcal{L})$ is zero-monotonic if its zero-normalization is monotonic.*

Proposition 3.2.3 *Let $v \in \Gamma(\mathcal{L})$ be a zero-monotonic game, where $\mathcal{L}$ is a convex geometry. Then, for every $C \in \mathcal{C}(\mathcal{L})$, the marginal worth vector associated to C is an imputation for the game v.*

Proof. Let $C \in \mathcal{C}(\mathcal{L})$. Since the vector $a^C(v)$ is efficient, we prove that $a_i^C(v) \geq v(\{i\})$, for all $\{i\} \in \mathcal{L}$. Indeed,

$$\begin{aligned} a_i^C(v) &= v(C(i)) - v(C(i) \setminus i) \\ &= v_0(C(i)) + \sum_{\{j \in C(i) : \{j\} \in \mathcal{L}\}} v(\{j\}) \\ &\quad - v_0(C(i) \setminus i) - \sum_{\{j \in C(i) \setminus i : \{j\} \in \mathcal{L}\}} v(\{j\}) \\ &= v_0(C(i)) - v_0(C(i) \setminus i) + v(\{i\}) \\ &\geq v(\{i\}), \end{aligned}$$

where the inequality follows the zero-monotonicity of the game v. □

In the following example, it is made clear that there are situations in which, although the core of the game is nonempty, this does not offer reasonable solutions. However, we can observe that there exist vectors in the Weber set which provide more appropriate distributions.

Example 3.2.1 *Consider $N = \{1,2,3,4\}$ and the convex geometry*

$$\mathcal{L} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, N\}.$$

For the game $v : \mathcal{L} \longrightarrow \mathbb{R}$ defined by

$$v(S) = \begin{cases} |S| - 1 & \text{if } S \neq N, \emptyset \\ 2 & \text{if } S = N, \end{cases}$$

we have that

$$Core(\mathcal{L}, v) = conv\{(1,1,0,0), (2,0,0,0)\}$$

$$Weber\left(\mathcal{L},v\right)=conv\left\{\left(1,1,0,0\right),\left(1,0,0,1\right),\left(1,0,1,0\right),\left(0,1,0,1\right),\left(0,1,1,0\right)\right\}.$$

Note that $Weber\left(\mathcal{L},v\right)\subseteq I\left(\mathcal{L},v\right),$ *and the distribution that provide the unique vector of the core seems less appropriate as a solution that, for example, the vector* $(1,1/2,1/4,1/4)\in Weber\left(\mathcal{L},v\right).$

Weber [67] and Derks [18] proved that if $\mathcal{L}=2^{N}$, the relation $Core\left(\mathcal{L},v\right)\subseteq Weber\left(\mathcal{L},v\right)$ is always verified. However, this inclusion is not true if $\mathcal{L}\neq 2^{N}$. In the following, we prove some examples in which this affirmation is made clear. In these, we consider $N=\{1,2,3\}$ and the convex geometry

$$Co\left(\{1,2,3\}\right)=\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{1,2,3\}\},$$

where its Hasse diagram is

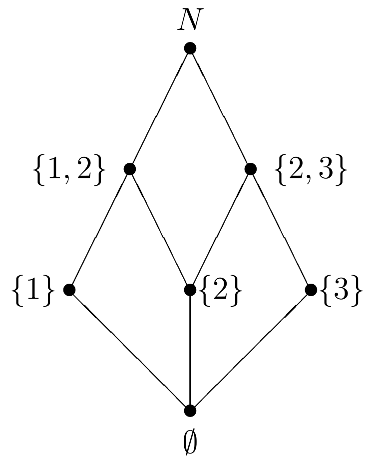

FIGURE 3.4

Note that there are four maximal chains in $\mathcal{L}$,

$$\begin{array}{ll} C_1: & \emptyset \subset \{1\} \subset \{1,2\} \subset \{1,2,3\}, \\ C_2: & \emptyset \subset \{2\} \subset \{1,2\} \subset \{1,2,3\}, \\ C_3: & \emptyset \subset \{2\} \subset \{2,3\} \subset \{1,2,3\}, \\ C_4: & \emptyset \subset \{3\} \subset \{2,3\} \subset \{1,2,3\}. \end{array}$$

Example 3.2.2 *Let $v : \mathcal{L} \longrightarrow \mathbb{R}$ be the game given by*

$$v(S) = \begin{cases} |S| & \text{if } |S| \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

The marginal worth vectors in the game v are

$$a^{C_1}(v) = (v(\{1\}) - v(\emptyset),\, v(\{1,2\}) - v(\{1\}),\, v(N) - v(\{1,2\})) = (0,2,1),$$
$$a^{C_2}(v) = (v(\{1,2\}) - v(\{2\}),\, v(\{2\}) - v(\emptyset),\, v(N) - v(\{1,2\})) = (2,0,1),$$
$$a^{C_3}(v) = (v(N) - v(\{2,3\}),\, v(\{2\}) - v(\emptyset),\, v(\{2,3\}) - v(\{2\})) = (1,0,2),$$
$$a^{C_4}(v) = (v(N) - v(\{2,3\}),\, v(\{2,3\}) - v(\{3\}),\, v(\{3\}) - v(\emptyset)) = (1,2,0),$$

and hence, we get that

$$Weber(\mathcal{L}, v) = \text{conv}\{(0,2,1), (2,0,1), (1,0,2), (1,2,0)\}.$$

On the other hand, the core of the game v is the set

$$\begin{aligned} Core(\mathcal{L}, v) &= \left\{x \in \mathbb{R}^n_+ : x_1 + x_2 + x_3 = 3,\ x_3 \leq 1,\ x_1 \leq 1\right\} \\ &= \text{conv}\{(0,2,1), (1,2,0), (0,3,0), (1,1,1)\}. \end{aligned}$$

In the following picture, the relation between both sets are shown.

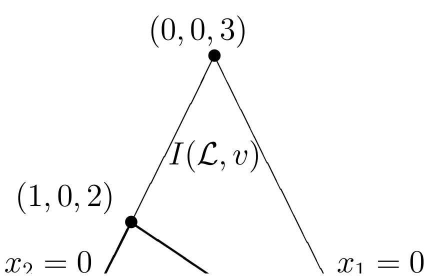

FIGURE 3.5

Note that $Core\,(\mathcal{L},v) \not\subseteq Weber\,(\mathcal{L},v)$ *and* $Weber\,(\mathcal{L},v) \not\subseteq Core\,(\mathcal{L},v)$.

Example 3.2.3 *Let* $v : \mathcal{L} \longrightarrow \mathbb{R}$ *the game given by* $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0,\ v(\{1,2\}) = 1,\ v(\{2,3\}) = 0$ *and* $v(N) = 3$.

The marginal worth vectors for this game v *are*

$$a^{C_1}(v) = (0,1,2)\,,\ a^{C_2}(v) = (1,0,2)\,,\ a^{C_3}(v) = (3,0,0)\,,\ a^{C_4}(v) = (3,0,0)\,,$$

and hence,

$$Weber\,(\mathcal{L},v) = \text{conv}\,\{(0,1,2)\,,(1,0,2)\,,(3,0,0)\}\,.$$

On the other hand, the core of the game is given by

$$\begin{aligned} Core\,(\mathcal{L},v) &= \left\{x \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 = 3,\ x_3 \le 2,\ x_1 \le 3\right\} \\ &= \text{conv}\,\{(0,1,2)\,,(1,0,2)\,,(3,0,0)\,,(0,3,0)\}\,. \end{aligned}$$

and the representation of both sets is

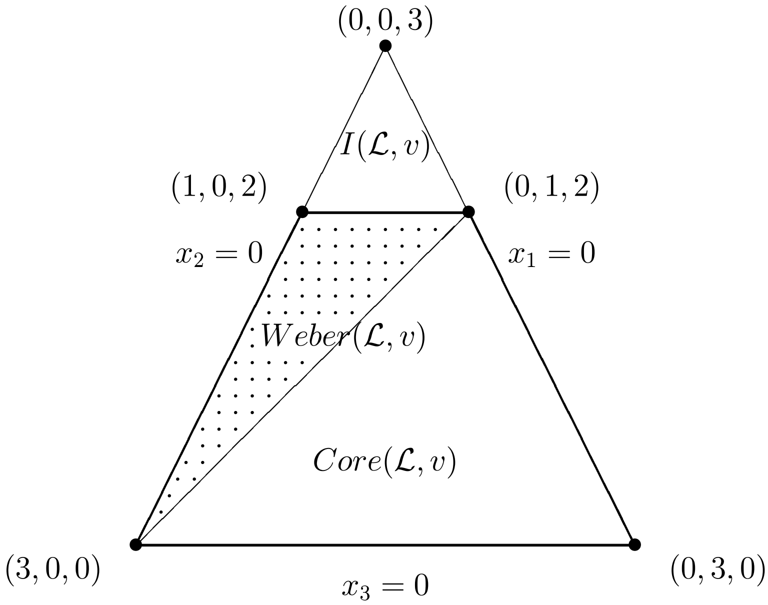

FIGURE 3.6

In this game, all marginal worth vectors coincide with some of the vertices of the core and, therefore, it holds

$$Weber(\mathcal{L}, v) \subset Core(\mathcal{L}, v).$$

As we show in the following, this inclusion is due to a characteristic of the game: its convexity.

3.3 Quasi-supermodular games

In 1971, Shapley [57] proved that for every cooperative game, the core coincides with the Weber set, and in 1981, Ichiishi [35] showed the converse; that is, a cooperative game for which the core coincides with the Weber set is a convex game. Therefore,

$$Core\left(2^N, v\right) = Weber\left(2^N, v\right) \text{ if and only if } v \text{ is a convex game.}$$

In the previous examples, we check that the inclusion

$$Core(\mathcal{L}, v) \subseteq Weber(\mathcal{L}, v)$$

does not hold in general if $\mathcal{L}$ is a convex geometry different to 2^N. Nevertheless, it makes sense to question it if the inverse inclusion is true in the case of supermodular games.

For the class of supermodular games, it is logical to think that the inclusion

$$Weber(\mathcal{L}, v) \subseteq Core(\mathcal{L}, v)$$

is true since we can argue this in the following way: if $v \in \Gamma(\mathcal{L})$ and we consider any extension $\tilde{v} \in \Gamma\left(2^N\right)$ of this game v; that is

$$\tilde{v} : 2^N \longrightarrow \mathbb{R} \quad \text{such that } \tilde{v}(S) = v(S) \text{ for all } S \in \mathcal{L},$$

then we have that

$$Core\left(2^N, \tilde{v}\right) \subseteq Core(\mathcal{L}, v),$$

because the vectors of $Core\left(2^N,\tilde{v}\right)$ verify all restrictions of $Core\left(\mathcal{L},v\right)$ and in general, some others. If we now compare the sets $Weber\left(\mathcal{L},v\right)$ and $Weber\left(2^N,\tilde{v}\right)$, it is clear that

$$Weber\left(\mathcal{L},v\right)\subseteq Weber\left(2^N,\tilde{v}\right).$$

This follows observing that the convex geometry 2^N consists of all maximal chains of $\mathcal{L}$ and, in general, some others. Applying the characterization of the convex cooperative games, if $\tilde{v}\in\Gamma\left(2^N\right)$ i a convex or supermodular game, we have that

$$Weber\left(\mathcal{L},v\right)\subseteq Weber\left(2^N,\tilde{v}\right)=Core\left(2^N,\tilde{v}\right)\subseteq Core\left(\mathcal{L},v\right).$$

In order to formalize this reasoning, we will previously generalize the definition of convex or supermodular game for games defined on convex geometries in a different direction to the aforementioned in the second chapter. Thus, with the idea of studying the relation between the core and the Weber set, we introduce the concept of quasi-supermodular game on a closure space. Later, we show that the two generalizations of the issue of supermodular cooperative game determine different classes of games.

Definition 3.3.1 *Let $\mathcal{L}$ be a closure space. A game $v\in\Gamma\left(\mathcal{L}\right)$ is called quasi-supermodular if, for all $S,T\in\mathcal{L}$ with $S\cup T\in\mathcal{L}$, it holds*

$$v(S\cup T)+v\left(S\cap T\right)\geq v\left(S\right)+v\left(T\right).$$

Similarly, we define the *strict quasi-supermodularity* using the strict inequality, whenever S and T are not comparable in the inclusion sense.

It is evident that every supermodular game is quasi-supermodular. The converse is not true; for example, consider $N=\{1,2,3\}$, $\mathcal{L}=Co\left(\{1,2,3\}\right)$ and the game $v\in\Gamma\left(\mathcal{L}\right)$ defined by $v\left(\{1\}\right)=v\left(\{3\}\right)=1$, $v\left(\{2\}\right)=-1$, $v\left(\{1,2\}\right)=v\left(\{2,3\}\right)=0$ and $v\left(\{1,2,3\}\right)=1$.

This game is quasi-supermodular but not supermodular since for $S=\{1\}$ and $T=\{3\}$, we have that $S\cap T=\emptyset$, $\overline{S\cup T}=N$, and

$$1=v\left(N\right)+v\left(\emptyset\right)<v\left(\{1\}\right)+v\left(\{3\}\right)=2.$$

Note that if $\mathcal{L}$ is a convex geometry with a unique maximal chain, given two coalitions $S, T \in \mathcal{L}$, then or $S \subseteq T$ either $T \subseteq S$ and so, every game defined on $\mathcal{L}$ is supermodular and quasi-supermodular because the equality

$$v(S \cup T) + v(S \cap T) = v(S) + v(T)$$

holds. That is, both notions coincide in this type of convex geometries.

The next proposition characterizes the quasi-supermodular games and will be used in the proofs of the following results.

Proposition 3.3.1 *Let $v \in \Gamma(\mathcal{L})$ be a game on a convex geometry. The game v is quasi-supermodular if and only if for all $S, T \in \mathcal{L}$ such that $T \subseteq S$ and for all $i \in ex(S) \cap T$, it holds*

$$v(S) - v(S \setminus i) \geq v(T) - v(T \setminus i).$$

Proof. *Necessary condition.* Let $S, T \in \mathcal{L}$ such that $T \subseteq S$ and let $i \in ex(S) \cap T$. If $S' = S \setminus i$ and $T' = T$, we have that

$$\begin{aligned} S' \cap T' &= (S \setminus i) \cap T = T \setminus i \in \mathcal{L}, \\ S' \cup T' &= (S \setminus i) \cup T = S \in \mathcal{L}. \end{aligned}$$

and applying the definition of quasi-supermodularity to S' and T', it follows

$$v(S) + v(T \setminus i) \geq v(S \setminus i) + v(T).$$

Sufficient condition. Let $S, T \in \mathcal{L}$ such that $S \cup T \in \mathcal{L}$. If $T \subseteq S$ or $S \subseteq T$, the equality trivially holds. So, we consider the case $S \cap T \neq S$ and $S \cap T \neq T$.

Let $C \in \mathcal{C}(\mathcal{L})$ be a maximal chain that contains the coalitions T and $S \cup T$. As $S \setminus T \neq \emptyset$, we suppose that $|S \setminus T| = k$ and so, we write $S \setminus T = \{i_1, i_2, \ldots, i_k\}$, where the players are ordered in the order of incorporation in the chain C, i.e.,

$$C(i_1) \subset C(i_2) \subset \cdots \subset C(i_k).$$

Then, the chain C is given by

$$\emptyset \subset \cdots \subset T \subset T \cup \{i_1\} \subset \cdots \subset T \cup \{i_1, i_2, \ldots, i_k\} = T \cup S \subset \cdots \subset N.$$

We denote $S_j = \{i_1, i_2, \ldots, i_j\}$, for all $1 \le j \le k$, and $S_0 = \emptyset$. Then, for all $1 \le j \le k$, we have that $T \cup S_j \in \mathcal{L}$. Furthermore, if $R = S \cap T$, then $R \cup S_j \in \mathcal{L}$ because it is an intersection of two coalitions of $\mathcal{L}$ (see figure 3.7). Indeed, $R \cup S_j = (S \cap T) \cup S_j = (S \cup S_j) \cap (T \cup S_j) = S \cap (T \cup S_j)$.

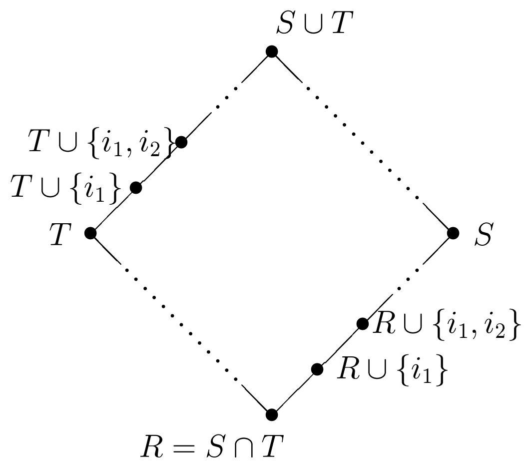

FIGURE 3.7

Applying the hypothesis to the convex sets $T \cup S_j$ and $R \cup S_j$, we get

$$v(R \cup S_j) - v(R \cup S_{j-1}) \le v(T \cup S_j) - v(T \cup S_{j-1}),$$

since $R \cup S_j \subset T \cup S_j$ and also $i_j \in ex(R \cup S_j) \cap (T \cup S_j)$. Hence,

$$\begin{aligned} v(S) - v(S \cap T) &= v(R \cup S_k) - v(R) \\ &= \sum_{j=1}^{k} [v(R \cup S_j) - v(R \cup S_{j-1})] \\ &\le \sum_{j=1}^{k} [v(T \cup S_j) - v(T \cup S_{j-1})] \end{aligned}$$

$$= v(T \cup S) - v(T).$$

□

The following result permits the identification of the games for which the marginal worth vectors are distributions of the core.

Theorem 3.3.2 *Let $\mathcal{L}$ be a convex geometry. A necessary and sufficient condition so that all marginal worth vectors of a game $v \in \Gamma(\mathcal{L})$ are vectors of the core is that the game v is quasi-supermodular.*

Proof. *Sufficient condition.* Let C be a maximal chain in $\mathcal{L}$. We know that $a^C(v)$ is efficient and for every coalition S in the chain C, we have that

$$\sum_{j \in S} a_j^C(v) = v(S).$$

In order to prove that $a^C(v) \in Core(\mathcal{L}, v)$, we prove that, for every coalition S, not in the chain C,

$$\sum_{j \in S} a_j^C(v) \geq v(S).$$

Indeed, let $S \in \mathcal{L}$ be a coalition that does not belong to C, and assume that $|S| = s$. We denote $S = \{i_1, i_2, \ldots, i_s\}$ where the elements are written following the order of incorporation in the chain C; that is,

$$C(i_1) \subset C(i_2) \subset \cdots \subset C(i_s).$$

If we denote by S_j the set $\{i_1, i_2, \ldots, i_j\}$, for all $1 \leq j \leq s$, and $S_0 = \emptyset$, note that, for all $1 \leq j \leq s$, we have that $S_j \in \mathcal{L}$ because $S_j = S \cap C(i_j)$. Moreover, $i_j \in ex(C(i_j))$, and also $i_j \in S_j$. As v is a quasi-supermodular game, the Proposition 3.19 implies that, for all $1 \leq j \leq s$,

$$v\left(C\left(i_j\right)\right) - v\left(C\left(i_j\right) \setminus i_j\right) \geq v\left(S_j\right) - v\left(S_{j-1}\right),$$

and we obtain

$$\begin{aligned}
\sum_{j \in S} a_j^C(v) &= \sum_{j=1}^{s} a_{i_j}^C(v) \\
&= \sum_{j=1}^{s} \left[v\left(C\left(i_j\right)\right) - v\left(C\left(i_j\right) \setminus i_j\right)\right] \\
&\geq \sum_{j=1}^{s} \left[v\left(S_j\right) - v\left(S_{j-1}\right)\right] \\
&= v(S).
\end{aligned}$$

Necessary condition. For all $S, T \in \mathcal{L}$ with $S \cup T \in \mathcal{L}$, consider a maximal chain $C \in \mathcal{C}(\mathcal{L})$ which contains $S \cap T$ and $S \cup T$. As the marginal worth vectors are elements of $Core(\mathcal{L}, v)$, we have that

$$\sum_{j \in S} a_j^C(v) \geq v(S) \qquad \text{and} \qquad \sum_{j \in T} a_j^C(v) \geq v(T).$$

By the election of the maximal chain C, it is also verified

$$\sum_{j \in S \cup T} a_j^C(v) = v(S \cup T) \qquad \text{and} \qquad \sum_{j \in S \cap T} a_j^C(v) = v(S \cap T).$$

Therefore,

$$\begin{aligned}
v(S) + v(T) &\leq \sum_{j \in S} a_j^C(v) + \sum_{j \in T} a_j^C(v) \\
&= \sum_{j \in S \cup T} a_j^C(v) + \sum_{j \in S \cap T} a_j^C(v) \\
&= v(S \cup T) + v(S \cap T).
\end{aligned}$$

□

As the core of the game $v \in \Gamma(\mathcal{L})$ is a convex set, an immediate consequence of this theorem is the following.

