J Optim Theory Appl (2012) 155:650–668
DOI 10.1007/s10957-012-0077-7

The Myerson Value and Superfluous Supports in Union Stable Systems

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Received: 8 September 2011 / Accepted: 30 April 2012 / Published online: 15 May 2012

Abstract In this paper, the set of feasible coalitions in a cooperative game is given by a union stable system. Well-known examples of such systems are communication situations and permission structures. Two games associated with a game on a union stable system are the restricted game (on the set of players in the game) and the conference game (on the set of supports of the system). We define two types of superfluous support property through these two games and provide new characterizations for the Myerson value. Finally, we analyze inheritance of properties between the restricted game and the conference game.

Keywords Union stable system · Myerson value · Superfluous support property · Restricted game · Conference game

Communicated by Irinel Chiril Dragan.

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1 Introduction

In the classical model of transferable utility games, it is generally assumed that any subset of a set N of players can form a coalition and cooperate. However, many real world situations appear, which require a more refined model which takes into account restrictions in cooperation. In Myerson's [1] model the feasible coalitions are induced by connected subgraphs. This line of research was continued by Owen [2], Borm et al. [3], van den Nouweland et al. [4] and Potters and Reijnierse [5]. However, as stated by Myerson himself, partial cooperation cannot always be modeled by a graph. So, the original communication model has been generalized in several directions, for instance towards *conference structures* by Myerson [6], *hypergraph communication situations* by van den Nouweland et al. [4], and *union stable systems* by Algaba et al. [7, 8].

In Algaba et al. [7, 8], it is assumed that if two feasible coalitions have common players, these players will act as intermediaries between the two coalitions in order to establish meaningful communication in the union of these coalitions. These feasible coalition systems are called *union stable systems*. This mathematical feature is essential and is the only requirement in these systems to establish the cooperation. In fact, different lines of research in the literature of cooperative games can be unified through these systems. For instance, the feasible coalitions coming from graph *communication situations* [1] and *permission structures* (see [9, 10]) are special union stable systems. Furthermore, these systems have a close relation with the hypergraph communication situations [11]. Important properties on the class of union stable systems have been studied by Faigle et al. [12], who in this framework find a meaningful notion of supermodularity that extends Shapley's convex cooperative game model.

The *basis* of a union stable system requires special attention, and it is formed by its *supports*, which in a communication situation are the edges of the graph and the singletons. In fact, the supports are those feasible coalitions that are not the union of two other non-disjoint feasible coalitions. All other feasible coalitions can be written as a union of non-disjoint supports. Two games that play an important role, in games on union stable systems, are the *restricted game* and the *conference game*, generalizing the corresponding games for communication situations. The restricted game is defined on the set of players and assigns to every coalition of players, the worth that they can earn given the cooperation restrictions. The conference game is defined on the set of non-unitary supports (i.e., supports containing at least two players) of the union stable system. This game assigns to every coalition of non-unitary supports, the worth that the 'grand coalition', consisting of all players, can earn in the union stable cooperation structure, generated by these supports. In this paper, we provide axiomatizations of the *Myerson value* for games on union stable systems, studied earlier in Algaba et al. [8], using properties related to superfluous supports. In particular, the new *strong superfluous support property*, inspired by the corresponding axiom for communication situations in van den Nouweland [13] and not satisfied by *Harsanyi power solutions*, in general, is defined using the restricted game. According to this property, the payoff distribution does not depend on the 'relations' of players who are null in the restricted game. Together with *component efficiency*, *component dummy, additivity* and *point unanimity*, we give a characterization of the Myerson value. After that, we show that these axioms also characterize the Myerson value on a special class of union stable systems, which contains those that arise from cycle-free graphs (and on which the position value was characterized in [7]). Moreover, in this case, we can use the weaker *superfluous support property* stating that the payoff allocation should not depend on supports that, in some sense, have no contribution.

Finally, we analyze relations between the restricted game and the conference game. In particular, we consider the inheritance of the properties of balancedness, superadditivity and convexity from one game to the other. Sufficient conditions on the conference game under which the Myerson value is in the core of the restricted game are also provided.

The paper is organized as follows. Section 2 recalls the general concepts on classical cooperation, the main definitions on restricted cooperation by means of union stable systems including the crucial driving notions of basis and supports, the restricted game, the conference game as well as the Myerson value and the position value. In Sect. 3, we first provide an axiomatization on the class of all union stable systems by means of the strong superfluous support property. Next, we give an axiomatization on the special class of union stable systems, which contains the sets of connected coalitions in a cycle-free graph, using the superfluous support property. In Sect. 4, we analyze the inheritance of properties between the restricted game and the conference game. Finally, a section of conclusions is given.

2 Preliminaries

2.1 Cooperative TU-Games

A *cooperative transferable utility (TU)-game* is a pair (N, v), where the set $N = \{1, \dots, n\}$ is a finite set of players, and $v : 2^N \to \mathbb{R}$, with $v(\emptyset) = 0$, is a characteristic function. A game (N, v) is non-negative iff $v(S) \geq 0$, for all $S \subseteq N$.

A distribution of the amount $v(N)$ among the players will be represented by a real-valued vector $x \in \mathbb{R}^n$. Here x_i represents the payoff to player i according to the involved payoff vector x. A *solution* is a real-valued function that assigns a payoff vector to every game (N, v). A solution f satisfies the *efficiency principle* iff $\sum_{j \in N} f_j(v) = v(N)$, for every game (N, v).

The best known solution is the *Shapley value* (Shapley [15]) given by

$$\Phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{(n - |S| - 1)!(|S|)!}{n!} \big(v\big(S \cup \{i\}\big) - v(S)\big), \quad \text{for all } i \in N.$$

A game (N, v) is *superadditive* iff $v(S \cup T) \geq v(S) + v(T)$, for all disjoint coalitions $S, T \in 2^N$, i.e., cooperation is profitable.

A game (N, v) is *convex* iff $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$, for all coalitions $S, T \in 2^N$.

For any non-empty T, $T \subseteq N$, the *unanimity game* on T is the game u_T, given by $u_T(S) = 1$, iff $T \subseteq S$ and $u_T(S) = 0$, otherwise.

2.2 Union Stable Systems

Let $N = \{1, \dots, n\}$ be a finite set of players. We refer to an arbitrary collection of sets $\mathcal{F} \subseteq 2^N$ as a *set system*. Note that N need not to be in $\mathcal{F}$. The set system $\mathcal{F} \subseteq 2^N$ is called *union stable* iff for all $A, B \in \mathcal{F}$ with $A \cap B \neq \emptyset$, it is satisfied that $A \cup B \in \mathcal{F}$.

Many real world situations find their natural framework in these structures. For instance, suppose that player 1 is a homeowner who wants to sell his/her house. Player 1 has signed a contract with a real estate agent that represents player 2. So, player 1 only can sell his/her house by means of player 2. There are two buyers, players 3 and 4. Note that in this economic application, the family of feasible coalitions that can generate a surplus are only those which make possible that the seller can sell his/her house. Therefore, the coalitions which can trade are

$$\mathcal{F} = \bigl\{\{1,2,3\},\{1,2,4\},\{1,2,3,4\}\bigr\}. \tag{1}$$

An important subclass of union stable systems is the class of communication situations, as considered in Myerson [1]. A *communication situation* is a triple (N, v, E), where (N, v) is a game and (N, E) is a simple graph. It is easy to see that the set system $\mathcal{F}$, defined by those coalitions which induce connected subgraphs, is a union stable system. However, in practice, a union stable system cannot always be modeled by a communication situation (see van den Brink [14], for a characterization of the set systems that can be obtained as connected coalitions in a communication graph). For example, the set system $\mathcal{F}$ pointed out above, with one seller/two buyers and a real estate agent as intermediary, is a union stable system which cannot be the set of connected coalitions in a communication graph. So, union stable systems not only allow for a generalization of set systems, derived from communication graphs, but also a better understanding and insight of them and their applications.

Let $\mathcal{F}$ be a union stable system and $\mathcal{G} \subseteq \mathcal{F}$. The following families are defined inductively:

$$\mathcal{G}^{(0)} = \mathcal{G}, \qquad \mathcal{G}^{(n)} = \bigl\{S \cup T : S, T \in \mathcal{G}^{(n-1)}, S \cap T \neq \emptyset\bigr\}, \quad n = 1, 2, \ldots.$$

Note that $\mathcal{G}^{(0)} \subseteq \mathcal{G}^{(n-1)} \subseteq \mathcal{G}^{(n)} \subseteq \mathcal{F}$, since $\mathcal{G} \subseteq \mathcal{F}$ and $\mathcal{F}$ is union stable. We define the collection $\bar{\mathcal{G}}$ by $\bar{\mathcal{G}} = \mathcal{G}^{(k)}$, where k is the smallest integer such that $\mathcal{G}^{(k+1)} = \mathcal{G}^{(k)}$.