Corollary 3.3.3 *Let $v \in \Gamma(\mathcal{L})$ be a game on a convex geometry. A necessary and sufficient condition so that $Weber(\mathcal{L}, v) \subseteq Core(\mathcal{L}, v)$ is that the game v is quasi-supermodular.*

We will show that the class of quasi-supermodular games is identified with the class of supermodular games inside the set of the monotonic games.

Theorem 3.3.4 *Let $v \in \Gamma(\mathcal{L})$ be a monotonic game on a convex geometry. Then, v is a supermodular game if and only if v is a quasi-supermodular game.*

Proof. It is sufficient to prove that every monotonic quasi-supermodular game is supermodular. Let $S, T \in \mathcal{L}$, and let $C \in \mathcal{C}(\mathcal{L})$ be a maximal chain that contains $\overline{S \cup T}$ and $S \cap T$. By the Theorem 3.20, the marginal worth vector associated to this chain, $a^C(v)$, is a vector which belongs to $Core(\mathcal{L}, v)$ and also $a_i^C(v) = v(C(i)) - v(C(i) \setminus i) \geq 0$ for all $i \in N$, because the game v is monotonic. Furthermore,

$$\begin{aligned}
\sum_{j \in S} a_j^C(v) &\geq v(S), \\
\sum_{j \in T} a_j^C(v) &\geq v(T), \\
\sum_{j \in \overline{S \cup T}} a_j^C(v) &= v\left(\overline{S \cup T}\right), \\
\sum_{j \in S \cap T} a_j^C(v) &= v(S \cap T).
\end{aligned}$$

Taking into account these relations, we get

$$\begin{aligned}
v(S) + v(T) &\leq \sum_{j \in S} a_j^C(v) + \sum_{j \in T} a_j^C(v) \\
&= \sum_{j \in S \cup T} a_j^C(v) + \sum_{j \in S \cap T} a_j^C(v) \\
&\leq \sum_{j \in \overline{S \cup T}} a_j^C(v) + \sum_{j \in S \cap T} a_j^C(v) \\
&= v\left(\overline{S \cup T}\right) + v(S \cap T).
\end{aligned}$$

□

The following example shows that there are quasi-supermodular games in which the core and the Weber set coincide.

Example 3.3.1 *Let* $v : \mathcal{L} \longrightarrow \mathbb{R}$ *the game defined by* $v(\{1\}) = v(\{3\}) = 1$, $v(\{2\}) = -1$, $v(\{1,2\}) = v(\{2,3\}) = 0$ *and* $v(N) = 1$.

This game is quasi-supermodular and the marginal worth vectors for v *are*

$$a^{C_1}(v) = a^{C_2}(v) = a^{C_3}(v) = a^{C_4}(v) = (1, -1, 1),$$

and hence $Weber(\mathcal{L}, v) = \{(1, -1, 1)\}$. *Moreover,* $Core(\mathcal{L}, v) = \{(1, -1, 1)\}$.

Proposition 3.3.5 *Let* $v \in \Gamma(\mathcal{L})$ *be a game on an atomic convex geometry. Then,* $Weber(\mathcal{L}, v) = \{(v(\{1\}), \ldots, v(\{n\}))\}$ *if and only if* $v(S) = \sum_{i \in S} v(\{i\})$, *for all* $S \in \mathcal{L}$.

Proof. If $v(S) = \sum_{i \in S} v(\{i\})$, for all $S \in \mathcal{L}$, all marginal worth vectors coincide and $a^C(v) = (v(\{1\}), \ldots, v(\{n\}))$ for all $C \in \mathcal{C}(\mathcal{L})$.

Conversely, we prove that $Weber(\mathcal{L}, v) = \{(v(\{1\}), \ldots, v(\{n\}))\}$ implies $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \in \mathcal{L}$, by induction on cardinal of S.

For $S \in \mathcal{L}$ with $|S| = 2$, we have that $S = \{i, j\}$ with $\{i\}, \{j\} \in \mathcal{L}$. Then, if C is a maximal chain that contains S, and $S = C(j)$, we have that $a_j^C(v) = v(C(j)) - v(C(j) \setminus j) = v(\{j\})$ and so, $v(S) = v(\{i\}) + v(\{j\})$.

Assume that the equality is true for all $T \in \mathcal{L}$ with $|T| = r$. For every $S \in \mathcal{L}$ with $|S| = r + 1$, there exists a coalition $T \in \mathcal{L}$ with $|T| = r$ such that $S = T \cup \{k\}$ with $k \notin T$. For a maximal chain C that contains S and T, we get $a_k^C(v) = v(S) - v(T)$. Also $a_k^C(v) = v(\{k\})$ for all $C \in \mathcal{C}(\mathcal{L})$. Therefore, $v(S) = v(T) + v(\{k\})$, and the hypothesis of induction implies that

$$v(T) = \sum_{i \in T} v(\{i\}).$$

Thus, we obtain that

$$v(S) = \sum_{i \in S} v(\{i\}).$$

□

Note that every game that verifies $v(S) = \sum_{i \in S} v(\{i\})$, for all $S \in \mathcal{L}$, is a quasi-supermodular game. So, we can affirm that these games are the unique games on atomic convex geometries for which the imputation set, core and Weber set coincide and they have as a unique distribution the vector $(v(\{1\}), \ldots, v(\{n\}))$.

As we have just proved, the marginal worth vectors associated to two or more maximal chains could coincide. The strict quasi-supermodularity of the game secures that all marginal worth vectors are different.

Proposition 3.3.6 *If $v \in \Gamma(\mathcal{L})$ is a strict quasi-supermodular game on a convex geometry, then $a^{C_1}(v) \neq a^{C_2}(v)$, for all $C_1, C_2 \in \mathcal{C}(\mathcal{L})$.*

Proof. Let $C_1, C_2 \in \mathcal{C}(\mathcal{L})$ be two maximal chains of the convex geometry $\mathcal{L}$, for example,

$$\begin{aligned} C_1 &: \emptyset \subset \{i_1\} \subset \{i_1, i_2\} \subset \ldots \subset \{i_1, \ldots, i_k\} \subset \ldots \subset N \\ C_2 &: \emptyset \subset \{j_1\} \subset \{j_1, j_2\} \subset \ldots \subset \{j_1, \ldots, j_k\} \subset \ldots \subset N \end{aligned}$$

For all k, $1 \leq k \leq n$, we denote $S_k = \{i_1, \ldots, i_k\}$ and $T_k = \{j_1, \ldots, j_k\}$ the convex sets of cardinal k in the maximal chains C_1 and C_2, respectively. That is, $S_k = C_1(i_k)$ and $T_k = C_2(j_k)$.

Let p be the lowest index such that $S_p \neq T_p$ and let $l > p$ be the lowest index such that $S_l = T_l$. As $S_l = C_1(i_l)$, $T_l = C_2(j_l)$ and $S_{l-1} \neq T_{l-1}$, then $i_l \neq j_l$. Hence $C_1(j_l) \subset C_2(j_l)$.

If we consider the marginal worth vectors associated to the chains C_1 and C_2 , by the strict quasi-supermodularity of v, we get

$$a_{j_l}^{C_2}(v) = v(C_2(j_j)) - v(C_2(j_{l-1})) > v(C_1(j_l)) - v(C_1(j_l) \setminus j_l) = a_{j_l}^{C_1}(v),$$

Thus, the marginal worth vectors are different as they have at least one distinct component.

□

3.4 Selections and marginal worth vectors

Remember that, in the previous chapter, we introduced a solution concept for games defined on atomic families, the selectope, and we established conditions so that the core was one subset of this. Now, with the objective of establishing the relation between the selectope and the Weber set, we consider, in this section, an atomic convex geometry $\mathcal{L}$. For every game $v \in \Gamma(\mathcal{L})$, we show a relation between the selections corresponding to the selectors on $\mathcal{L}$ and the marginal worth vectors associated to the maximal chains in the geometry $\mathcal{L}$.

Proposition 3.4.1 *Let $C \in \mathcal{C}(\mathcal{L})$ be a maximal chain of the atomic convex geometry $\mathcal{L}$ and let $\alpha : \mathcal{L} \setminus \{\emptyset\} \longrightarrow N$ defined by*

$$\alpha(S) = j, \quad \textit{where } j \in S \textit{ and } S \subseteq C(j).$$

Then α is a selector and $m^{\alpha}(v) = a^{C}(v)$, for every $v \in \Gamma(\mathcal{L})$.

Proof. First of all, we prove that α is well-defined. Indeed, for every nonempty coalition $S \in \mathcal{L}$, it is clear that there exists a unique $j \in S$ such that $S \subseteq C(j)$ where j is the last element of S that is incorporated in the maximal chain $C \in \mathcal{C}(\mathcal{L})$, i.e., this element $j \in S$ is such that $C(k) \subseteq C(j)$ for all $k \in S$. Moreover, $j \in ex(S)$ since

$$\left.\begin{array}{l} C(j) \setminus j \in \mathcal{L} \\ S \in \mathcal{L} \end{array}\right\} \Longrightarrow (C(j) \setminus j) \cap S = S \setminus j \in \mathcal{L}.$$

Therefore, α is well-defined and it is the selector that associates, to every nonempty coalition $S \in \mathcal{L}$, the element which is the last to be incorporated in the maximal chain C.

Now, for all $i \in N$ and for every $v \in \Gamma(\mathcal{L})$, we get

$$\begin{aligned} a_i^C(v) &= v(C(i)) - v(C(i) \setminus i) \\ &= \sum_{\{T \in \mathcal{L}, T \subseteq C(i)\}} \triangle_v(T) - \sum_{\{T \in \mathcal{L}, T \subseteq C(i) \setminus i\}} \triangle_v(T) \\ &= \sum_{\{T \in \mathcal{L}, T \subseteq C(i), i \in T\}} \triangle_v(T) \\ &= \sum_{\{T \in \mathcal{L}, T \subseteq C(i), i \in ex(T)\}} \triangle_v(T) \\ &= \sum_{\{T \in \mathcal{L} : \alpha(T) = i\}} \triangle_v(T) \\ &= m_i^{\alpha}(v). \end{aligned}$$

□

As the selectope for a game $v \in \Gamma(\mathcal{L})$ is a convex set, one immediate consequence of this proposition is the following result.

Theorem 3.4.2 *If $v \in \Gamma(\mathcal{L})$ is a game on an atomic convex geometry, then*

$$Weber(\mathcal{L}, v) \subseteq Sel(\mathcal{L}, v).$$

In order to secure the converse of the above proposition, we have to establish a condition on the selector.

Definition 3.4.1 *A selector $\alpha \in \mathcal{A}(\mathcal{L})$ is called consistent if it satisfies the following conditions:*

(1) For all $S \in \mathcal{L} \setminus \{\emptyset\}$, $\alpha(S) \in ex(S)$.

(2) If $S \subset T$ and $\alpha(T) \in ex(S)$, then $\alpha(S) = \alpha(T)$.

Note that if we take a maximal chain $C \in \mathcal{C}(\mathcal{L})$ and we can define the selector α as in the above proposition, it is easy to check that α is consistent. Moreover, different chains correspond to different selectors.

Theorem 3.4.3 *Let $\mathcal{L}$ be an atomic convex geometry and let $\alpha \in \mathcal{A}(\mathcal{L})$. Then, there exists a maximal chain $C \in \mathcal{C}(\mathcal{L})$ such that $m^{\alpha}(v) = a^{C}(v)$ for every $v \in \Gamma(\mathcal{L})$ if and only if α is consistent. Moreover, the maximal chain*

$$C : \emptyset \subset \{i_1\} \subset \{i_1, i_2\} \subset \cdots \subset \{i_1, \ldots, i_{n-1}\} \subset \{i_1, \ldots, i_n\} = N$$

is unique and it is recursively defined as

$$\begin{aligned} &i_n = \alpha(N) \\ &i_k = \alpha(N \setminus \{i_n, i_{n-1}, \ldots i_{k+1}\}) \quad \text{for all } 1 \leq k < n. \end{aligned}$$

Proof. First of all, note that for every unanimity game $\zeta_S \in \Gamma(\mathcal{L})$, with $S \in \mathcal{L} \setminus \{\emptyset\}$, we have that

$$\begin{aligned} &\triangle_{\zeta_S}(S) = 1, \\ &\triangle_{\zeta_S}(R) = 0 \quad \text{for } R \in \mathcal{L}, R \neq S, \end{aligned}$$

because $\{\zeta_T : T \in \mathcal{L}, T \neq \emptyset\}$ is a basis of $\Gamma(\mathcal{L})$ and, for every $v \in \Gamma(\mathcal{L})$,

$$v = \sum_{\{T \in \mathcal{L}, T \neq \emptyset\}} \triangle_v(T)\, \zeta_T.$$

Sufficient condition. Let $\alpha \in \mathcal{A}(\mathcal{L})$ be a selector such that there exists a maximal chain $C \in \mathcal{C}(\mathcal{L})$ satisfying $m^\alpha(v) = a^C(v)$, for every $v \in \Gamma(\mathcal{L})$. We show that α is consistent. In order to prove the first condition of consistency; that is, for all $T \in \mathcal{L}$, $T \neq \emptyset$ it holds $\alpha(T) \in ex(T)$, consider the unanimity game $\zeta_T \in \Gamma(\mathcal{L})$. If $\alpha(T) = i$, we have that

$$m_i^\alpha(\zeta_T) = \sum_{\{R \in \mathcal{L} : \alpha(R) = i\}} \triangle_{\zeta_T}(R) = 1$$

and as, by hypothesis, $m_i^\alpha(\zeta_T) = a_i^C(\zeta_T)$, then

$$\zeta_T(C(i)) - \zeta_T(C(i) \setminus i) = 1.$$

Hence, $T \subseteq C(i)$ and $T \not\subseteq C(i) \setminus i$. Thus, $\alpha(T) \in T$ and $T \subseteq C(\alpha(T))$, and therefore, $\alpha(T) \in ex(T)$ since $\alpha(T) \in ex(C(\alpha(T)))$.

Now, we prove the second condition of consistency. Let $S \subset T$ and $\alpha(T) = i \in ex(S)$, then, $S \subset C(i)$ because $T \subseteq C(i)$ and so,

$$a_i^C(\zeta_S) = \zeta_S(C(i)) - \zeta_S(C(i) \setminus i) = 1,$$

then, by hypothesis

$$m_i^\alpha(\zeta_S) = \sum_{\{R \in \mathcal{L} : \alpha(R) = i\}} \triangle_{\zeta_S}(R) = 1.$$

For this, it must be verified that $i = \alpha(S)$, and therefore, α is consistent.

Necessary condition. Let $\alpha \in \mathcal{A}(\mathcal{L})$ be a consistent selector. First of all, we prove that there exists a maximal chain $C \in \mathcal{C}(\mathcal{L})$ for which $a^C(v) =$

$m^{\alpha}(v)$ for every $v \in \Gamma(\mathcal{L})$. Consider the maximal chain C in the following way

$$C : \emptyset \subset \{i_1\} \subset \{i_1, i_2\} \subset \cdots \subset \{i_1, \ldots, i_{n-1}\} \subset \{i_1, \ldots, i_n\} = N,$$

that is,

$$C : \emptyset \subset C(i_1) \subset C(i_2) \subset \cdots \subset C(i_{n-1}) \subset C(i_n) = N,$$

where

$$\begin{array}{l} i_n = \alpha(N) \\ i_k = \alpha(N \setminus \{i_n, i_{n-1}, \ldots i_{k+1}\}) \quad \text{for all } 1 \leq k < n. \end{array}$$

Note that $N, N \setminus \{i_n\}, \ldots, N \setminus \{i_n, i_{n-1}, \ldots i_{k+1}\} \in \mathcal{L}$ because the selector α is consistent.

Moreover, the selector α coincides with the vector defined in the above proposition for this chain, i.e., if $\beta \in \mathcal{A}(\mathcal{L})$ is a selector such that, for all $S \in \mathcal{L}, S \neq \emptyset$, is defined by

$$\beta(S) = j, \quad \text{where } j \in S \text{ and } S \subseteq C(j),$$

then $\alpha = \beta$. Indeed, it is evident that $\alpha(S) = \beta(S)$ for every nonempty coalition S in the chain C. If S is not in the chain C, then $\beta(S) = i \in ex(S)$ and $S \subset C(i)$ where $C(i)$ is the smallest convex set in the chain that contains i. Since $S \subset C(i)$, $\alpha(C(i)) = i \in ex(S)$ and because the selector α is consistent it follows that $\alpha(S) = i$. Therefore, $\beta(S) = \alpha(S)$ for every nonempty coalition $S \in \mathcal{L}$. Applying the above proposition to the selector α, we have that $a^C(v) = m^{\alpha}(v)$ for every $v \in \Gamma(\mathcal{L})$.

Finally, we prove that this chain is the unique chain for which the equality $a^C(v) = m^{\alpha}(v)$ is satisfied for every $v \in \Gamma(\mathcal{L})$. Consider the unanimity game $\zeta_N \in \Gamma(\mathcal{L})$. For each $j \in N$, we have that

$$m_j^{\alpha}(\zeta_N) = \sum_{\{R \in \mathcal{L} : \alpha(R) = j\}} \triangle_{\zeta_N}(R) = \begin{cases} 1 & \text{if } \alpha(N) = j, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$a_j^C(\zeta_N) = \zeta_N(C(j)) - \zeta_N(C(j) \setminus j) = \begin{cases} 1 & \text{if } C(j) = N, \\ 0 & \text{otherwise,} \end{cases}$$

Then, as $\alpha(N) = i_n$, both vectors coincide if and only if $C(i_n) = N$.

We now consider $\zeta_{N\setminus\{i_n\}} \in \Gamma(\mathcal{L})$. For every $j \in N$, we have that

$$m_j^{\alpha}\left(\zeta_{N\setminus\{i_n\}}\right) = \sum_{\{R\in\mathcal{L}:\alpha(R)=j\}} \triangle_{\zeta_{N\setminus\{i_n\}}}(R) = \begin{cases} 1 & \text{if } \alpha(N\setminus\{i_n\}) = j, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} a_j^{C}\left(\zeta_{N\setminus\{i_n\}}\right) &= \zeta_{N\setminus\{i_n\}}(C(j)) - \zeta_{N\setminus\{i_n\}}(C(j)\setminus j) \\ &= \begin{cases} 1 & \text{if } C(j) \supseteq N\setminus\{i_n\} \text{ and } C(j)\setminus j \not\supseteq N\setminus\{i_n\}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Note that if $C(j) \supseteq N\setminus\{i_n\}$ and $C(j)\setminus j \not\supseteq N\setminus\{i_n\}$, then $C(j)$ is a convex set where there must be n or $n-1$ elements, but in every maximal chain there is only one convex set with k elements, for all $1 \leq k \leq n$, and the convex set of n elements is excluded. Therefore,

$$a_j^{C}\left(\zeta_{N\setminus\{i_n\}}\right) = \begin{cases} 1 & \text{if } C(j) = N\setminus\{i_n\}, \\ 0 & \text{otherwise,} \end{cases}$$

then, since $\alpha(N\setminus\{i_n\}) = i_{n-1}$, it holds that $a^{C}\left(\zeta_{N\setminus\{i_n\}}\right) = m^{\alpha}\left(\zeta_{N\setminus\{i_n\}}\right)$ if and only if $C(i_{n-1}) = N\setminus\{i_n\}$. By repeating this argument, the maximal chain C is obtained. □

From the previous results, there is one to one correspondence between maximal chains of the convex geometry $\mathcal{L}$ and the consistent selectors on $\mathcal{L}$.