For each union stable family $\mathcal{F}$, it is interesting to find a minimal subset $\mathcal{B}(\mathcal{F})$ such that $\overline{B(\mathcal{F})} = \mathcal{F}$. So, the following set is well-defined:

$$\mathcal{E}(\mathcal{F}) = \{G \in \mathcal{F} : G = A \cup B, A \neq G, B \neq G, A, B \in \mathcal{F}, A \cap B \neq \emptyset\}.$$

The set $\mathcal{B}(\mathcal{F}) = \mathcal{F} \setminus \mathcal{E}(\mathcal{F})$, is called the *basis* of $\mathcal{F}$, and the elements of $\mathcal{B}(\mathcal{F})$ are called *supports* of $\mathcal{F}$. We remark that the basis $B(\mathcal{F})$ is the minimal subset of the union stable system $\mathcal{F}$ such that $\overline{B(\mathcal{F})} = \mathcal{F}$ (see Algaba et al. [7]).

Let $\mathcal{G} \subseteq 2^N$ be a set system and let $S \subseteq N$. A set $T \subseteq S$ is called a $\mathcal{G}$-*component* of S iff $T \in \mathcal{G}$ and there exists no $T' \in \mathcal{G}$ such that $T \subset T' \subseteq S$. Therefore, the $\mathcal{G}$-components of S are the maximal feasible coalitions that belong to $\mathcal{G}$ and are contained in S. We denote by $C_{\mathcal{G}}(S)$, the collection of the $\mathcal{G}$-components of S. Union stable systems can be characterized in terms of the $\mathcal{F}$-components of a coalition, in the following way: The set system $\mathcal{F} \subseteq 2^N$ is union stable if and only if for any $S \subseteq N$, with $C_{\mathcal{F}}(S) \neq \emptyset$, the $\mathcal{F}$-components of S are a collection of pairwise disjoint subsets of S.

Let (N, v) be a cooperative game and $\mathcal{F} \subseteq 2^N$ a union stable system. Let $\mathcal{B}(\mathcal{F})$ be the basis of $\mathcal{F}$ and $\mathcal{C}(\mathcal{F}) = \{B \in \mathcal{B}(\mathcal{F}) : |B| \geq 2\}$ be the collection of non-unitary supports, where a feasible coalition is called *non-unitary* when it contains at least two players. If there is no confusion, we will just write $\mathcal{B}$ and $\mathcal{C}$. The *$\mathcal{F}$-restricted game* $v^{\mathcal{F}}: 2^N \to \mathbb{R}$, is defined on the player set and is given by $v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T)$. On the other hand, the *conference game* is defined on the set of supports, with at least two players, of a union stable system and it is the game $(\mathcal{C}, v^{\mathcal{C}})$ where $v^{\mathcal{C}}: 2^{\mathcal{C}} \to \mathbb{R}$, is given by $v^{\mathcal{C}}(\mathcal{A}) = v^{\bar{\mathcal{A}}}(N)$.[1] The term conference game comes from the conference structures model by Myerson [6] who was the first one to point out the necessity of extending communication situations towards more general models. Myerson introduced the term conference to refer to any coalition of two or more players, who can meet to discuss their cooperation plans. Note that the game $(\mathcal{C}, v^{\mathcal{C}})$ is well defined since for each $\mathcal{A} \subseteq \mathcal{C}$, $\bar{\mathcal{A}}$ is a union stable subsystem on N, where there may be some players of the grand coalition, who do not to contribute in the conference game. In fact, this union stable subsystem defines those specific parts of the cooperation structure which control the worth of the grand coalition.

The $\mathcal{F}$-restricted game focuses on the role of a player in creating economic possibilities and establishing meaningful communication among the players, whereas the conference game measures the economic value of the grand coalition, when only specific parts of the cooperation structure are considered.

The two above definitions extend the *point game* (introduced by Myerson [1]) and the *arc game* (introduced by Borm et al. [3]) for communication situations, where for a communication situation (N, v, E) we find that the set of non-unitary supports is given by $\mathcal{C} = \{\{i, j\} : \{i, j\} \in E\}$.