Example 3.4.1 *Consider the game defined in the example 3.16 where we studied the relation between the core and the Weber set. The selections cor-*

responding to this game are

α	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$	$m_1^\alpha(v)$	$m_2^\alpha(v)$	$m_3^\alpha(v)$
1	1	2	1	1	2	0
2	1	2	2	2	1	0
3	1	2	3	2	2	−1
4	1	3	1	1	0	2
5	1	3	2	2	−1	2
6	1	3	3	2	0	1
7	2	2	1	−1	4	0
8	2	2	2	0	3	0
9	2	2	3	0	4	−1
10	2	3	1	−1	2	2
11	2	3	2	0	1	2
12	2	3	3	0	2	1

TABLE 3.1

Note that the selectors 1,4,6,12 are consistent and the selections corresponding to these selectors coincide with the marginal worth vectors associated to the four maximal chains of $\mathcal{L}$. In the next figure, we show the position of the Weber set and the selectope for this game.

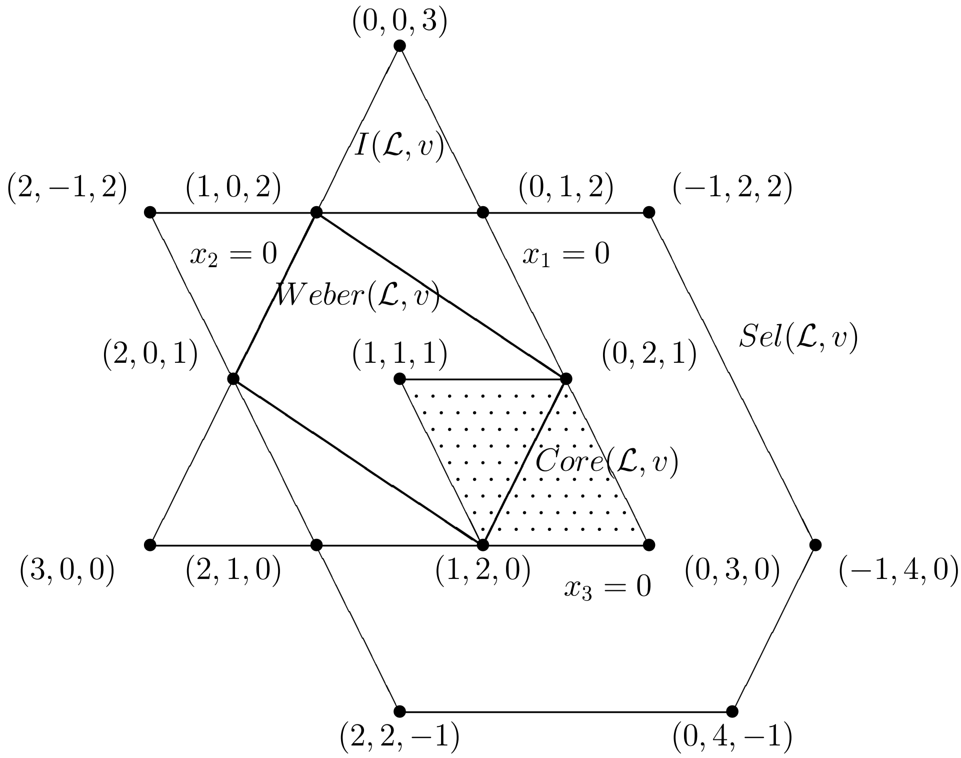

FIGURE 3.8

In this example, we can observe that the inclusion of the Weber into the selectope is a strict inclusion

$$Weber(\mathcal{L}, v) \neq Sel(\mathcal{L}, v).$$

In the following result, we show a sufficient and necessary condition so that the equality holds and both sets contain a unique distribution.

Proposition 3.4.4 *Let $v \in \Gamma(\mathcal{L})$ be a game on an atomic convex geometry. Then, $Sel(\mathcal{L}, v) = \{(v(\{1\}), \ldots, v(\{n\}))\}$ if and only if $v(S) = \sum_{i \in S} v(\{i\})$, for every $S \in \mathcal{L}$.*

Proof. If the game v verifies $v(S) = \sum_{i \in S} v(\{i\})$ for every $S \in \mathcal{L}$, then $\triangle_v(\{i\}) = v(\{i\})$ and $\triangle_v(S) = 0$, for each $S \in \mathcal{L}$, $S \neq \{i\}$ for all $i \in N$. Therefore, $Sel(\mathcal{L}, v) = \{(v(\{1\}), \ldots, v(\{n\}))\}$.

Conversely, if we suppose that $Sel(\mathcal{L}, v) = \{(v(\{1\}), \ldots, v(\{n\}))\}$, by Theorem 3.27, $Weber(\mathcal{L}, v) = \{(v(\{1\}), \ldots, v(\{n\}))\}$ and by the Proposition 3.24, the game v satisfies $v(S) = \sum_{i \in S} v(\{i\})$ for every $S \in \mathcal{L}$. □

In the preceding chapter, we showed that the almost positive games are the unique games for which $Sel(\mathcal{L}, v) \subseteq Core(\mathcal{L}, v)$. Thus, as always $Weber(\mathcal{L}, v) \subseteq Sel(\mathcal{L}, v)$, it is obvious that for the almost positive games on atomic convex geometries, the Weber set is a subset of the core and so, these games are quasi-supermodular games. This is true, although the family $\mathcal{L}$ is not a convex geometry.

In the remainder of this chapter, we relate the almost positive games and the quasi-supermodular games.

Proposition 3.4.5 *Let $\mathcal{L}$ be an atomic closure space and let $v \in \Gamma(\mathcal{L})$. If the game v is almost positive, then v is quasi-supermodular.*

Proof. Let $v \in \Gamma(\mathcal{L})$ be an almost positive game. We distinguish two cases:

a) Let $A, B \in \mathcal{L}$ be two nonempty coalitions such that $A \cup B \in \mathcal{L}$ and $A \cap B = \emptyset$. Then, $v(A \cup B) - v(A) - v(B)$ is equal to

$$\sum_{\{T \in \mathcal{L}, T \subseteq A \cup B\}} \triangle_v(T) - \sum_{\{T \in \mathcal{L}, T \subseteq A\}} \triangle_v(T) - \sum_{\{T \in \mathcal{L}, T \subseteq B\}} \triangle_v(T)$$

and simplifying, we obtain

$$\sum_{\{T\in\mathcal{L},T\subseteq A\cup B,T\not\subseteq A,T\not\subseteq B\}} \triangle_v(T) \geq 0,$$

since all coalitions $T \in \mathcal{L}$ such that $T \subseteq A \cup B$, $T \not\subseteq A$, $T \not\subseteq B$, verify that $|T| \geq 2$ and so, $\triangle_v(T) \geq 0$.

b) Let $A, B \in \mathcal{L}$ be two nonempty coalitions such that $A \cup B \in \mathcal{L}$, $A \cap B \neq \emptyset$, $A \not\subseteq B$ and $B \not\subseteq A$, then $v(A \cup B) + v(A \cap B) - v(A) - v(B)$ is equal to

$$\sum_{\{T\in\mathcal{L},T\subseteq A\cup B\}} \triangle_v(T) + \sum_{\{T\in\mathcal{L},T\subseteq A\cap B\}} \triangle_v(T) - \sum_{\{T\in\mathcal{L},T\subseteq A\}} \triangle_v(T) - \sum_{\{T\in\mathcal{L},T\subseteq B\}} \triangle_v(T).$$

Note that the collections $\{T \in \mathcal{L} : T \subseteq A \cup B\}$ and $\{T \in \mathcal{L} : T \subseteq A \cap B\}$, contain all coalitions of $\{T \in \mathcal{L} : T \subseteq A\}$ and $\{T \in \mathcal{L} : T \subseteq B\}$, and so, the dividends are simplified obtaining in this expression only those coalitions $T \in \mathcal{L}$ such that $T \subseteq A \cup B$ and containing at least two players i, j such that $i \in A \setminus B$ and $j \in B \setminus A$. Hence, $|T| \geq 2$, and so, all dividends are nonnegative. Therefore,

$$v(A \cup B) + v(A \cap B) - v(A) - v(B) \geq 0.$$

In the remainder of the cases, it is evident that the game v is quasi-supermodular. □

The converse, in general, is not true. There are quasi-supermodular games for which there exist coalitions where their cardinals are greater or equal to 2 and their dividends are negative. Take the game $v \in \Gamma(\mathcal{L})$, where $\mathcal{L} = 2^{\{1,2,3\}}$ and the characteristic function is given by $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1,2\}) = v(\{1,3\}) = 1$, $v(\{2,3\}) = 5$ and $v(\{1,2,3\}) = 6$. This game is quasi-supermodular but however $\triangle_v(N) = -1$.

Using the expression of the dividends for the coalitions of $\mathcal{L}$, we can establish conditions on $\mathcal{L}$ which secure that every quasi-supermodular game is almost positive.

Proposition 3.4.6 *Let $\mathcal{L}$ be a convex geometry and let $v \in \Gamma(\mathcal{L})$. For each $S \in \mathcal{L}$, it holds*

$$\triangle_v(S) = \sum_{\{T : S \setminus ex(S) \subseteq T \subseteq S\}} (-1)^{|S|-|T|} v(T).$$

Proof. As the collection of unanimity games $\{\zeta_T : T \in \mathcal{L}, T \neq \emptyset\}$ is a basis of $\Gamma(\mathcal{L})$, we know that

$$v = \sum_{\{T \in \mathcal{L}, T \neq \emptyset\}} \triangle_v(T) \zeta_T,$$

and, for every $S \in \mathcal{L}$, we have

$$v(S) = \sum_{\{T \in \mathcal{L}, T \subseteq S\}} \triangle_v(T).$$

Since $(\mathcal{L}, \subseteq)$ is a finite partially ordered set, and considering the functions $v : \mathcal{L} \longrightarrow \mathbb{R}$ and $\triangle_v : \mathcal{L} \longrightarrow \mathbb{R}$, we can apply the Möbius inversion formula. Then, for all $S \in \mathcal{L}$,

$$v(S) = \sum_{\{T \in \mathcal{L}, T \subseteq S\}} \triangle_v(T) \Longleftrightarrow \triangle_v(S) = \sum_{\{T \in \mathcal{L}, T \subseteq S\}} \mu(T,S) v(T).$$

If $\mathcal{L}$ is a convex geometry, the Möbius function [21, Theorem 4.3] is given by

$$\mu(T,S) = \begin{cases} (-1)^{|S|-|T|} & \text{if } S \setminus T \subseteq ex(S), \\ 0 & \text{otherwise.} \end{cases}$$

If $T \subseteq S$, then $S \setminus T \subseteq ex(S)$ if and only if $S \setminus ex(S) \subseteq T$. Therefore,

$$\triangle_v(S) = \sum_{\{T \in \mathcal{L} : S \setminus ex(S) \subseteq T \subseteq S\}} (-1)^{|S|-|T|} v(T).$$

□

Proposition 3.4.7 *Let $\mathcal{L}$ be an atomic convex geometry such that $|ex(S)| = 2$, for every coalition $S \in \mathcal{L}$ with $|S| \geq 2$. Then, every game $v \in \Gamma(\mathcal{L})$ quasi-supermodular is almost positive.*

Proof. Let $v \in \Gamma(\mathcal{L})$ be a quasi-supermodular game and let $S \in \mathcal{L}$ with $|S| \geq 2$. If $ex(S) = \{i_1, i_2\}$, then

$$\begin{aligned} \triangle_v(S) &= \sum_{\{T: S \setminus \{i_1, i_2\} \subseteq T \subseteq S\}} (-1)^{|S|-|T|} v(T) \\ &= v(S) - v(S \setminus \{i_1\}) - v(S \setminus \{i_2\}) + v(S \setminus \{i_1, i_2\}) \geq 0 \end{aligned}$$

because the game v is quasi-supermodular. □

As one immediate consequence of this proposition, Theorem 2.22, Theorem 2.17 and Proposition 3.32, we deduce that for those convex geometries in which any non-unitary coalition have two extreme points and are also atomic intersecting families, the unique games in which the selectope coincides with the core, are the quasi-supermodular games. In the next example this observation is made clear.

Example 3.4.2 *Let* $N = \{1, 2, 3\}$ *and consider the convex geometry*

$$Co(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}.$$

Note that $|ex(S)| = 2$, *for all* $S \in \mathcal{L}$ *with* $|S| \geq 2$. *Moreover,* $\mathcal{L}$ *is an intersecting family. Let* $v \in \Gamma(\mathcal{L})$ *be a quasi-supermodular game define by* $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1,2\}) = 1$, $v(\{2,3\}) = 0$ *and* $v(\{1,2,3\}) = 3$.

The selections for this game are given in the following table:

α	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$	$m_1^\alpha(v)$	$m_2^\alpha(v)$	$m_3^\alpha(v)$
1	1	2	3	1	0	2
2	2	2	3	0	1	2
3	1	3	3	1	0	2
4	2	3	3	0	1	2
5	1	2	1	3	0	0
6	2	2	1	2	1	0
7	1	3	1	3	0	0
8	2	3	1	2	1	0
9	1	2	2	1	2	0
10	2	2	2	0	3	0
11	1	3	2	1	2	0
12	2	3	2	0	3	0

TABLE 3.2

In this case, $Core\,(\mathcal{L}, v) = Sel\,(\mathcal{L}, v)$.

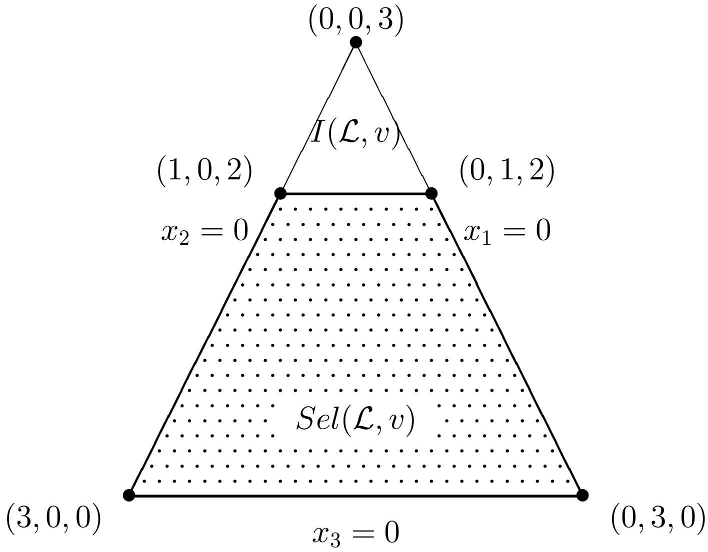

FIGURE 3.9

Chapter 4

Values on convex geometries

The first part of this chapter centres on the study of the individual valuation of the prospects of the players from their participation in a game. The probabilistic values for games on convex geometries are defined and a characterization of these values are obtained. This characterization is given by a family of axioms which are sequentially introduced with the purpose of showing the repercussion of each one in the expression of the probabilistic value.

The second part is devoted to the study of the group values. In a cooperative game, it is supposed that the total saving $v(N)$ of the grand coalition is distributed among the players. When a group value carries out this objective, it is said that it verifies the efficiency axiom. The existing relation between the fact that a group value is efficient and that its components are probabilistic values is shown. This will lead to the definition of the compatible-order values and so prove that they are the efficient group values where their components are probabilistic values. The family of compatible-order values associates a set of preimputations with each game. The set of preimputations associated with the game v for the family of compatible-order values is the Weber set.

In the last section, a compatible-order value is pointed out: the Shapley value for games on convex geometries. From the Shapley value, a different axiomatic characterization to the one obtained by Edelman and Bilbao [7] will be shown.

4.1 Individual values and group values

In this chapter, we consider that the family of feasible coalitions $\mathcal{L}$ is a convex geometry on the set of players N, and in the following, we omit this in the enunciation of the results.

For every player $i \in N$, a *value* for i on $\Gamma(\mathcal{L})$ is a function

$$\Phi_i : \Gamma(\mathcal{L}) \longrightarrow \mathbb{R},$$

which assigns to each game v defined on $\mathcal{L}$ a real number. The number $\Phi_i(v)$ associated to a game v can represent the value that the game v has for the player (for instance, the saving that the player i would obtain in the game v). In this way, the function Φ_i allows player i to evaluate his participation in the different games.

A value for the set of players or *group value* on $\Gamma(\mathcal{L})$ is a function

$$\Phi : \Gamma(\mathcal{L}) \longrightarrow \mathbb{R}^n,$$

which associates to each game v a vector $(\Phi_1(v), \ldots, \Phi_n(v))$. This vector represents the value that every player has in the participation in the game v.

4.2 Probabilistic values

The probabilistic values are individual values which have the aim of making that each player evaluates his own prospects in the participation in a game, that is, his expectations in the different games in which he can participate.

Definition 4.2.1 *A value Φ_i for player i on $\Gamma(\mathcal{L})$ is a probabilistic value if there exists a collection of non-negative real numbers $\{p_S^i : S \in \mathcal{L}, i \in ex(S)\}$ satisfying $\sum_{\{S\in\mathcal{L} : i\in ex(S)\}} p_S^i = 1$, such that*

$$\Phi_i(v) = \sum_{\{S\in\mathcal{L} : i\in ex(S)\}} p_S^i \left[v(S) - v(S \setminus i)\right],$$

for every game $v \in \Gamma(\mathcal{L})$.

In the following this formulation for the probabilistic values will be used. Nevertheless, if $T = S \setminus i$ is written in the above expression,

$$\Phi_i(v) = \sum_{\{T\in\mathcal{L}\,:\,i\notin T,\,T\cup\{i\}\in\mathcal{L}\}} p_T^i \left[v(T\cup\{i\}) - v(T)\right],$$

is obtained and all the following results could be formulated making use of this expression.

In the definition of the probabilistic value, there must be understood that the player i estimates his participation in the game, evaluating his marginal contributions $v(S) - v(S\setminus i)$ in those feasible coalitions S that are formed from others when i is incorporated. For every $S \in \mathcal{L}$ such that $i \in ex(S)$, p_S^i is the subjective probability that the player i has of joining the coalition $S \setminus i$. Thus, $\Phi_i(v)$ is the value that the player i can expect in the game v.

With the objective of obtaining an axiomatic characterization of the probabilistic values, in the following we start from a value Φ_i and introduce reasonable conditions that a value must satisfy. For instance, it seems reasonable that a player might well consider the prospective gain from playing the game $v+w$ to be the sum of his prospective gains from playing the games v and w. Similarly, if the game cv is considered, the player must expect the rescaling of his prospective gain in the game v. For these considerations, if Φ_i is a value for the player i, the following axiom must be verified.

Linearity axiom. For all $\alpha, \beta \in \mathbb{R}$, and $v, w \in \Gamma(\mathcal{L})$,

$$\Phi_i(\alpha v + \beta w) = \alpha\Phi_i(v) + \beta\Phi_i(w).$$

Theorem 4.2.1 *Let Φ_i be a value for player i on $\Gamma(\mathcal{L})$ which satisfies the linearity axiom. Then, there is a unique set of real numbers $\{a_S^i : S \in \mathcal{L},\ S \neq \emptyset\}$ such that*

$$\Phi_i(v) = \sum_{S\in\mathcal{L}} a_S^i v(S),$$

for every game $v \in \Gamma(\mathcal{L})$.

Proof. Being that the collection of identity games $\{\delta_S : S \in \mathcal{L},\ S \neq \emptyset\}$ is a basis of $\Gamma(\mathcal{L})$, and each game $v \in \Gamma(\mathcal{L})$ can be written as

$$v = \sum_{\{S\in\mathcal{L}, S\neq\emptyset\}} v(S)\delta_S,$$

then

$$\Phi_i(v) = \sum_{\{S\in\mathcal{L}, S\neq\emptyset\}} \Phi_i(\delta_S)v(S),$$

using the linearity of Φ_i. Therefore, upon taking $a_S^i = \Phi_i(\delta_S)$ for all nonempty $S \in \mathcal{L}$. □

We now introduce the concept of *dummy player*, understanding that a player is a dummy player when his contribution to each coalition formed with his incorporation to another coalition is exactly the value that he has in the game.

Definition 4.2.2 *A player $i \in N$ is a dummy player in the game $v \in \Gamma(\mathcal{L})$ if, for every $T \in \mathcal{L}$ such that $i \in ex(T)$, it holds*

$$v(T) - v(T \setminus i) = \begin{cases} v(\{i\}) & \text{if } \{i\} \in \mathcal{L}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that it also makes sense to define the dummy player in a game, as the player that all marginal contributions to the different feasible coalitions which form with his incorporation coincide, that is, a player $i \in N$ is a dummy in $v \in \Gamma(\mathcal{L})$ if, for all $S, T \in \mathcal{L}$ such that $i \in ex(S) \cap ex(T)$, it holds that

$$v(T) - v(T \setminus i) = v(S) - v(S \setminus i).$$

Note that both definitions are equivalent when the player i participate individually in the game, that is, $\{i\} \in \mathcal{L}$.

In order to prove the following results, we need some properties of the dummy players in the unanimity and identity games. In the following, for simplifying the notation, we write $S \cup i$ instead of $S \cup \{i\}$.

Proposition 4.2.2 *Let $\mathcal{L}$ be a convex geometry and let $S \in \mathcal{L}$ a nonempty convex set. Then,*

(1) If $i \notin ex(S)$, then i is a dummy player in the unanimity game ζ_S.

(2) If $i \in S \setminus ex(S)$, then i is a dummy player in the identity game δ_S.

(3) If $i \notin S$ and $S \cup i \notin \mathcal{L}$, then i is a dummy player in the identity game δ_S.

(4) If $\{i\} \in \mathcal{L}$, then i is a dummy player in the unanimity game $\zeta_{\{i\}}$.

Proof. First of all, note that if $S \in \mathcal{L}$ is a nonempty convex set and the player i is such that $i \notin ex(S)$, then if $\{i\} \in \mathcal{L}$, we have that $S \neq \{i\}$, and hence $\zeta_S(\{i\}) = 0$ and $\delta_S(\{i\}) = 0$.

(1) Let $i \notin ex\,(S)$. We suppose that there exists a coalition $T \in \mathcal{L}$ with $i \in ex\,(T)$ and satisfying $\zeta_S(T) \neq \zeta_S(T \setminus i)$. Then $\zeta_S(T) = 1$, $\zeta_S(T \setminus i) = 0$ and hence $S \subseteq T$ and $S \not\subseteq T \setminus i$. Thus $i \in S$. Moreover, $S \setminus i = S \cap (T \setminus i)$ and therefore, $S \setminus i \in \mathcal{L}$, but this is a contradiction because $i \notin ex(S)$.

(2) Let $i \in S \setminus ex\,(S)$. For every $T \in \mathcal{L}$ with $i \in ex\,(T)$, we have that $T \neq S$ and $T \setminus i \neq S$, and hence, $\delta_S(T) = \delta_S(T \setminus i) = 0$.

(3) Let $i \notin S$ and $S \cup i \notin \mathcal{L}$. If $T \in \mathcal{L}$ such that $i \in ex\,(T)$, it is clear that $\delta_S(T) = \delta_S(T \setminus i) = 0$.

(4) Let $\{i\} \in \mathcal{L}$. In this case, if $T \in \mathcal{L}$ with $i \in ex\,(T)$ then $\zeta_{\{i\}}\,(T) = 1$ and $\zeta_{\{i\}}\,(T \setminus i) = 0$. Moreover, $\zeta_{\{i\}}\,(\{i\}) = 1$. □

Denominating a player i as a dummy player in a game v is to consider that this player has no meaningful strategic role in the game, since his contributions to the feasible coalitions formed with his incorporation coincide. Therefore, the value that this player should expect in the game v must exactly be his contribution. This consideration justifies the introduction to the following axiom.

Dummy axiom. If the player $i \in N$ is dummy in $v \in \Gamma(\mathcal{L})$, then

$$\Phi_i(v) = \begin{cases} v(\{i\}) & \text{if } \{i\} \in \mathcal{L}, \\ 0 & \text{otherwise.} \end{cases}$$

In the following result, we can observe that if we add the dummy axiom to the linearity axiom, the value for the player i can be expressed as a linear combination of his contributions.

Theorem 4.2.3 *Let Φ_i be a value for player i on $\Gamma(\mathcal{L})$ defined by $\Phi_i(v) = \sum_{S\in\mathcal{L}} a_S^i v(S)$ for every game $v \in \Gamma(\mathcal{L})$, and satisfying the dummy axiom. Then,*

$$\Phi_i(v) = \sum_{\{S\in\mathcal{L}\,:\,i\in ex(S)\}} a_S^i \left[v(S) - v(S \setminus i)\right].$$

Moreover, if $\{i\} \in \mathcal{L}$ then $\sum_{\{S\in\mathcal{L}\,:\,i\in ex(S)\}} a_S^i = 1.$

Proof. Let $E_i : \Gamma(\mathcal{L}) \longrightarrow \mathbb{R}$ be an operator defined, for all $v \in \Gamma(\mathcal{L})$, by

$$E_i(v) = \sum_{\{S\in\mathcal{L}\,:\,i\in ex(S)\}} a_S^i \left[v(S) - v(S \setminus i)\right].$$

The operators E_i and Φ_i are linear operators and the unanimity games form a basis of $\Gamma(\mathcal{L})$. Thus, it will be enough to show that $\Phi_i(\zeta_T) = E_i(\zeta_T)$, for every nonempty coalition $T \in \mathcal{L}$. Consider a coalition $T \in \mathcal{L}$, $T \neq \emptyset$, we distinguish two cases:

1. If $i \notin ex(T)$, by the above proposition, i is a dummy player in the game ζ_T. Then $\zeta_T(S) - \zeta_T(S \setminus i) = 0$, for all $S \in \mathcal{L}$ such that $i \in ex(S)$ because $\zeta_T(\{i\}) = 0$, even when $\{i\} \in \mathcal{L}$. Therefore, the definition of E_i yields $E_i(\zeta_T) = 0$, and furthermore, the dummy axiom implies that $\Phi_i(\zeta_T) = 0$.

2. If $i \in ex(T)$, then $i \in T$, and hence $\zeta_T(S \setminus i) = 0$ for every $S \in \mathcal{L}$ with $i \in ex(S)$. Thus, we obtain the following equivalence

$$\zeta_T(S) - \zeta_T(S \setminus i) = 1 \quad \text{if and only if} \quad S \supseteq T$$

Then, we have

$$\begin{aligned} E_i(\zeta_T) &= \sum_{\{S\in\mathcal{L}\,:\,i\in ex(S),\, S\supseteq T\}} a_S^i \\ &= \sum_{\{S\in\mathcal{L}\,:\,i\in ex(S),\, S\supseteq T\}} \Phi_i(\delta_S) \\ &= \Phi_i\left(\sum_{\{S\in\mathcal{L}\,,\, S\supseteq T\}} \delta_S\right) \\ &= \Phi_i(\zeta_T), \end{aligned}$$

where the last but one equality follows from $\Phi_i(\delta_S) = 0$ when $i \in S \setminus ex(S)$, applying the dummy axiom.

Finally, if $\{i\} \in \mathcal{L}$, then the player i is dummy in the game $\zeta_{\{i\}}$. As Φ_i satisfies the dummy axiom, we deduce from the expression of Φ_i that

$$\sum_{\{S \in \mathcal{L} \,:\, i \in ex(S)\}} a_S^i = \Phi_i(\zeta_{\{i\}}) = \zeta_{\{i\}}(\{i\}) = 1.$$

□

If $i \in N$ is not an element of the convex geometry $\mathcal{L}$, then the unanimity game $\zeta_{\overline{\{i\}}} : \mathcal{L} \longrightarrow \mathbb{R}$, is defined by

$$\zeta_{\overline{\{i\}}}(T) = \begin{cases} 1 & \text{if } \ i \in T, \\ 0 & \text{otherwise,} \end{cases}$$

and in this game, the player i is not a dummy. Hence, if we have a value Φ_i for this player that satisfies the hypothesis of the above theorem, then

$$\sum_{\{S \in \mathcal{L} \,:\, i \in ex(S)\}} a_S^i = \Phi_i\left(\zeta_{\overline{\{i\}}}\right),$$

and thus, we obtain the following result.

Theorem 4.2.4 *Let Φ_i be a value for player i on $\Gamma(\mathcal{L})$ that satisfies the linearity and dummy axioms. Then, for every game v, there is a collection of real numbers $\{a_S^i \,:\, S \in \mathcal{L},\ i \in ex(S)\}$ such that*

$$\Phi_i(v) = \sum_{\{S \in \mathcal{L} \,:\, i \in ex(S)\}} a_S^i \left[v(S) - v(S \setminus i)\right].$$

for all $v \in \Gamma(\mathcal{L})$. Moreover, if $\Phi_i\left(\zeta_{\overline{\{i\}}}\right) = 1$ then $\sum_{\{S \in \mathcal{L} : i \in ex(S)\}} a_S^i = 1.$

If we suppose that a game $v \in \Gamma(\mathcal{L})$ is a monotonic game, that is, the value of each coalition is less or equal to the value of anyone that contains it, then the repercussion of every player to any coalition is never unfavorable, hence the player can expect to be valued non-negatively in each monotonic game.

Monotonicity axiom. If $v \in \Gamma(\mathcal{L})$ is a monotonic game, then $\Phi_i(v) \geq 0$.

If we introduce this new axiom in the hypothesis of the above theorem, we can affirm that the coefficients a_S^i are non-negative.

Theorem 4.2.5 *Let Φ_i be a value for player i on $\Gamma(\mathcal{L})$ defined, for every game v, by*

$$\Phi_i(v) = \sum_{\{S \in \mathcal{L} \,:\, i \in ex(S)\}} a_S^i \left[v(S) - v(S \setminus i) \right].$$

If the value Φ_i satisfies the monotonicity axiom, then $a_S^i \geq 0$, for all $S \in \mathcal{L}$ such that $i \in ex(S)$.

Proof. For every $T \in \mathcal{L}$, we consider the game

$$\widehat{\zeta}_T(S) = \begin{cases} 1 & \text{if } T \subset S, \\ 0 & \text{otherwise.} \end{cases}$$

The game $\widehat{\zeta}_T$ is monotonic, hence $\Phi_i\left(\widehat{\zeta}_T\right) \geq 0$. On the other hand, for every $R \in \mathcal{L}$ such that $i \in ex\left(R\right)$, we obtain

$$\Phi_i\left(\widehat{\zeta}_{R \setminus i}\right) = \sum_{\{S \in \mathcal{L} \,:\, i \in ex(S)\}} a_S^i \left[\widehat{\zeta}_{R \setminus i}(S) - \widehat{\zeta}_{R \setminus i}(S \setminus i) \right] = a_R^i,$$

and then $a_R^i \geq 0$ for every $R \in \mathcal{L}$ such that $i \in ex\left(R\right).$

□

It is easy to check that every probabilistic value satisfies the previous axioms. Therefore, we can give a characterization of the probabilistic value for a player i from the combination of the above results.

Theorem 4.2.6 *Let Φ_i be a value for player i on $\Gamma\left(\mathcal{L}\right)$. If $\{i\} \in \mathcal{L}$, then, the value Φ_i is a probabilistic value if and only if Φ_i satisfies the linearity, dummy and monotonicity axioms.*

Theorem 4.2.7 *Let Φ_i be a value for player i on $\Gamma(\mathcal{L})$. Then, the value Φ_i is a probabilistic value if and only if Φ_i satisfies the linearity, dummy and monotonicity axioms and $\Phi_i\left(\zeta_{\overline{\{i\}}}\right) = 1$.*

4.3 Efficient values

In this section, we study those group values

$$\Phi : \Gamma(\mathcal{L}) \longrightarrow \mathbb{R}^n, \quad \Phi(v) = (\Phi_1(v), \ldots, \Phi_n(v)),$$

that provide an equitable distribution of the value of the grand coalition among the players. From this perspective, Φ must satisfy the following axiom:

Efficiency axiom. For every $v \in \Gamma(\mathcal{L})$, it holds

$$\sum_{i \in N} \Phi_i(v) = v(N).$$

Theorem 4.3.1 *Let $\Phi = (\Phi_1, \ldots, \Phi_n)$ be a group value on $\Gamma(\mathcal{L})$ satisfying the efficiency axiom. Then, if its component Φ_i satisfies the linearity, dummy and monotonicity axioms, then Φ_i is a probabilistic value.*

Proof. By Theorem 4.9, it is sufficient to prove that $\Phi_i\left(\zeta_{\overline{\{i\}}}\right) = 1$, for all $i \in N$. Note that if $i \in N$, $ex\left(\overline{\{i\}}\right) = \{i\}$ holds, because in a convex geometry we have that $ex\left(\overline{\{i\}}\right) \neq \emptyset$, and moreover, if there exists $j \in ex\left(\overline{\{i\}}\right)$ with $j \neq i$ then $\overline{\{i\}} \setminus j \in \mathcal{L}$. Thus, we obtain $i \in \overline{\{i\}} \setminus j$, and hence, by definition of the closure operator, $\overline{\{i\}} \subseteq \overline{\{i\}} \setminus j$, which is a contradiction.

From Proposition 4.4, it follows that for each j distinct to i, $\Phi_j\left(\zeta_{\overline{\{i\}}}\right) = 0$ is satisfied, since $j \notin ex\left(\overline{\{i\}}\right)$. Therefore, the efficiency axiom implies that

$$\sum_{j \in N} \Phi_j\left(\zeta_{\overline{\{i\}}}\right) = \Phi_i\left(\zeta_{\overline{\{i\}}}\right) = 1.$$

□

The following theorem characterizes the group values which are efficient.

Theorem 4.3.2 *Let $\Phi = (\Phi_1, \ldots, \Phi_n)$ be a group value on $\Gamma(\mathcal{L})$ defined, for every game v and for all $i \in N$, by*

$$\Phi_i(v) = \sum_{\{S \in \mathcal{L} : i \in ex(S)\}} a_S^i \left[v(S) - v(S \setminus i) \right].$$

Then, the group value Φ satisfies the efficiency axiom if and only if

$$\sum_{i \in ex(N)} a_N^i = 1, \quad \text{and} \quad \sum_{i \in ex(S)} a_S^i = \sum_{\{j \notin S : S \cup j \in \mathcal{L}\}} a_{S \cup j}^j, \ \ \forall S \in \mathcal{L}, \ \ S \neq \emptyset, N.$$

Proof. For every $v \in \Gamma(\mathcal{L})$ we have

$$\sum_{i \in N} \Phi_i(v) = \sum_{i \in N} \sum_{\{S \in \mathcal{L} : i \in ex(S)\}} a_S^i \left[v(S) - v(S \setminus i) \right]$$

Note that in the preceding expression, we consider the values of the game v over all nonempty coalitions, since every nonempty coalition has some extreme point. By this, the above expression can be written as a linear combination of the values $v(S)$ for all nonempty coalition $S \in \mathcal{L}$. We have

$$\sum_{i \in N} \Phi_i(v) = \sum_{\{S \in \mathcal{L}, S \neq \emptyset, N\}} v(S) \left[\sum_{i \in ex(S)} a_S^i - \sum_{\{j \notin S : S \cup j \in \mathcal{L}\}} a_{S \cup j}^j \right] + v(N) \sum_{i \in ex(N)} a_N^i.$$

If the coefficients satisfy the relations of the hypothesis of theorem, then $\sum_{i \in N} \Phi_i(v) = v(N)$, and therefore, Φ satisfies the efficiency axiom.

Reciprocally, fix a nonempty convex set $T \in \mathcal{L}$, and applying the preceding equality to the identity game δ_T, we have

$$\sum_{i \in N} \Phi_i(\delta_T) = \begin{cases} \displaystyle\sum_{i \in ex(N)} a_N^i & \text{if } T = N, \\ \displaystyle\sum_{i \in ex(T)} a_T^i - \sum_{\{j \notin T : T \cup j \in \mathcal{L}\}} a_{S \cup j}^j & \text{if } T \neq N. \end{cases}$$

Thus, if Φ satisfies the efficiency axiom, the relations for the coefficients are obtained. □

Note that in the case that the convex geometry $\mathcal{L}$ is formed by a unique maximal chain C, each player is extreme in only one coalition of $\mathcal{L}$. This coalition will precisely be the smallest convex in the chain that contains it, denoted by $C(i)$. Thus, for a probabilistic value

$$\Phi_i(v) = v(C(i)) - v(C(i) \setminus i),$$

and we can observe that this value corresponds to the i-th component of the marginal worth vector $a^C(v)$ associated to the maximal chain C.

4.4 Compatible-order values

The characterization of the efficient group values given in the previous section establishes that even when the players have distinct perceptions of their participation in the different coalitions, they can get to be efficient together.

We now consider group values which result from a common perception for all players. It is assumed that all of them estimate that the grand coalition N is formed as a sequential process where in each step a different player is incorporated. These sequential processes are reflected considering the different maximal chains in the convex geometry $\mathcal{L}$.

Definition 4.4.1 *A compatible-order value on $\Gamma(\mathcal{L})$ is a group value $\Psi = (\Psi_1, \dots, \Psi_n)$, such that there exists a collection of non-negative real numbers $\{p_C : C \in \mathcal{C}(\mathcal{L})\}$ verifying $\sum_{C\in\mathcal{C}(\mathcal{L})} p_C = 1$ so that*

$$\Psi_i(v) = \sum_{C\in\mathcal{C}(\mathcal{L})} p_C \left[v(C(i)) - v(C(i) \setminus i)\right].$$

for all $i \in N$ and for every $v \in \Gamma(\mathcal{L})$.

A compatible-order value is a group value where each player evaluates his marginal contributions in the different processes of formation of the grand coalition. Moreover, all players have a commom perception of the probability of these processes. The relation between the compatible-order values and the probabilistic values that satisfies the efficiency axiom is stated in the following theorems.

Theorem 4.4.1 *Let $\Psi = (\Psi_1, \ldots, \Psi_n)$ be a compatible-order value on $\Gamma(\mathcal{L})$. Then Ψ satisfies the efficiency axiom and each component of Ψ is a probabilistic value.*

Proof. Let $\{p_C : C \in \mathcal{C}(\mathcal{L})\}$ be a collection of constants associated to Ψ so, for every $i \in N$, and $v \in \Gamma(\mathcal{L})$, we have

$$\Psi_i(v) = \sum_{C \in \mathcal{C}(\mathcal{L})} p_C \left[v\left(C(i)\right) - v\left(C(i) \setminus i\right)\right].$$

Being that changing C in $\mathcal{C}(\mathcal{L})$, the convex sets $C(i)$ determine all coalitions of $\mathcal{L}$ which have the player i as extreme, we can write

$$\Psi_i(v) = \sum_{\{S \in \mathcal{L} : i \in ex(S)\}} \left(\sum_{\{C \in \mathcal{C}(\mathcal{L}) : C(i) = S\}} p_C \right) \left[v(S) - v(S \setminus i)\right].$$

Thus, for each $i \in N$, and for all $S \in \mathcal{L}$ with $i \in ex(S)$, and if we call

$$p_S^i = \sum_{\{C \in \mathcal{C}(\mathcal{L}) : C(i) = S\}} p_C,$$

we have that $p_S^i \geq 0$, and also

$$\sum_{\{S \in \mathcal{L} : i \in ex(S)\}} p_S^i = \sum_{C \in \mathcal{C}(\mathcal{L})} p_C = 1,$$

since, fix i, in the expression

$$\sum_{\{S \in \mathcal{L} : i \in ex(S)\}} \left(\sum_{\{C \in \mathcal{C}(\mathcal{L}) : C(i) = S\}} p_C \right),$$

we examine all the convex sets S which have player i as extreme point and for each S, the chains where $C(i) = S$ are considered. When S is varied, different chains are taken which are all the chains of $\mathcal{L}$, being that in all of them there is a coalition which has i as extreme.

This proves that Ψ_i is a probabilistic value. Moreover, for every game $v \in \Gamma(\mathcal{L})$, we have

$$\begin{aligned}
\sum_{i\in N} \Psi_i(v) &= \sum_{i\in N} \sum_{C\in\mathcal{C}(\mathcal{L})} p_C \left[v\left(C(i)\right) - v\left(C(i)\setminus i\right)\right] \\
&= \sum_{C\in\mathcal{C}(\mathcal{L})} p_C \left(\sum_{i\in N} \left[v\left(C(i)\right) - v\left(C(i)\setminus i\right)\right] \right) \\
&= \sum_{C\in\mathcal{C}(\mathcal{L})} p_C \left[v(N) - v(\emptyset)\right] \\
&= v(N).
\end{aligned}$$

□

Theorem 4.4.2 *Let $\Phi = (\Phi_1, \dots, \Phi_n)$ be a group value on $\Gamma(\mathcal{L})$ that satisfies the efficiency axiom and such that each component of Φ is a probabilistic value. Then, Φ is a compatible-order value.*

Proof. By hypothesis, for each component of Φ, we have

$$\Phi_i(v) = \sum_{\{S\in\mathcal{L}\,:\, i\in ex(S)\}} p_S^i \left[v(S) - v(S\setminus i)\right], \quad \text{for all } i \in N.$$

For any $T \in \mathcal{L}$ and $i \notin T$ such that $T \cup i \in \mathcal{L}$, we define

$$A(i,T) = \begin{cases} \dfrac{p^i_{T\cup i}}{\displaystyle\sum_{\{j\notin T: T\cup j\in\mathcal{L}\}} p^j_{T\cup j}} & \text{if the denominator is not equal to } 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that in this expression we are considering a quotient between the assigned probability for the player from his union to the coalition T and the sum up of the probabilities of the union to this coalition, of all the players that can form a coalition with T.

For any maximal chain $C \in \mathcal{C}(\mathcal{L})$ associated to a compatible order $i_1 < i_2 < \cdots < i_n$, that is

$$C : \emptyset \subset C(i_1) \subset C(i_2) \subset \cdots \subset C(i_n),$$

we define

$$p_C = p^{i_1}_{\{i_1\}} A(i_2, \{i_1\})\ A(i_3, \{i_1, i_2\}) \cdots A(i_n, \{i_1, i_2, \ldots, i_{n-1}\}),$$

where $C = (i_1, i_2, \ldots, i_n)$. Note that the last factor is equal to 1. The collection $\{p_C : C \in \mathcal{C}(\mathcal{L})\}$ satisfies, besides being $p_C \geq 0$, that

$$\begin{aligned}
\sum_{C\in\mathcal{C}(\mathcal{L})} p_C &= \sum_{\{i_1 : \{i_1\}\in\mathcal{L}\}} \sum_{\{i_2\notin\{i_1\} : \{i_1,i_2\}\in\mathcal{L}\}} \cdots \sum_{\{i_n : i_n\notin\{i_1,\ldots,i_{n-1}\}\}} p_{(i_1,\ldots,i_n)} \\
&= \sum_{\{i_1 : \{i_1\}\in\mathcal{L}\}} p^{i_1}_{\{i_1\}} \\
&= \sum_{i\in N} \Phi_i\left(\widehat{\zeta}_\emptyset\right) \\
&= 1,
\end{aligned}$$

where the last equality follows from the efficiency axiom applied to the game $\widehat{\zeta}_\emptyset$, defined by $\widehat{\zeta}_\emptyset(S) = 1$, for every nonempty coalition $S \in \mathcal{L}$. Hence $\{p_C : C \in \mathcal{C}(\mathcal{L})\}$ is a probability distribution. Let Ψ be the compatible-order value associated to this probability distribution, that is

$$\Psi_i(v) = \sum_{C\in\mathcal{C}(\mathcal{L})} p_C \left[v(C(i)) - v(C(i) \setminus i)\right],$$

for all $i \in N$ and for every $v \in \Gamma(\mathcal{L})$. Since

$$\Psi_i(v) = \sum_{\{S\in\mathcal{L} : i\in ex(S)\}} \left(\sum_{\{C\in\mathcal{C}(\mathcal{L}) : C(i)=S\}} p_C \right) \left[v(S) - v(S \setminus i)\right],$$

for all $i \in N$, we have $\Phi_i = \Psi_i$ if for every $S \in \mathcal{L}$ such that $i \in ex(S)$, the coefficients satisfy

$$p^i_S = \sum_{\{C\in\mathcal{C}(\mathcal{L}) : C(i)=S\}} p_C.$$

We will write $C_1 = (i_1, \ldots, i_{s-1})$, with $s = |S|$, any chain C_1 from $\emptyset$ to $S \setminus i$ and $C_2 = (i_{s+1}, \ldots, i_n)$ any chain C_2 from S to N. These chains can be concatenated with i to make a maximal chain of $\mathcal{L}$,

$$C = (i_1, \ldots, i_{s-1}, i, i_{s+1}, \ldots, i_n).$$

Then, we obtain

$$\sum_{\{C\in\mathcal{C}(\mathcal{L}):C(i)=S\}} p_C$$

$$\begin{aligned}
&= \sum_{i_{s-1}\in ex(S\setminus i)} \sum_{i_{s-2}\in ex(S\setminus\{i,i_{s-1}\})} \cdots \sum_{i_1\in ex(S\setminus\{i,i_{s-1},\dots,i_2\})} \sum_{\{i_{s+1}\notin S:\, S\cup i_{s+1}\in\mathcal{L}\}} \cdots \\
&\qquad \sum_{\{i_n\notin S\cup\{i_{s+1},\dots,i_{n-1}\}:S\cup\{i_{s+1},\dots,i_n\}\in\mathcal{L}\}} p_{(i_1,\dots,i_{s-1},i,i_{s+1},\dots,i_n)} \\
&= A(i,S\setminus i) \sum_{i_{s-1}\in ex(S\setminus i)} A\left(i_{s-1},S\setminus\{i,i_{s-1}\}\right)\cdots \\
&\qquad \sum_{i_1\in ex(S\setminus\{i,i_{s-1}\dots,i_2\})} p^{i_1}_{\{i_1\}} \sum_{\{i_{s+1}\notin S:\, S\cup i_{s+1}\in\mathcal{L}\}} A(i_{s+1},S)\cdots \\
&\qquad \sum_{\{i_n\notin S\cup\{i_{s+1},\dots,i_{n-1}\}:\, S\cup\{i_{s+1},\dots,i_n\}\in\mathcal{L}\}} A\left(i_n,\ S\cup\{i_{s+1},\dots,i_{n-1}\}\right).
\end{aligned}$$

From right to left, the first $n-s$ sums each, in turn, have value 1. Indeed, if we write $T_k = S\cup\{i_{s+1},\dots,i_k\}$,

$$\sum_{\{i_k\notin T_{k-1}:T_{k-1}\cup i_k\in\mathcal{L}\}} A(i_k,T_{k-1}) = \sum_{\{i_k\notin T_{k-1}:\, T_{k-1}\cup i_k\in\mathcal{L}\}} \left(\frac{p^{i_k}_{T_{k-1}\cup i_k}}{\sum_{\{j\notin T_{k-1}:T_{k-1}\cup j\in\mathcal{L}\}} p^{j}_{T_{k-1}\cup j}}\right)$$

is equal to 1.

Continuing leftward, each numerator of one factor is equal to the previous denominator, applying the equations of Theorem 4.11, and thus,

$$\frac{p_S^i}{\sum_{\{j\notin S\setminus i:(S\setminus i)\cup j\in\mathcal{L}\}} p^j_{(S\setminus i)\cup j}}$$
$$\cdot \sum_{i_{s-1}\in ex(S\setminus i)} \left(\frac{p^{i_{s-1}}_{S\setminus i}}{\sum_{\{j\notin S\setminus\{i,i_{s-1}\}:\,(S\setminus\{i,i_{s-1}\})\cup j\in\mathcal{L}\}} p^j_{(S\setminus\{i,i_{s-1}\})\cup j}} \right)$$
$$\cdots \sum_{i_1\in ex(S\setminus\{i,i_{s-1},\ldots,i_2\})} p^{i_1}_{\{i_1\}} = p_S^i.$$

□

The family of compatible-order values associates a set of preimputations with each game. Remember that, for any game $v \in \Gamma(\mathcal{L})$, and for any maximal chain $C \in \mathcal{C}(\mathcal{L})$, the marginal worth vector associated to the chain C is defined by $a^C(v) = v(C(i)) - v(C(i)\setminus i)$ for all $i \in N$. Therefore, being that the Weber set of the game v is the convex hull of these vectors, we can affirm that $Weber(\mathcal{L}, v)$ is the set of preimputations associated with the game v by the family of compatible-order values.

4.5 The Shapley value

The classical characterization of the Shapley value [56] for cooperative games, states that it is the only group value $\Phi = (\Phi_1, \ldots, \Phi_n)$ on $\Gamma(2^N)$ whose components satisfy the following axioms:

1. Carrier axiom. If $U \in 2^N$ is a carrier of $v \in \Gamma(2^N)$, that is, $v(S) = v(S\cap U)$, for all $S \subseteq N$, then
$$\sum_{i\in U} \Phi_i(v) = v(U).$$
2. Symmetry axiom. For every permutation π of the set of players N,
$$\Phi_{\pi i}(\pi v) = \Phi_i(v),$$

where πv is the game defined by $\pi v(\pi S) = v(S)$, for all $S \subseteq N$.

3. Linearity axiom. For all $\alpha, \beta \in \mathbb{R}$ and $u, w \in \Gamma(2^N)$, it holds

$$\Phi_i(\alpha u + \beta w) = \alpha \Phi_i(u) + \beta \Phi_i(w).$$

For the distributive lattice $\mathcal{L} = J(P)$ of all order ideals of a partially ordered set $(P, \leq)$, Faigle and Kern [25] define the concept of the *hierarchical strength* of a player i in the coalition $S \in \mathcal{L}$ such that $i \in S$, as

$$h_S(i) = \frac{|\{C \in \mathcal{C}(\mathcal{L}) : C(i) \cap S = S\}|}{|\mathcal{C}(\mathcal{L})|};$$

that is, $h_S(i)$ is the average of compatible orderings of $\mathcal{L}$ in which the player i is the last member of S in the ordering. Note that $h_S(i) \neq 0$ if and only if $i \in ex(S)$. Faigle and Kern propose the following axiom for games defined on the distributive lattice $\mathcal{L} = J(P)$.

Hierarchical strength axiom. For any $S \in \mathcal{L}$ and $i, j \in S$,

$$h_S(i)\,\Phi_j(\zeta_S) = h_S(j)\,\Phi_i(\zeta_S).$$

Faigle and Kern [25] showed the following extension of the Shapley value to a game $v : J(P) \longrightarrow \mathbb{R}^n$.

Theorem 4.5.1 *There is a unique function $\Phi : \Gamma(J(P)) \longrightarrow \mathbb{R}^n$ satisfying the linearity, carrier and hierarchical strength axioms. Moreover, for every $i \in P$,*

$$\Phi_i(v) = \sum_{\{T \in J(P) : i \in Max(T)\}} \frac{e(T \setminus i)\, e(P \setminus T)}{e(P)} \left[v(T) - v(T \setminus i)\right],$$

where $e(\cdot)$ is the number of linear extensions of the corresponding subposets of $(P, \leq)$.

Bilbao and Edelman [9] generalized this formula for the Shapley value for games on convex geometries. In this context, the number of linear extensions is the number of maximal chains of $\mathcal{L}$.

For any $S, T \in \mathcal{L}$, $T \subset S$, we denote by $c([T, S])$ the number of maximal chains from T to S. Furthermore, we write $c(S)$ instead of $c([\emptyset, S])$ by simplifying the notation and thus, $c(N)$ is the total number of maximal chains.

Definition 4.5.1 *Let $v \in \Gamma(\mathcal{L})$ be a game on the convex geometry $\mathcal{L}$. The Shapley value, for player $i \in N$ and the game v, is given by*

$$\Phi_i(v) = \sum_{\{S \in \mathcal{L}: i \in ex(S)\}} \frac{c(S \setminus i)\, c([S, N])}{c(N)} [v(S) - v(S \setminus i)],$$

where $c([N, N]) = c(\emptyset) = 1$.

The results of the previous sections allow us to obtain a new axiomatic characterization of the Shapley value for games on convex geometries. Note that when we require that the components of an efficient group value verify the linearity and dummy axioms, these components are not uniquely determined and, these components could or could not be probabilistic values. However, the next axiom leads to a unique group value that is the Shapley value. In this new axiom, the value of the player depends on the position which he has in the structure of the convex geometry.

Chain axiom. For any nonempty coalition $S \in \mathcal{L}$ and $i, j \in ex(S)$,

$$c(S \setminus i)\Phi_j(\delta_S) = c(S \setminus j)\Phi_i(\delta_S).$$

In convex geometries, the Shapley value is characterized by four axioms. Three of these appear in the classic characterization of the Shapley value when every coalition is feasible. The chain axiom replaces the classic symmetry axiom.

Theorem 4.5.2 *The Shapley value is the unique efficient group value $\Phi = (\Phi_1, \ldots, \Phi_n)$ on $\Gamma(\mathcal{L})$ where its components Φ_i satisfy the linearity, dummy and chain axioms.*

Proof. It it obvious that the Shapley value satisfies the properties. Conversely, it follows from Theorems 4.6 and 4.11 that, for every $i \in N$, there exists a collection $\{a_S^i : \ S \in \mathcal{L}, \ i \in ex(S)\}$ such that

$$\Phi_i(v) = \sum_{\{S \in \mathcal{L} \,:\, i \in ex(S)\}} a_S^i \left[v(S) - v(S \setminus i)\right],$$

and the coefficients satisfy

$$\begin{aligned} \sum_{i \in ex(N)} a_N^i &= 1, \\ \sum_{i \in ex(S)} a_S^i &= \sum_{\{j \notin S \,:\, S \cup j \in \mathcal{L}\}} a_{S \cup j}^j, \quad \text{for all } S \in \mathcal{L}, \quad S \neq \emptyset, N. \end{aligned}$$

Therefore, it suffices to show that

$$a_S^i = \frac{c(S \setminus i)\, c([S, N])}{c(N)}, \quad \text{for all} \ \ S \in \mathcal{L} \ \ \text{and} \ \ i \in ex(S).$$

Note that the coefficients are $a_S^i = \Phi_i(\delta_S)$ and, hence the chain axiom implies that $a_S^j c(S \setminus i) = a_S^i c(S \setminus j)$, for all $i, j \in ex(S)$. If we fix $i \in ex(S)$, then we obtain

$$\begin{aligned} \sum_{j \in ex(S)} a_S^j &= a_S^i + \sum_{\{j \in ex(S) \,:\, j \neq i\}} \frac{c(S \setminus j)}{c(S \setminus i)} a_S^i \\ &= \frac{a_S^i}{c(S \setminus i)} \sum_{j \in ex(S)} c(S \setminus j) \\ &= a_S^i \frac{c(S)}{c(S \setminus i)}. \end{aligned}$$

Taking $S = N$ in the above expression, and using the relations which the coefficients verify, we have,

$$c(N \setminus i) = a_N^i \, c(N), \quad \text{for every } i \in ex(N).$$

Thus,

$$a_N^i = \frac{c(N \setminus i)\, c([N, N])}{c(N)}, \quad \text{for all } i \in ex(N).$$

We assume the following induction hypothesis: For every $T \in \mathcal{L}$, with $|T| = k \geq 2$, it holds

$$a_T^i = \frac{c(T \setminus i)\ c([T, N])}{c(N)}, \quad \text{for all } i \in ex(T).$$

The case $k = n$; that is, $T = N$ has just been proved. Let $S \in \mathcal{L}$, such that $|S| = k - 1 < n$. Then,

$$\begin{aligned}
\sum_{i \in ex(S)} a_S^i &= \sum_{\{j \notin S \,:\, S \cup j \in \mathcal{L}\}} a_{S \cup j}^j \\
&= \sum_{\{j \notin S \,:\, S \cup j \in \mathcal{L}\}} \frac{c(S)\ c([S \cup j, N])}{c(N)} \\
&= \frac{c(S)}{c(N)} \sum_{\{j \notin S \,:\, S \cup j \in \mathcal{L}\}} c([S \cup j, N]) \\
&= \frac{c(S)\, c([S, N])}{c(N)},
\end{aligned}$$

where the second equality follows from the induction hypothesis for $T = S \cup j$.

Finally, for every $i \in ex(S)$, the identity

$$a_S^i \frac{c(S)}{c(S \setminus i)} = \frac{c(S)\, c([S, N])}{c(N)}$$

implies

$$a_S^i = \frac{c(S \setminus i)\, c([S, N])}{c(N)}.$$

□

Note that, if we take $p_C = 1/c(N)$, for all $C \in \mathcal{C}(\mathcal{L})$, the distribution $\{p_C : C \in \mathcal{C}(\mathcal{L})\}$ is a probability distribution, and the compatible-order value Ω associated to this distribution is precisely the Shapley value. This leads to the conclusion that the Shapley value for a game $v \in \Gamma(\mathcal{L})$ is always an element of its Weber set.

Chapter 5

Simple games

In this chapter, we study the different solution concepts already defined in the previous chapters for an interesting particular class of games: the *simple games*. This type of games arises, for instance, from modeling voting situations in which the result reflects two possibilities (win-lose, accept-reject, etc.) and they have been the object of a specific study like the initial ones of Isbell [36] and Shapley [58]; of Rafels and Marin-Solano [42], and Einy and Wettstein [23] which have a particular incidence in this chapter; and the work of Carreras [15] that studies the simple games restricted by cooperation graphs.

5.1 Preliminaries

In the following, we assume that the family of feasible coalitions $\mathcal{L} \subseteq 2^N$ is such that $\emptyset, N \in \mathcal{L}$, and only in certain occasions, other requirements are needed for this family.

Definition 5.1.1 *A game $v \in \Gamma(\mathcal{L})$ is called simple if it satisfies the following conditions:*

1. *For every coalition $S \in \mathcal{L}$, $v(S) \in \{0,1\}$ and $v(N) = 1$.*
2. *If $S, T \in \mathcal{L}$ with $S \subseteq T$, then $v(S) \leq v(T)$.*

We denote by $\Omega(\mathcal{L})$ the class of all simple games defined on the family $\mathcal{L}$.

In a simple game, a coalition $S \in \mathcal{L}$ is *winning* if $v(S) = 1$; otherwise, is *losing*. A winning coalition S is *minimal* if it is winning, and there does not exist any winning coalition contained in S. We denote by $\mathcal{W}$ the set of all minimal winning coalitions. This set is nonempty and, in general, we write $\mathcal{W} = \{S_1, \ldots, S_r\}$ with $r \geq 1$. Note that every simple game is totally characterized by the minimal winning coalitions. If a game has only one minimal winning coalition, that is, $\mathcal{W} = \{S_1\}$, then v is the unanimity game ζ_{S_1}.

There is a certain type of players that, in the simple games, perform a very important role, and which are so-called veto players. A player $i \in N$ is a *veto player* in the game v if he belongs to each winning coalition $S \in \mathcal{L}$, that is,

$$v(S) = 1 \Longrightarrow i \in S.$$

We denote by $\mathcal{V}$ the set of all veto players in the game $v \in \Omega(\mathcal{L})$, i.e.,

$$\mathcal{V} = \bigcap_{\{S \in \mathcal{L} \,:\, v(S)=1\}} S.$$

If there exist $\{i\}, \{j\} \in \mathcal{L}$, $i \neq j$ such that $v(\{i\}) = v(\{j\}) = 1$, then $\mathcal{V} = \emptyset$. This means that there are no veto players in all simple games. Because of this, to distinguish between the simple games which have veto players, and those games that do not, we introduce the following definition. A simple game is called *weak* if it has at least one veto player, that is, $\mathcal{V} \neq \emptyset$. Each unanimity game ζ_T is weak and the veto players are the members of the coalition T.

In the following proposition we show that, in the class of simple games on closure spaces, the set of supermodular games is the set of unanimity games. This result is the key to the proofs of some theorems in the last two sections of this chapter.

Proposition 5.1.1 *If $\mathcal{L}$ is a closure space and $v \in \Omega(\mathcal{L})$, the following statements are equivalent:*

(a) The game v is supermodular.

(b) The game v is a unanimity game.

Moreover, when the family $\mathcal{L}$ is a convex geometry, then (a) and (b) are equivalent to

(c) The game v is quasi-supermodular.

Proof. First of all, we show that if v is a unanimity game, then v is a supermodular game. Indeed, consider the unanimity game ζ_T, for any nonempty coalition $T \in \mathcal{L}$, and let $A, B \in \mathcal{L}$. We distinguish three cases:

1. If $A \supseteq T$ and $B \supseteq T$, then
$$\zeta_T(A) = 1,\ \zeta_T(B) = 1,\ \zeta_T\left(\overline{A \cup B}\right) = 1 \text{ and } \zeta_T(A \cap B) = 1.$$
2. If $A \supseteq T$ and $B \not\supseteq T$, then
$$\zeta_T(A) = 1,\ \zeta_T(B) = 0,\ \zeta_T\left(\overline{A \cup B}\right) = 1 \text{ and } \zeta_T(A \cap B) = 0.$$
3. If $A \not\supseteq T$ and $B \not\supseteq T$, then
$$\zeta_T(A) = 0,\ \zeta_T(B) = 0,\ \zeta_T\left(\overline{A \cup B}\right) \geq 0 \text{ and } \zeta_T(A \cap B) = 0.$$

Therefore, the supermodularity condition holds in all cases.

Conversely, let $v \in \Omega(\mathcal{L})$ be a supermodular game and let $T \in \mathcal{L}$ such that
$$|T| = \min\{|S| : v(S) = 1\} = \alpha.$$

This coalition T is unique because if there are two coalitions $A, T \in \mathcal{L}$, $A \neq T$ which satisfy $v(T) = v(A) = 1$ and $|T| = |A| = \alpha$ then, the inequality
$$v(T) + v(A) \leq v\left(\overline{T \cup A}\right) + v(T \cap A),$$
lead to $v(T \cap A) = 1$. This contradicts the election of T.

Now, taking the coalition T, for any other $A \in \mathcal{L}$ such that $A \not\supseteq T$, as it holds that
$$v(A) + v(T) \leq v\left(\overline{A \cup T}\right) + v(A \cap T) = 1,$$
we deduce that $v(A) = 0$. Thus, the game v is the unanimity game corresponding to the coalition T, that is, $v = \zeta_T$.

Finally, note that when the game is defined on a convex geometry, the equivalence between (a) and (c) is already proved in the Theorem 3.22. $\square$

The equivalence of (a) and (b) with the affirmation (c) is not true when $\mathcal{L}$ is not a convex geometry. If we consider $N = \{1,2,3,4\}$, the family $\mathcal{L} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{3,4\}, \{1,2,3\}, N\}$ and the game $v : \mathcal{L} \longrightarrow \mathbb{R}$ which has the coalitions $\{1\}, \{2\}, \{3\}$ as minimal winning coalitions, this game is quasi-supermodular but is not a unanimity game.

5.2 Imputations and core

Throughout this section, we assume that the family of feasible coalitions $\mathcal{L}$ is atomic. If we denote by $\{e_i\}_{i=1}^n$ the vectors of the canonical basis of $\mathbb{R}^n$, it is easy to check that there are three possibilities for the imputation set $I(\mathcal{L}, v)$ in a game $v \in \Omega(\mathcal{L})$:

1. $I(\mathcal{L}, v) = \emptyset$, if there are two winning coalitions of cardinality one.
2. $I(\mathcal{L}, v) = \{e_k\}$, if there exists $k \in N$, with $v(\{k\}) = 1$, and $v(\{i\}) = 0$ for all $i \in N$, $i \neq k$.
3. $I(\mathcal{L}, v) = \text{conv}\{e_1, \ldots, e_n\}$, if $v(\{i\}) = 0$ for all $i \in N$.

Thus, the imputation set is a larger set when no other minimal winning coalition is a unitary coalition.

The core of a simple game is completely determined by the veto players. As, in general, the core is

$$Core(\mathcal{L}, v) = \{x \in \mathbb{R}^n : x(N) = v(N), \quad x(S) \geq v(S) \text{ for all } S \in \mathcal{L}\},$$

note that, if $v \in \Omega(\mathcal{L})$, there are some inequalities $x(S) \geq v(S)$ which are redundant. To be exact, the inequalities corresponding to non-unitary coalitions that are losing coalitions or winning coalitions but not minimal are not needed in the description of the core. Indeed, the vectors of the core must verify the inequalities $x_i \geq 0$ for all $i \in N$, and therefore, the corresponding restrictions to non-unitary losing coalitions are redundant. Moreover, if $S \in$

$\mathcal{L}$ is winning but not minimal, there exists a minimal winning coalition S^* such that $S^* \subset S$. In this case, we have

$$x(S) \geq x(S^*) \geq v(S^*) = v(S)$$

and hence, the condition $x(S) \geq v(S)$ is a consequence of $x(S) \geq v(S^*)$ and therefore, it is redundant. From observing this, we can conclude that

$$Core(\mathcal{L}, v) = \{x \in \mathbb{R}^n : x \geq 0,\ x(N) = x(S) = 1 \text{ for all } S \in \mathcal{W}\}.$$

Theorem 5.2.1 *Let $v \in \Omega(\mathcal{L})$ be a simple game on an atomic family. A necessary and sufficient condition so that $Core(\mathcal{L}, v) \neq \emptyset$ is that the game v is weak. Furthermore, in this case,*

$$Core(\mathcal{L}, v) = \{x \in \mathbb{R}^n : x \geq 0,\ x(N) = x(\mathcal{V}) = 1\},$$

where $\mathcal{V}$ is the set of veto players in v.

Proof. *Sufficient condition.* Assume that v is weak, then $\mathcal{V} \neq \emptyset$ and for each $i \in \mathcal{V}$, we consider the corresponding vector e_i of the canonical basis of $\mathbb{R}^n$. Then, we have that $e_i(N) = 1 = v(N)$ and $e_i(S) = v(S)$ for all $S \in \mathcal{W}$ and therefore, $e_i \in Core(\mathcal{L}, v)$.

Necessary condition. We construct a $r \times n$ matrix $A = (a_{ij})$ such that

$$a_{ij} = \begin{cases} 1 & \text{if } j \in S_i, \\ 0 & \text{otherwise,} \end{cases}$$

where $S_1, \ldots, S_r$ are the minimal winning coalitions in the game v. Then $Core(\mathcal{L}, v)$ is the set of vectors satisfying

$$\sum_{j=1}^{n} x_j = 1, \quad \sum_{j=1}^{n} a_{ij} x_j = 1, \quad 1 \leq i \leq r, \qquad x_j \geq 0, \ \ j = 1, \ldots, n.$$

If this set is nonempty, there must be one element $x \neq 0$. Then, if $\mathcal{V} = \emptyset$, each column of the matrix A has at least one entry equal to 0, and taking the sum of the equations corresponding to the rows of Ax we obtain

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = |\mathcal{W}|, \ \text{ with } \alpha_j < |\mathcal{W}| \ \text{ for } 1 \leq j \leq n.$$

Therefore, $(|\mathcal{W}| - \alpha_1)\, x_1 + \cdots + (|\mathcal{W}| - \alpha_n)\, x_n = 0$, and this is a contradiction because $x_j \geq 0$ for $1 \leq j \leq n$ and any other $x_k > 0$ with $1 \leq k \leq n$. Thus, the game v is weak.

Finally, note that if the game is weak and take $i \notin \mathcal{V}$, then $x_i = 0$ for every vector of the core, and hence

$$Core\,(\mathcal{L}, v) = \{x \in \mathbb{R}^n : x \geq 0,\ x\,(N) = x\,(\mathcal{V}) = 1\}\,.$$

□

The previous theorem generalizes an anologous result of Curiel [16] and permits the identification of the extreme points of the polyhedron $Core\,(\mathcal{L}, v)$. As a consequence of this, the following proposition is immediate.

Proposition 5.2.2 *For every weak game $v \in \Omega\,(\mathcal{L})$ on an atomic family,*

$$Core\,(\mathcal{L}, v) = \text{conv}\,\{e_i : i \in \mathcal{V}\}$$

where $\mathcal{V}$ is the set of veto players in v.

Although in this section, we have considered simple games on atomic families, it is easy to check that every weak simple game on any family of feasible coalitions has a nonempty core. However, if we consider weak games on non-atomic families, the Proposition 5.4 does not hold in general. Indeed, if we consider $N = \{1, 2, 3\}$, the family $\mathcal{L} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$ and the unanimity game $\zeta_{\{1,2\}}$, the core is not bounded.

On the other hand, if the family is not atomic, then the core can be nonempty and the game not weak. For example, by taking $N = \{1, 2, 3, 4, 5\}$, $\mathcal{L} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 2, 3, 4, 5\}\}$ and the simple game v for which its minimal winning coalitions are $\{1, 2\}$ and $\{3\}$. In this case, the vector $(1/2, 1/2, 1, -1, 0) \in Core\,(\mathcal{L}, v)$ and v is not weak.

5.3 The Weber set

By considering the Weber set in this section, it is necessary to require, in the following, that the family of feasible coalitions $\mathcal{L}$ is a convex geometry on N, not necessarily atomic.

First of all, we prove that the Weber set is a convex set where its extreme points are determined by the players that by joining a losing coalition is then converted into a winning coalition.

Proposition 5.3.1 *If $v \in \Omega(\mathcal{L})$ is a game on a convex geometry, then*

$$Weber(\mathcal{L}, v) = \text{conv}\left\{ e_i \,:\, i \in \bigcup_{l=1}^{r} ex(S_l) \right\}.$$

where $\{S_1, \ldots, S_r\}$ is the set of minimal winning coalitions in v.

Proof. Let $C \in \mathcal{C}(\mathcal{L})$ be a maximal chain and let $a^C(v) \in \mathbb{R}^n$ be the marginal worth vector associated with this chain. For every $i \in N$ we have that $a_i^C(v) = v(C(i)) - v(C(i) \setminus i) \in \{0, 1\}$ since v is a monotonic game. Moreover $a^C(v)$ is efficient and hence $\sum_{i \in N} a_i^C(v) = 1$. Therefore, the vector $a^C(v) \in \mathbb{R}^n$ has only one of its components equal to 1 and the rest of them equal to 0. If we assume that the component j is equal to 1, i.e., $v(C(j)) = 1$ and $v(C(j) \setminus j) = 0$, then $a^C(v) = e_j$.

On the other hand, $C(j)$ is a winning coalition and so, there exists a minimal winning coalition S_k such that $S_k \subseteq C(j)$. Note that $j \in S_k$, since if $S_k \subseteq C(j) \setminus j$ then $C(j) \setminus j$ is a winning coalition but it is not possible as $v(C(j) \setminus j) = 0$. Moreover, $j \in ex(S_k)$ because $S_k \setminus j = (C(j) \setminus j) \cap S_k \in \mathcal{L}$. Therefore, we have proved that $Weber(\mathcal{L}, v) \subseteq \text{conv}\{e_i : i \in \cup_{l=1}^{r} ex(S_l)\}$ because $Weber(\mathcal{L}, v)$ is the convex hull of marginal worth vectors.

In order to prove the inverse inclusion, if $i \in \bigcup_{l=1}^{r} ex(S_l)$. Then there exists a minimal winning coalition S_l such that $i \in ex(S_l)$. Let $C \in \mathcal{C}(\mathcal{L})$ be a maximal chain such that $S_l = C(i)$. For this chain, we have that $a^C(v) = e_i$. □

Note that if $\mathcal{L} = 2^N$ then

$$Weber(\mathcal{L}, v) = \text{conv}\left\{ e_i \,:\, i \in \bigcup_{l=1}^{r} S_l \right\}$$

and it contains the core. However, when $\mathcal{L}$ is not this convex geometry, this inclusion is not always verified for simple games. Indeed, let $N = \{1, 2, 3, 4\}$,

$\mathcal{L} = Co(\{1,2,3,4\})$, and let $v : \mathcal{L} \longrightarrow \mathbb{R}$ be the simple game which has only one minimal winning coalition $\{1,2,3\}$, then

$$Core(\mathcal{L}, v) = \text{conv}\{e_1, e_2, e_3\} \quad \text{and} \quad Weber(\mathcal{L}, v) = \text{conv}\{e_1, e_3\}.$$

The following result shows that each efficient distribution that satisfies the individual rationality principle and the dummy axiom (see p. 94) is an element of the Weber set.

Theorem 5.3.2 *Let $v \in \Omega(\mathcal{L})$ be a simple game on an atomic convex geometry. If $J \subseteq I(\mathcal{L}, v)$ and the components of the distributions $x = (x_1, \ldots, x_n)$ of J satisfy the dummy axiom, then $J \subseteq Weber(\mathcal{L}, v)$. The converse is true when $I(\mathcal{L}, v) = \text{conv}\{e_1, \ldots, e_n\}$.*

Proof. If $I(\mathcal{L}, v) = \{e_i\}$ the result follows easily. Thus, we assume that $I(\mathcal{L}, v) = \text{conv}\{e_1, \ldots, e_n\}$, i.e., there is no winning coalition of cardinal one and so $v(\{i\}) = 0$ for all $i \in N$.

If $x \in J$, then x is an imputation and so, $x \in \text{conv}\{e_1, \ldots, e_n\}$, and hence

$$x = \sum_{i=1}^{n} \mu_i e_i$$

with $\sum_{i=1}^{n} \mu_i = 1$ and $\mu_i \geq 0$ for $i = 1, \ldots, n$.

By the above theorem, to prove that $x \in Weber(\mathcal{L}, v)$ is equivalent to prove that $\mu_k = 0$ when k is not extreme for any minimal winning coalition. For this, let $k \in N \setminus \bigcup_{j=1}^{r} ex(S_j)$. As the game v is monotonic, we have that $v(S) - v(S \setminus k) \geq 0$ for all $S \in \mathcal{L}$ with $k \in ex(S)$. Moreover, in this case, we assert that $v(S) - v(S \setminus k) = 0$. Indeed, if $v(S) - v(S \setminus k) = 1$, then necessarily $v(S) = 1$ and $v(S \setminus k) = 0$, and hence, there would be a minimal winning coalition $S_j \subseteq S$ with $k \in S_j$, because if $S_j \subseteq S \setminus k$ we would obtain $v(S \setminus k) = 1$, but it is not possible. Therefore, k is a dummy player in v and so, $\mu_k = x_k = v(\{k\}) = 0$.

Reciprocally, let $v \in \Omega(\mathcal{L})$ such that $I(\mathcal{L}, v) = \text{conv}\{e_1, \ldots, e_n\}$ and let $J \subseteq Weber(\mathcal{L}, v)$. Since $Weber(\mathcal{L}, v) \subseteq I(\mathcal{L}, v)$, then we have $J \subseteq I(\mathcal{L}, v)$. Moreover, if $k \in N$ is a dummy player in v and $x \in J$, then $x_k = v(S) - v(S \setminus k)$ for any $S \in \mathcal{L}$ with $k \in ex(S)$, because $x \in Weber(\mathcal{L}, v)$. Therefore, $x_k = v(\{k\})$. □

5.4 Stable sets

In the second chapter, we introduced the notion of dominance and the concept of stable set. In this section, we study some results regarding the stability for simple games $v \in \Omega(\mathcal{L})$ where $\mathcal{L}$ is an atomic family.

First of all, we show that every simple game in which the imputation set is nonempty has at least one stable set.

Theorem 5.4.1 *Every simple game $v \in \Omega(\mathcal{L})$ on an atomic family with nonempty imputation set, has at least one stable set.*

Proof. As $\mathcal{L}$ is atomic and the imputation set is nonempty, we have only two possibilities for $I(\mathcal{L}, v)$. When the imputation set is reduced to a vector, it can be immediately deduced that this set is the unique stable set of the game. For this, we assume that $I(\mathcal{L}, v) = \text{conv}\{e_1, \ldots, e_n\}$. In this case, for each minimal winning coalition S in v, the set formed by the imputations that distribute the unit among the players of S is a stable set, as we prove in the following. If we denote by E_S this set, that is

$$E_S = \left\{ x \in \mathbb{R}^n_+ : \sum_{i \in S} x_i = 1,\ x_j = 0 \text{ for all } j \notin S \right\},$$

we have the following properties:

i) *Internal stability.* No imputation in E_S dominates another. Assume that there are $x, y \in E_S$ such that x *dom* y. Then, there exists a nonempty coalition $T \in \mathcal{L}$, such that

$$x_i > y_i \quad \text{for all } i \in T, \qquad \sum_{j \in T} x_j \leq v(T).$$

These conditions can only be verified if $T \subset S$ and $v(T) = 1$, since $x, y \in E_S$. But it is impossible because S is a minimal winning coalition.

ii) *External stability.* If $x \in I(\mathcal{L}, v) \setminus E_S$, then there exists $y \in E_S$ such that y *dom* x. Indeed, taking into account that $v(S) - x(S) > 0$, upon taking the vector y where its components are

$$y_i = \begin{cases} 0 & \text{if } i \notin S, \\ x_i + \dfrac{v(S) - x(S)}{\mid S \mid} & \text{if } i \in S. \end{cases}$$

□

The following result affirms that the unanimity games are the unique simple games on atomic convex geometries for which the core is a stable set.

Theorem 5.4.2 *Let $v \in \Omega(\mathcal{L})$ be a simple game on an atomic convex geometry. Then, $Core(\mathcal{L}, v)$ is stable if and only if v is a unanimity game.*

Proof. If $Core(\mathcal{L}, v)$ is a stable set, then it must be nonempty and we also know that $Core(\mathcal{L}, v) = \text{conv}\left\{e_i : i \in \bigcap_{j=1}^{r} S_j\right\}$ where $\{S_1, \ldots, S_r\}$ is the set of all minimal winning coalitions with $|S_1| \leq |S_2| \leq \ldots \leq |S_r|$. We consider two cases:

1. If $|S_1| = 1$ or $|S_1| = n$, then $v = \zeta_{S_1}$.

2. Let $1 < |S_1| < n$, and assume that $Weber(\mathcal{L}, v) \not\subseteq Core(\mathcal{L}, v)$. Then there exists $k \in \bigcup_{j=1}^{r} ex(S_j) \setminus \bigcap_{j=1}^{r} S_j$ and so $e_k \notin Core(\mathcal{L}, v)$. As $Core(\mathcal{L}, v)$ is a stable set, there exists $x \in Core(\mathcal{L}, v)$ such that $x \operatorname{dom} e_k$ using a coalition $T \in \mathcal{L}$. Since $x(T) = v(T)$, $x_i > 0$ for all $i \in T$, $i \neq k$, and $x_k > 1$ if $k \in T$, there must be $v(T) = 1$ and $k \notin T$. Thus, there exists a minimal winning coalition $S_{j^*} \in \mathcal{L}$ such that $S_{j^*} \subseteq T$ and $x(S_{j^*}) = v(S_{j^*}) = 1$. So, $x_i > 0$ for all $i \in S_{j^*}$ and as $x(N \setminus S_{j^*}) = 0$, it implies that $x(S_p \setminus S_{j^*}) = 0$ for every minimal winning coalition $S_p \in \mathcal{L}$, $S_p \neq S_{j^*}$. However, as $x \in Core(\mathcal{L}, v)$, $v(S_p) = 1$ holds and hence $x(S_p \cap S_{j^*}) = 1$ for every minimal winning coalition $S_p \in \mathcal{L}$. Then, we assert that $S_{j^*} \subseteq \bigcap_{j=1}^{r} S_j$. Assume not, there exists a minimal winning coalition $S_p \in \mathcal{L}$ such that $S_{j^*} \not\subseteq S_p$ and we have that

$$x(N) = x(S_{j^*} \cap S_p) + x(S_{j^*} \setminus S_p) + \ldots > 1,$$

but it is impossible. Therefore, $\bigcap_{j=1}^{r} S_j = S_{j^*}$ and so $v = \zeta_{S_{j^*}}$.

In order to prove the converse, let $v = \zeta_T$ be a unanimity game. Then, $Core(\mathcal{L}, v) = \text{conv}\{e_i : i \in T\}$ because T is the unique minimal winning coalition. By the above theorem, $E_T = Core(\mathcal{L}, v)$ is a stable set. □

Example 5.4.1 *Consider $N = \{1, 2, 3\}$, and the game $v : 2^N \longrightarrow \mathbb{R}$ given by*

$$v(S) = \begin{cases} 1 & \text{if} \mid S \mid \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $I(2^N, v) = conv\{e_1, e_2, e_3\}$ and $Core(2^N, v) = \emptyset$.

Futhermore (see Driessen [20]), there is an infinity of stable sets. Every set formed by imputations that assign a constant value c to a player i, and determine all the possible distributions of the amount $1 - c$ among the other two players, are stables sets. Each one of these stable sets contains infinite imputations and the following finite set is also stable

$$E = \left\{ \left(\frac{1}{2}, \frac{1}{2}, 0 \right), \left(\frac{1}{2}, 0, \frac{1}{2} \right), \left(0, \frac{1}{2}, \frac{1}{2} \right) \right\}.$$

In this example, we observe that every stable set can be interpreted by a model of behaviour for which the imputations are selected. Furthermore, it suggests that each stable set delimits forms of distribution that the player must negotiate, or indicates a smaller game between the coalitions.

If we now consider the family $\mathcal{L} = Co(\{1, 2, 3\})$, and the game $v \in \Omega(\mathcal{L})$ defined by

$$v(S) = \begin{cases} 1 & \text{if} \mid S \mid \geq 2, \\ 0 & \text{otherwise,} \end{cases}$$

it is interesting to observe which are its stable sets and compare these with the case of $\mathcal{L} = 2^N$. Taking into account that the sets E_S, with $S \in \mathcal{W}$, are stable, in this case, the sets

$$E_{\{1,2\}} = conv\{e_1, e_2\}, \quad E_{\{2,3\}} = conv\{e_2, e_3\}$$

are stable. However, the other stable sets of the case $\mathcal{L} = 2^N$ are now not stable. Furthermore, we can observe that, although the core is not empty, this neither offers reasonable solutions to the game, nor is a stable set.

5.5 Bargaining sets

Although in the literature several concepts of bargaining sets exist, we consider only three of them in the context of simple games, and we show some results that relate them to the core and the Weber set.

The different solution concepts called bargaining sets have in common the study of how to distribute the gain of the cooperation (it is supposed that the cooperation has already been decided) taking into account the negotiation processes that can be established among the different coalitions. That is, it bears in mind that certain players can offer pacts to others with the aim of obtaining more advantageous distributions and at the same time other pacts that could put the ones that do this in danger.

We consider that the family of feasible coalitions $\mathcal{L}$ is an atomic closure space on N. One of the three different bargaining sets considered is the Aumann-Maschler bargaining set. Its definition is based in the *objections* and *counterobjections* of some players against others, and they are introduced when the imputation set is nonempty.

- *The Aumann-Maschler bargaining set (1964)* [1].

Let $i, j \in N$ with $i \neq j$. We denote by Λ_{ij} the set of all feasible coalitions of $\mathcal{L}$ that contains player i but not player j, i.e.,

$$\Lambda_{ij} = \{S \in \mathcal{L} : i \in S,\ j \notin S\}.$$

Note that the set Λ_{ij} is nonempty for all $i, j \in N$ because the family of feasible coalitions $\mathcal{L}$ is atomic.

An *objection* of player i against player j with respect to the imputation x in game v is a pair (y, S), where $S \in \Lambda_{ij}$ and the vector $y = (y_k)_{k \in S}$ satisfy

$$y(S) = v(S) \quad \text{and} \quad y_k > x_k \text{ for each } k \in S.$$

A *counterobjection* of j against the pair (y, S) is a pair (z, T), where $T \in \Lambda_{ji}$ and $z = (z_k)_{k \in T}$ satisfy

$$z(T) = v(T), \text{ and } \begin{cases} z_k \geq x_k & \text{for } k \in T \setminus S, \\ z_k \geq y_k & \text{for } k \in T \cap S. \end{cases}$$

An objection is said to be *justified* if there does not exist a counterobjection.

An imputation $x \in I(\mathcal{L}, v)$ is said that belongs to the bargaining set $\mathcal{B}(\mathcal{L}, v)$ if for each objection of a player against another with respect to x, there exists a counterobjection; that is,

$$\mathcal{B}(\mathcal{L}, v) = \{x \in I(\mathcal{L}, v) : \text{there does not exist a justified objection to } x\}.$$

Note that $Core\,(\mathcal{L}, v) \subseteq \mathcal{B}\,(\mathcal{L}, v)$ as there can not be an objection with respect to an element of the core.

Example 5.5.1 *Consider the game* $v \in \Gamma\left(2^N\right)$ *defined in the example 5.9. For this game, we have* $I(2^N, v) = conv\{e_1, e_2, e_3\}$ *and* $Core(2^N, v) = \emptyset$*. From this, in the case that the cooperation among the players is already decided, how do the players share out the worth of the grand coalition* $v(N) = 1$*? In this case, the core does not determine any distribution. Furthermore, it seems logical to think that not any imputation is accepted as distribution by all players. For example, the imputation* $(1, 0, 0)$ *must not satisfy the players* 2 *and* 3*, since their coalition is evaluated by* 1 *in the game. On the other hand, taking into account the symmetry of the players in this game, it is reasonable to expect that the distribution that is adopted treats each one equally, that is,* $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$*. In this case,* $\mathcal{B}\,(\mathcal{L}, v) = \left\{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\right\}$*. To check this affirmation, observe that each player* i *can object against another player* j *with respect to this imputation using the coalition that he forms with the other player* k*. However, every objection presented has a counterobjection by the player* j *using the coalition* $\{j, k\}$*.*

Note that when the imputation set is empty, then the Aumann-Maschler bargaining set is empty . This occurs, for example, in the case of the game $v : 2^{\{1,2,3\}} \longrightarrow \mathbb{R}$, given by

$$v(S) = \begin{cases} 0, & \text{if } S = \emptyset, \{3\} \\ 1, & \text{otherwise.} \end{cases}$$

In cases like this, in which the imputation set is empty, how the sharing out should be done when the players have decided to form the grand coalition? The following bargaining sets can offer solutions.

- *The Mas-Colell bargaining set (1989)* [43].

Consider the set $X(\mathcal{L},v)=\left\{x\in\mathbb{R}^n_+ : x(N)=v(N)\right\}$ and let $x\in X(\mathcal{L},v)$. An *objection* to x *in the sense of Mas-Colell* is a pair (y,S) with $y\in\mathbb{R}^n_+,\ S\in\mathcal{L}$, satisfying

$$\begin{aligned}&y(S)\leq v(S),\\&y_k\geq x_k \text{ for each } k\in S,\quad \text{with at least one strict inequality.}\end{aligned}$$

A *counterobjection* to the pair (y,S) is a pair (z,T), where $z\in\mathbb{R}^n_+,\ T\in\mathcal{L}$, satisfy

$$\begin{aligned}&z(T)\leq v(T),\\&z_k\geq y_k \text{ for } k\in T\cap S, z_k\geq x_k \text{ for } k\in T\setminus S, \text{ with at least one}\\&\text{strict inequality.}\end{aligned}$$

An objection (y,S) to x is said *to be justified in the sense of Mas-Colell* if there does not exist any counterobjection to (y,S). The *Mas-Colell bargaining set* of v, $\mathcal{MB}(\mathcal{L},v)$, consists of all vectors $x\in X(\mathcal{L},v)$ for which there does not exist (y,S) which is a justified objection to x in the sense of Mas-Colell.

- *The Greenberg bargaining set (1992)* [29].

Consider the set $X^*(\mathcal{L},v)=\left\{x\in\mathbb{R}^n_+ :\ x(N)\leq v(N)\right\}$, and let $x\in X^*(\mathcal{L},v)$. An *objection* to x *in the sense of Greenberg* is a pair (y,S) with $y\in\mathbb{R}^n_+$, $S\in\mathcal{L}$, satisfying

$$\begin{aligned}&y(S)=v(S),\\&y_k>x_k \text{ for all } k\in S,\\&y_k=x_k \text{ for all } k\in N\setminus S.\end{aligned}$$

A *counterobjection* to the pair (y,S) is a pair (z,T), where $z\in\mathbb{R}^n_+$, $T\in\mathcal{L}$, satisfy

$$\begin{aligned}&z(T)=v(T),\\&z_k>y_k \text{ for } k\in T,\\&z_k=y_k \text{ for } k\in N\setminus T.\end{aligned}$$

An objection (y, S) is said *to be justified in the sense of Greenberg* if there does not exist any counterobjection to (y, S). The *modified bargaining set*, $\mathcal{MBS}(\mathcal{L}, v)$, consists of all vectors $x \in X^*(\mathcal{L}, v)$ for which there exist no justified objections in the sense of Greenberg.

We can check that for the last example, we have that the vector $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ belongs to the two last bargaining sets. Moreover, this vector is a reasonable distribution since, in this case, we can understand that the player 3 does not contribute anything to the coalitions in which he participates, but however, players 1 and 2 have the same role in the game.

The three bargaining sets introduced contain the core, as with these definitions, there does not exist objection to vectors of the core. We now study a certain class of simple games in which these sets coincide with the core.

Theorem 5.5.1 *Let $\mathcal{L}$ be an atomic closure space and $v \in \Omega(\mathcal{L})$ a weak simple game. Then*

$$\mathcal{B}(\mathcal{L}, v) = \mathcal{MB}(\mathcal{L}, v) = Core(\mathcal{L}, v).$$

Proof. If $\mathcal{V}$ is the set of veto player, we know, by Theorem 5.3, that

$$Core(\mathcal{L}, v) = \{x \in \mathbb{R}^n : x \geq 0,\ x(N) = x(\mathcal{V}) = 1\}.$$

First, we show that $\mathcal{B}(\mathcal{L}, v) = Core(\mathcal{L}, v)$. Since one inclusion is evident, assume that there is $x \in \mathcal{B}(\mathcal{L}, v) \setminus Core(\mathcal{L}, v)$. Then $x(\mathcal{V}) < 1$ and hence, $\mathcal{V} \neq N$. So, there exists $i \in N \setminus \mathcal{V}$ such that $x_i > 0$ because $x(N) = 1$. If $j \in \mathcal{V}$, then j can object against i in x. Indeed, if $S \in \mathcal{L}$ a winning coalition such that $j \in S$ and $i \notin S$, then $x(S) < 1$. We define $y = (y_k)_{k \in S}$ by

$$y_k = x_k + \frac{1 - x(S)}{|S|}, \quad \text{for all } k \in S.$$

This vector y satisfies that $y(S) = v(S)$ and $y_k > x_k$ for each $k \in S$. Therefore, (y, S) is an objection of j against i in x. We suppose that there exists a counterobjection (z, T) to (y, S). Then $z(T) = v(T)$, $v(T) = 0$ and $i \in T$, and hence $z_i = 0$. But this is a contradiction because $z_i \geq x_i > 0$.

We now prove that $\mathcal{MB}(\mathcal{L}, v) = Core(\mathcal{L}, v)$. If there exists $x \in \mathcal{MB}(\mathcal{L}, v)$ but $x \notin Core(\mathcal{L}, v)$, then $x(\mathcal{V}) < 1$ and hence $\mathcal{V} \neq N$. Therefore, there exists

$i \in N \setminus \mathcal{V}$ such that $x_i > 0$ and $x(N \setminus i) < 1$. Let $T \in \mathcal{L}$ be a winning coalition such that $x(T) = \min\{x(S) : v(S) = 1\}$. Then $x(T) < 1$. We define $y \in \mathbb{R}^n_+$ by

$$y_k = \begin{cases} x_k + \dfrac{1 - x(T)}{|\mathcal{V}|} & \text{if } k \in \mathcal{V}, \\ x_k & \text{if } k \in N \setminus \mathcal{V}. \end{cases}$$

Thus $y_k > x_k$ for each $k \in \mathcal{V}$ and

$$y(T) = x(\mathcal{V}) + 1 - x(T) + x(T \setminus \mathcal{V}) = 1 = v(T).$$

Therefore (y, T) is an objection to x in the sense of Mas-Colell. We show that (y, T) is justified. Assume not, let (z, R) be a counterobjection to (y, T). Then $z(R) \leq v(R) = 1$ and $z_k \geq y_k$ for each $k \in R \cap T$ and $z_k \geq x_k$ for $k \in R \setminus T$ with at least one strict inequality. Therefore,

$$\begin{aligned} z(R) &= z(R \setminus T) + z(R \cap T) > x(R \setminus T) + y(R \cap T) \\ &= y(R \setminus T) + y(R \cap T) = y(R) \\ &= 1 + x(R) - x(T) \geq 1, \end{aligned}$$

which is a contradiction. □

The above result is not true if the game $v \in \Omega(\mathcal{L})$ is not weak. It is enough to see the example 5.9.

Theorem 5.5.2 *Let $\mathcal{L}$ be an atomic convex geometry and let $v \in \Omega(\mathcal{L})$ a weak simple game. Then $\mathcal{MBS}(\mathcal{L}, v) = Core(\mathcal{L}, v)$ if and only if v is a unanimity game.*

Proof. Let $T \in \mathcal{L}$ be a nonempty coalition such that $v = \zeta_T$ and let $x \in \mathcal{MBS}(\mathcal{L}, v)$ such that $x \notin Core(\mathcal{L}, v)$. Then $x(T) < 1$. Define $y \in \mathbb{R}^n_n$ by

$$y_i = \begin{cases} x_i + \dfrac{1 - x(T)}{|T|} & \text{if } i \in T, \\ x_i & \text{if } i \notin T. \end{cases}$$

Then (y, T) is an objection to x in the sense of Greenberg and this objection is justified. Indeed, if (z, S) is a counterobjection to (y, T), then

$v(S) = 1$ and hence $S \supset T$. Therefore, $1 = z(S) > y(S) \geq y(T) = 1$, which is impossible.

To obtain the converse, we will show that $\mathcal{MBS}(\mathcal{L}, v) = Core(\mathcal{L}, v)$ implies that v is a quasi-supermodular game, since the Proposition 5.2 affirms that v is a unanimity game. If v is not a quasi-supermodular game (see Theorem 3.20), then there exists $C \in \mathcal{C}(\mathcal{L})$ a maximal chain such that $a^C(v) \notin Core(\mathcal{L}, v)$. Note that $a_j^C(v) \in \{0, 1\}$ for all $j \in N$ and $\sum_{i \in N} a_i^C(v) = 1$. Then, the vector $a^C(v) \in \mathbb{R}^n$ is equal to a vector of the canonical basis of $\mathbb{R}^n$.

On the other hand, if $\mathcal{V}$ is the set of veto players in v, then $\mathcal{V} \neq \emptyset, N$. Note that $\mathcal{V} \neq \emptyset$ because the game is weak and $\mathcal{V} \neq N$ since we assume that $v \neq \zeta_N$. Futhermore, $Core(\mathcal{L}, v) = \text{conv}\{e_j : j \in \mathcal{V}\}$ and as $a^C(v) \notin Core(\mathcal{L}, v)$ then there exists a player $i \in N \setminus \mathcal{V}$ such that $a^C(v) = e_i$. As $a^C(v) \notin \mathcal{MBS}(\mathcal{L}, v)$, there exists a justified objection (x, S) to $a^C(v)$ in the sense of Greenberg, where $x \in \mathbb{R}^n$, $S \in \mathcal{L}$ satisfy $x \geq 0$, $x(S) = v(S)$, $x_k > a_k^C(v)$ for each $k \in S$ and $x_k = a_k^C(v)$ for $k \in N \setminus S$. Therefore, $x(S) = v(S) = 1$ and $i \notin S$. Let

$$A = \{G \in \mathcal{L} : i \notin G \text{ and there is } j \in S \text{ such that } j \notin G\}.$$

This set is nonempty, and we can distinguish two cases:

1. There exists $G \in A$ such that $v(G) = 1$. In this case

$$\begin{aligned} x(G) &= x(G \cap S) + x(G \cap (N \setminus S)) = x(G \cap S) \\ &= x(S) - x(S \setminus G) < 1. \end{aligned}$$

Define $y \in \mathbb{R}^n$ by

$$y_k = \begin{cases} x_k + \dfrac{x(S \setminus G)}{|G|} & \text{if } k \in G, \\ x_k & \text{if } k \in N \setminus G, \end{cases}$$

then $y \geq 0$, $y_k > x_k$ for $k \in G$, $y_k = x_k$ for $k \in N \setminus G$, and

$$y(G) = x(G) + x(S \setminus G) = x(S) = 1 = v(G).$$

Thus, (y, G) is a counterobjection to (x, S) and this is a contradiction.

2. For all $G \in A$, $v(G) = 0$. In this case $v(T) = 1$ implies $i \in T$ or $S \subseteq T$. Since $v \in \Omega(\mathcal{L})$ is not a quasi-supermodular game, then $v(\mathcal{V}) = 0$. Let $j \in S \setminus \mathcal{V}$. Define $y \in \mathbb{R}^n_+$ by

$$y_k = \begin{cases} 1/(2|\mathcal{V}|) & \text{if } k \in \mathcal{V}, \\ 1/4 & \text{if } k \in \{i, j\}, \\ 0 & \text{otherwise,} \end{cases}$$

then $y \notin Core(\mathcal{L}, v)$. We show that $y \in \mathcal{MBS}(\mathcal{L}, v)$ and will get a contradiction. Let (z, T) be an objection to (y, S). Then $z \in \mathbb{R}^n$ and $T \in \mathcal{L}$ satisfy $z \geq 0$, $z(T) = v(T)$, $z_k > y_k$ for $k \in T$, $z_k = y_k$ for $k \in N \setminus T$. Hence $z(T) = v(T) = 1$ and $\mathcal{V} \subset T$. Consider two possibilities:

(a) If $i \in T$ then $j \notin T$ because $z(T) = 1$. We have $z(N \setminus T) = y(N \setminus T) = 1/4$, $z(N) = 5/4$ and $z_i > y_i = 1/4$ and then $z(N \setminus i) < 1$. Thus, we can construct a counterobjection to (z, T) using a winning coalition T^* such that $i \notin T^*$.

(b) If $i \notin T$ then $S \subseteq T$ and so $j \in T$, hence $z_j > y_j = 1/4$. Therefore, a counterobjection to (z, T) using a winning coalition T^* such that $j \notin T^*$. $\square$

The unanimity games on atomic convex geometries are weak games. If we consider the unanimity game $v = \zeta_T$, with $T \in \mathcal{L}$, $T \neq \emptyset$, then:

(1) $\mathcal{V} = T$,

(2) $Core(\mathcal{L}, v) = \text{conv}\{e_i : i \in T\} = \mathcal{B}(\mathcal{L}, v) = \mathcal{MB}(\mathcal{L}, v) = \mathcal{MBS}(\mathcal{L}, v)$,

(3) $Weber(\mathcal{L}, v) = \text{conv}\{e_i : i \in ex(T)\}$,

(4) $Core(\mathcal{L}, v) = Weber(\mathcal{L}, v) \Longleftrightarrow T = ex(T) \Longleftrightarrow 2^T \subseteq \mathcal{L}$.

Theorem 5.5.3 *Let $v \in \Omega(\mathcal{L})$ be a simple game on an atomic convex geometry. If $\mathcal{B}(\mathcal{L}, v) = Weber(\mathcal{L}, v)$, then v is a unanimity game.*

Proof. If v is not a unanimity game, then it is not quasi-supermodular and so $Weber(\mathcal{L}, v) \not\subseteq Core(\mathcal{L}, v)$. Let $k \in \bigcup_{j=1}^r ex(S_j) \setminus \bigcap_{j=1}^r S_j$, or equivalently $e_k \in Weber(\mathcal{L}, v) \setminus Core(\mathcal{L}, v)$. Then, there exists a minimal winning coalition $S_{j^*} \in \mathcal{L}$ such that $k \notin S_{j^*}$. A player $i \in S_{j^*}$ can object against player k with respect to e_k using the coalition S_{j^*}. Define $y = (y_k)_{k \in S_{j^*}}$ by

$$y_t = \frac{1}{|S_{j^*}|} \qquad \text{if } t \in S_{j^*}.$$

It is clear that $y(S_{j^*}) = v(S_{j^*}) = 1$ and $y_t > 0$ for all $t \in S_{j^*}$. Then, the pair (y, S_{j^*}) is an objection of i against k with respect to $e_k \in Weber(\mathcal{L}, v) = \mathcal{B}(\mathcal{L}, v)$. We show that there exists no counterobjection to this objection. Indeed, the player k cannot make a counterobjection using $R = \{k\}$ because, in this case, he needs $v(\{k\}) = 1$ and then $I(\mathcal{L}, v) = Weber(\mathcal{L}, v) = \mathcal{B}(\mathcal{L}, v)$ and it implies that $v = \zeta_{\{k\}}$.

Moreover, k cannot make a counterobjection using $R \in \mathcal{L}$ with $|R| > 1$ and such that $k \in R,\ i \notin R$ because in this case, there must be a vector $z = (z_t)_{t \in R}$ such that $z(R) = v(R),\ z_t \geq y_t > 0$ for $t \in R \cap S_{j^*},\ z_t \geq 0$ for $t \in R \setminus S_{j^*},\ t \neq k$ and $z_k \geq 1$ but it is impossible. □

Theorem 5.5.4 *Let $v \in \Omega(\mathcal{L})$ be a simple game on an atomic convex geometry. If $Core(\mathcal{L}, v) \neq \emptyset$ and $\mathcal{MB}(\mathcal{L}, v) = Weber(\mathcal{L}, v)$, then v is a unanimity game.*

Proof. If v is not a unanimity game, then $Weber(\mathcal{L}, v) \not\subseteq Core(\mathcal{L}, v)$ and since $Core(\mathcal{L}, v) \neq \emptyset$, then $Core(\mathcal{L}, v) = \mathcal{MB}(\mathcal{L}, v)$. Hence, $Weber(\mathcal{L}, v) \neq \mathcal{MB}(\mathcal{L}, v)$. □

The converses of Theorem 5.13 and 5.14 are not true. If $v = \zeta_T$, with $T \in \mathcal{L}$, $T \neq \emptyset$ such that $2^T \not\subseteq \mathcal{L}$, then $Weber(\mathcal{L}, v) \subset \mathcal{B}(\mathcal{L}, v)$ and $Weber(\mathcal{L}, v) \subset \mathcal{MB}(\mathcal{L}, v)$.

5.6 The Shapley value

In this section, we prove that the only simple games on atomic convex geometries where the Shapley value is a vector of the core are the unanimity games.

For a simple game $v \in \Omega(\mathcal{L})$, where $\mathcal{L}$ is an atomic convex geometry, the expression of the Shapley value $\Phi(v) \in \mathbb{R}^n$ is given, for all $i \in N$, by

$$\Phi_i(v) = \sum_{\{S \in \mathcal{W} : i \in ex(S)\}} \frac{c(S \setminus i)\, c([S, N])}{c(N)} \left[v(S) - v(S \setminus i) \right].$$

where $\mathcal{W}$ is the set of minimal winning coalitions in the game v.

Theorem 5.6.1 *Let $v \in \Omega(\mathcal{L})$ be a simple game on an atomic convex geometry. The game v is a unanimity game if and only if the Shapley value $\Phi(v) \in Core(\mathcal{L}, v)$.*

Proof. Suppose that $\Phi(v) \in Core(\mathcal{L}, v) = \operatorname{conv}\{e_i : i \in \mathcal{V}\}$, then $\Phi_i(v) = 0$ for all $i \in N \setminus \mathcal{V}$. Moreover, the formula of the Shapley value implies that $\Phi_i(v) \neq 0$ for all $i \in \bigcup_{j=1}^{r} ex(S_j)$. In order to prove that v is a unanimity game, we show that v is quasi-supermodular. Assume not, then $Weber(\mathcal{L}, v) \not\subseteq Core(\mathcal{L}, v)$ and there exists $i \in \bigcup_{j=1}^{r} ex(S_j) \setminus \mathcal{V}$. For this player i, we have $\Phi_i(v) = 0$ and $\Phi_i(v) \neq 0$ but it is impossible.

The converse is obvious because $\Phi(v) \in Weber(\mathcal{L}, v)$ and when v is a quasi-supermodular game on a convex geometry, $Weber(\mathcal{L}, v) \subseteq Core(\mathcal{L}, v)$.

□

Chapter 6

Appendix

6.1 Selectope Algorithm

Input: List L of feasible coalitions in increasing order of their cardinals and list V of their corresponding values.

Output: All selectors with their corresponding selections.

Step 0. Initialization: Let N be the number of players, C the number of coalitions and S the number of possible selectors.

Let $L2$ be a list equal to list L including 0 as the last element of each coalition.

Step 1. Calculation of dividends: We create a list D with the dividends for each coalition.

```
For i = 1 until C
 If cardinal of L(i) = 1
    Append V(i) to list D
 Else
  dividend = V(i)
    For j = 1 until i - 1
       If L(i) ∩ L(j) = L(j)
          dividend = dividend - D(j)
       EndIf
    EndFor
  Append dividend to list D
 EndIf
EndFor
```

Step 2. Procedure: We create a list $L3$ containing the possible selectors and a list $M2$ with their corresponding selections.

```
For i = 1 until S
   Laux = {}
   Append to Laux first player of each coalition of L2
   Append Laux to list L3
   M1 = {}
   For j = 1 until N
      acumulator = 0
      For k = 1 until C
         If Laux(k) = j
            acumulator = acumulator + D(k)
         EndIf
      EndFor
      Append acumulator to list M1
   EndFor
   Append M1 to list M2
   Rotate to left last coalition of L2
   If i < S and first player of last coalition of L2 = 0
      index = C
      Do while first player of L2(index) = 0
         Rotate to left coalition L2(index)
         index = index - 1
         Rotate to left coalition L2(index)
      EndDo
   EndIf
EndFor
```

This algorithm have been implemented with the program of symbolic calculus MATHEMATICA. We give an example of the output, where the option 1 shows all the possible selectors and their corresponding selections and the option 2 shows only the non repeated selections.

```
coal := {{1},{2},{3},{1,2},{2,3},{1,2,3}}
values := {0,1,1,2,3,4}
Selectope[1,coal,values]
1. {1, 2, 3, 2, 3, 1} -> {0, 2, 2}
2. {1, 2, 3, 2, 3, 2} -> {0, 2, 2}
3. {1, 2, 3, 2, 3, 3} -> {0, 2, 2}
4. {1, 2, 3, 2, 2, 1} -> {0, 3, 1}
```

5. {1, 2, 3, 2, 2, 2} –> {0, 3, 1}
6. {1, 2, 3, 2, 2, 3} –> {0, 3, 1}
7. {1, 2, 3, 1, 3, 1} –> {1, 1, 2}
8. {1, 2, 3, 1, 3, 2} –> {1, 1, 2}
9. {1, 2, 3, 1, 3, 3} –> {1, 1, 2}
10. {1, 2, 3, 1, 2, 1} –> {1, 2, 1}
11. {1, 2, 3, 1, 2, 2} –> {1, 2, 1}
12. {1, 2, 3, 1, 2, 3} –> {1, 2, 1}

Selectope[2,coal,values]
1. {0, 2, 2}
2. {0, 3, 1}
3. {1, 1, 2}
4. {1, 2, 1}

6.2 Weber Algorithm

Input: List L of the feasible coalitions in increasing order of their cardinals and list of their corresponding values.

Output: All maximal chains and their corresponding marginal worth vectors.

Step 0. Initialization: Let N be the number of players and C the number of coalitions. Let $L2$ be a list whose elements are constituted by coalitions which represent each one of the maximal chains. Initially, we start with the same number of chains as unitary coalitions exist.

Step 1. Calculation of maximal chains: We append to list $L2$ all possible maximal chains and we complete this with what we had initially.

```
For i = N + 1 until C
  Laux = {}
   For j = 1 until last chain of L2
     If last coalition of chain L2(j) ⊂ L(i)
       Append L(i) to chain L2(j)
     Else
       If length of chain L2(j) = cardinal of L(i) and the
       last but one coalition of chain L2(j) ⊂ L(i)
         Laux2 = L2(j) − last coalition of L2(j)
         Append L(i) to chain Laux2
         If Laux2 ∉ Laux
           Append Laux2 to Laux
         EndIf
       EndIf
     EndIf
   EndFor
  If Laux ≠ {}
    Append Laux to list L2
  EndIf
EndFor
```

Step 2: Calculation of marginal worth vectors: We create a list V with the marginal worth vectors corresponding to the maximal chains of $L2$.

```
For i = 1 until last chain of L2
   Laux = {}
   For j = 1 until N
      k = 1
      Do while j ∉ coalition k of L2(i)
         k = k + 1
      EndDo
      If cardinal of coalition k of L2(i) = 1
            Append Value of coalition k to list Laux
      Else
            Append Value of coalition k − Value
            of coalition k − 1 to list Laux
      EndIf
   EndFor
   Append Laux to list V
EndFor
```

In the following example, we see the output when this algorithm is implemented with the program MATHEMATICA. The option 1 shows all maximal chains in the convex geometry so the marginal worth vectors associated; the option 2 shows only the non repeated vectors.

```
coal := {{1},{2},{3},{4},{1,2},{1,3},{1,4},{1,2,3},{1,2,4},{1,2,3,4}}
val := {0,0,0,1,2,1,2,3,1,4}
Weber[1,coal,val]
1. {{1}, {1, 2}, {1, 2, 3}, {1, 2, 3, 4}} -> {0, 2, 1, 1}
2. {{2}, {1, 2}, {1, 2, 3}, {1, 2, 3, 4}} -> {2, 0, 1, 1}
3. {{3}, {1, 3}, {1, 2, 3}, {1, 2, 3, 4}} -> {1, 2, 0, 1}
4. {{4}, {1, 4}, {1, 2, 4}, {1, 2, 3, 4}} -> {1, -1, 3, 1}
5. {{1}, {1, 3}, {1, 2, 3}, {1, 2, 3, 4}} -> {0, 2, 1, 1}
6. {{1}, {1, 4}, {1, 2, 4}, {1, 2, 3, 4}} -> {0, -1, 3, 2}
7. {{1}, {1, 2}, {1, 2, 4}, {1, 2, 3, 4}} -> {0, 2, 3, -1}
8. {{2}, {1, 2}, {1, 2, 4}, {1, 2, 3, 4}} -> {2, 0, 3, -1}

Weber[2,coal,val]
1. {0, -1, 3, 2}
```

2. {0, 2, 1, 1}
3. {0, 2, 3, -1}
4. {1, -1, 3, 1}
5. {1, 2, 0, 1}
6. {2, 0, 1, 1}
7. {2, 0, 3, -1}

Bibliography

[1] Aumann, R., Maschler, M. (1964) The bargaining set for cooperative games, *Advances in Game Theory,* M. Dresher, L. Shapley, A. Tucker (eds.), Princeton University Press, Princeton, New Jersey, 443–471.

[2] Aumann, R., Myerson, R (1988) Endogenous formation of links between players and coalitions: an application of the Shapley value. *The Shapley value* (Ed. A.Roth), Cambridge University Press, Cambridge, United Kingdom, 175–191.

[3] Aumann, R., Dréze, J.H. (1974) Cooperative Games with Coalition Structures, *International Journal of Game Theory* 3, 217–237.

[4] Axelrod, R., (1970) *Conflict of Interest*, Markham, Chicago.

[5] Bergantiños, G., Carreras, F., García-Jurado, I. (1993) Cooperation when Some Players are Incompatible, *ZOR–Methods and Models of Operations Research* 38, 187–201.

[6] Bilbao, J.M. (1998) Axioms for the Shapley value on convex geometries, *European Journal of Operational Research* 110, 368–376.

[7] Bilbao, J.M. (1998) Values and potential of games with cooperation structure, *International Journal of Game Theory* 27, 131–145.

[8] Bilbao, J.M. (1998) Closure spaces and restricted games, *Mathematical Methods of Operations Research* 48 (1) 57–69.

[9] Bilbao, J.M., Edelman P.H. (1998) The Shapley value on convex geometries, to appear in *Discrete Applied Mathematics.*

[10] Bilbao, J.M., Lebrón, E., Jiménez, N. (1998) Probabilistic values on convex geometries, *Annals of Operations Research* 84, 79–95.

[11] Birkhoff, G., Bennett, M. K. (1985) The Convexity Lattice of a Poset, *Order* 2, 223–242.

[12] Birkhoff, G. (1948) *Lattice Theory,* American Mathematical Society, Providence, Rhode Island.

[13] Borm, P., Nouweland, A., Owen, G., Tijs, S. (1993) Cost Allocation and Communication, *Naval Research Logistics* 40, 733–744.

[14] Calvo, E., Lasaga, J. (1997) Probabilistic Graphs and Power Indices: An Application to the Spanish Parliament, *Journal of Theorical Politics* 9 (4), 477–501.

[15] Carreras, F. (1991) Restriction of Simple Games, *Mathematical Social Sciences* 21, 245–260.

[16] Curiel, I. J. (1988) Cooperative game theory and applications, *Ph. Dissertation*, University of Nijmegen, The Netherlands.

[17] Davis, M., Maschler, M. (1965) The kernel of a cooperative game. *Naval Research Logistics Quarterly* 12, 223–259.

[18] Derks, J. (1992) A Short Proof of the Inclusion of the Core in the Weber Set, *International Journal of Game Theory,* 21, 140–150.

[19] Derks, J., Haller, H., Peters, H. (1997) The selectope for cooperative games, *METEOR Research Memoranda* RM/97/016, University of Maastricht, The Netherlands.

[20] Driessen, T. (1988) *Cooperative Games, Solutions and Applications,* Kluwer, Dordrecht, The Netherlands.

[21] Edelman, P.H., Jamison, R.E. (1985) The theory of convex geometries, *Geometriae Dedicata* 19, 247–270.

[22] Edelman, P.H. (1997) A note on voting, *Mathematical Social Sciences.* 34, 37–50

[23] Einy, E., Wettstein, D. (1996). Equivalence between Bargaining sets and the Core in Simple games. *International Journal of Game Theory* 25, 65–71.

[24] Faigle, U. (1989) Cores of Games with Restricted Cooperation, *ZOR–Methods and Models of Operations Research* 33, 405–402.

[25] Faigle, U., Kern, W. (1992) The Shapley Value for Cooperative Games under Precedence Constraints, *International Journal of Game Theory* 21, 249–266.

[26] Faigle, U., Kern, W. (1995) Partition Games and the Core of Hierarchically Convex Cost Games, *preprint.*

[27] Gillies, D.B. (1953) Some Theorems on $n-$Person Games, *Ph. D. Thesis,* Princeton University Press, Princeton, New Jersey.

[28] Grafe, F., Mouleon, A., Iñarra, E. (1995) A Simple Procedure to Compute the Nucleolus of Γ-Component Additive Games, *TOP,* Vol 3, n.2, 235–245.

[29] Greenberg, J. (1992) On the sensitivity of von Neumann and Morgenstern abstract stable sets. The stable set and the individual stable bargaining set, *International Journal of Game Theory* 21, 41–55.

[30] Grötschel, L., Lovász, L., Schrijver, A. (1988) *Geometric Algorithms and Combinatorial Optimization,* Springer–Verlag, New York.

[31] Hammer, P.L., Peled, U.N., Sorensen, S. (1977) Pseudo-Boolean Functions and Game Theory. I. Core Elements and Shapley Value, *Cahiers du CERO,* 19, 159–176.

[32] Harsanyi, J.C. (1963) A Simplified Bargaining Model for the n–Person Cooperative Game, *International Economic Review* 4, 194–220.

[33] Hart, S., Kurz, M. (1983) Endogenous Formation of Coalitions, *Econometrica* 51, 1047–1064.

[34] Hart, S., Kurz, M. (1984) Stable Coalition Structures, en *Coalitions and Collective Action,* M. Holler (ed.), Physica–Verlag, Wuerzburg, Alemania, 235–258.

[35] ICHIISHI, T. (1981) Supermodularity: applications to convex games and to the greedy algorithm for LP, *J. Economic Theory* 25, 283–286.

[36] ISBELL, J. R. (1958) A class of simple games, *Duke Math.J.* 25, 423–429.

[37] KUIPERS, J. (1994) Combinatorial Methods in Cooperative Game Theory, *Ph. D. Thesis,* University of Maastricht, The Netherlands.

[38] KURZ, M. (1988) Coalitional Value, en *The Shapley Value,* A. Roth (ed.), Cambridge University Press, Cambridge, 155–173.

[39] LEVY, A., MCLEAN, R. (1989) Weighted Coalition Structure Values, *Games and Economic Behavior* 1, 234–249.

[40] LÓPEZ, J. (1996) Cooperación parcial en juegos de n-personas. *Ph. D. Thesis,* Universidad de Sevilla.

[41] LUCAS, W.F. (1968) A game with no solution. *Bull. Amer. Math. Soc.* 74, 237–239.

[42] MARIN-SOLANO, J., RAFELS, C. (1997) Convexity versus Average Convexity: Potential, PMAS, the Shapley Value and Simple Games. *preprint.*

[43] MAS-COLELL, A. (1989) An equivalence theorem for a bargaining set. *Journal of Mathematical Economics* 18, 129–138.

[44] MASCHLER, M., PELEG, B., SHAPLEY, L.S. (1979) Geometric properties of the kernel, nucleolus, and related solution concepts. *Methods of Operations Research*, 4, 303–338.

[45] MASCHLER, M., PELEG, B., SHAPLEY, L.S. (1972) The kernel and bargaining set for convex games. *International Journal of Game Theory* 1, 73–93.

[46] MCLEAN, R. (1991) Random Order Coalition Structure Values, *International Journal of Game Theory* 20, 109–127.

[47] MYERSON, R.B. (1977) Graphs and Cooperation in Games, *Mathematics of Operations Research* 2, 225–229.

[48] Myerson, R.B. (1980) Conference Structures and Fair Allocation Rules, *International Journal of Game Theory* 9, 169–182.

[49] Nouweland, A. van den, Borm, P. (1991) On the Convexity of Communication Games, *International Journal of Game Theory* 19, 421–430.

[50] Nouweland, A. van den, Borm, P., Tijs, S. (1992) Allocation Rules for Hypergraph Communication Situations, *International Journal of Game Theory* 20, 255–268.

[51] Nouweland, A. van den, (1993) Games and Graphs in Economic Situations, *Ph. D. Thesis,* Tilburg University, The Netherlands.

[52] Owen, G. (1977) Values of Games with a Priori Unions, *Essays in Mathematical Economics and Game Theory,* R. Henn, O. Moeschlin (eds.), Springer–Verlag, Belin, 76–88.

[53] Owen, G. (1986) Values of Graph–Restricted Games, *SIAM Journal of Algebraic and Discrete Methods* 7, 210–220.

[54] Potters, J., Reijnierse, H. (1995) Γ-Component Additive Games, *International Journal of Game Theory* 24, 49–56.

[55] Rockafellar, R. (1970) *Convex Analysis*, centon University Press, Princeton, New Yersey.

[56] Shapley, L. (1953) A Value for n–Person Games, *Contributions to the Theory of Games,* vol. II, H.W. Kuhn, A.W. Tucker (eds.), Princeton, New Jersey, 307–317.

[57] Shapley, L. (1971) Cores of convex games, *International Journal of Game Theory* 1, 11–26.

[58] Shapley, L. S. (1962) Simples games: an outline of the descriptive theory, *Behavioral Sci.* 7, 59–66.

[59] Shenoy, P. (1979) On Coalition Formation: A Game Theoretic Approach, *International Journal of Game Theory* 8, 133–164.

[60] Schmeidler, D. (1969) The nucleolus of a characteristic function game. *SIAM Journal of Applied Mathematics*, 17, 1163-1170.

[61] Schrijver, A. (1986) *Theory of Linear and Integer Programming,* John Wiley & Sons, New York.

[62] Stanley, R.P. (1986) *Enumerative Combinatorics,* Vol I, Wadsworth.

[63] Tijs, S.H. (1981) Bounds for the core and τ-value, *Game Theory and Mathematical Economics,* Eds. O. Moeschlin and D. Pallaschke North-Holland Publishing company, Amsterdam, 123–132.

[64] Von Neumann, J. (1928) Zur Theorie der Gesellschaftsspiele, *Mathematische Annalen* 100, 295–320.

[65] Von Neumann, J., Morgenstern, O. (1944) *Theory of Games and Economic Behaviour,* Princeton University Press, Princeton, New Jersey.

[66] Weber, R.J. (1978) Probabilistic Values for Games, *Cowles Foundation Discussion Paper* no. 471, Yale University, New Haven, Connecticut.

[67] Weber, R.J. (1988) Probabilistic Values for Games, *The Shapley Value,* A. Roth (ed.), Cambridge University Press, Cambridge, 101–119.

[68] Winter, E. (1988) A Value for Cooperative Games with Level Structure of Cooperation, *International Journal of Game Theory* 18, 227-240.

[69] Winter, E. (1991) On Non–Tranferable Utility Games with Coalition Structure, *International Journal of Game Theory* 20, 53–63.

[70] Winter, E. (1992) The Consistency and Potential for Games with Coalition Structure, *Games and Economic Behaviour* 4,132–144.

[71] Wolfram, S. (1991) *Mathematica: A System for Doing Mathematics by Computer,* Addison–Wesley, Massachusetts.