A *union stable cooperation structure* is a triple $(N, v, \mathcal{F})$, where the set $N = \{1, \dots, n\}$ is the set of players, (N, v) is a game with $v : 2^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$, and $\mathcal{F}$ is a union stable system. For convenience, we assume, from now on, that the underlying game (N, v) is zero-normalized, i.e., $v(\{i\}) = 0$, for all $i \in N$.

In the following, the set of all union stable cooperation structures on N will be denoted by $\mathrm{US}^N = \{(N, v, \mathcal{F}) : \mathcal{F} \text{ is union stable}\}$.

We will denote by USI^N a special subclass of US^N, where the following two conditions are satisfied:

(1) For all $S, T \in \mathcal{F}$, with $|S \cap T| \geq 2$ we have $S \cap T \in \mathcal{F}$.
(2) All non-unitary feasible coalitions can be written in a unique way as a union of non-unitary supports.

Note that this subclass of union stable cooperation structures generalizes those communication situations for which the graphs do not contain cycles.

2.3 Allocation Rules for Union Stable Structures

An *allocation rule* on US^N is a map γ that assigns to each union stable cooperation structure, $(N, v, \mathcal{F})$, a payoff vector, $\gamma(N, v, \mathcal{F}) \in \mathbb{R}^n$.

Both the Myerson value and the position value are defined from the Shapley value [15] of the two games, which were defined above, the $\mathcal{F}$-restricted game and the

[1] Although in the beginning of this section, we mentioned that we always take as player set $N = \{1, \dots, n\}$, the definition of the conference game is the only occasion where we deviate from this. Note that the player set in a conference game is still derived from a structure on N.

conference game, respectively. The Myerson value was introduced in Myerson [1], and later extended in [6]. Myerson pointed out the need to generalize this model towards restricted cooperation situations, which cannot be modeled by a graph. This idea has been studied by van den Nouweland et al. [4], and Algaba et al. [8]. So, given a union stable cooperation structure $(N, v, \mathcal{F})$, the *Myerson value* denoted by $\mu(N, v, \mathcal{F}) \in \mathbb{R}^n$ is defined by

$$\mu(N, v, \mathcal{F}) = \Phi\big(N, v^{\mathcal{F}}\big).$$

The position value for graph communication situations was first introduced in Meessen [16], and studied in Borm et al. [3]. This value was extended to hypergraph communication situations in [4], and it is defined in union stable cooperation structures in [7]. Let $(N, v, \mathcal{F})$ be a union stable cooperation structure. The *position value*, denoted by $\pi(N, v, \mathcal{F}) \in \mathbb{R}^n$, is given by

$$\pi_i(N, v, \mathcal{F}) = \sum_{C \in \mathcal{C}_i(\mathcal{F})} \frac{1}{|C|} \Phi_C\big(\mathcal{C}, v^{\mathcal{C}}\big), \quad \text{for } i \in N,$$

where $\mathcal{C}_i(\mathcal{F}) = \{C \in \mathcal{C} : i \in C\}$. When there is no confusion, we will often write $\mathcal{C}_i$ instead of $\mathcal{C}_i(\mathcal{F})$.

The following example illustrates the concepts of Myerson and position values for the economic application introduced in (1).

Let $N = \{1, 2, 3, 4\}$ and $\mathcal{F} = \{\{1, 2, 3\}, \{1, 2, 4\}, N\}$ which is a union stable system. Let $v : 2^N \to \mathbb{R}$ be defined by $v(S) = |S| - 1$, if $|S| \geq 1$. Then,

$$\mathcal{B} = \mathcal{C} = \big\{\{1, 2, 3\}, \{1, 2, 4\}\big\}.$$

In this case,

$$v^{\mathcal{F}}(S) = \begin{cases} |S| - 1, & \text{if } S \in \mathcal{F}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the Myerson value for this situation is

$$\mu(N, v, \mathcal{F}) = \left(\frac{13}{12}, \frac{13}{12}, \frac{5}{12}, \frac{5}{12}\right).$$

The conference game is given by

$\mathcal{A} \subseteq \mathcal{C}$	$\bar{\mathcal{A}}$	$C_{\bar{\mathcal{A}}}(N)$	$v^{\mathcal{C}}(\mathcal{A})$
$\{\{1, 2, 3\}\}$	$\{\{1, 2, 3\}\}$	$\{\{1, 2, 3\}\}$	2
$\{\{1, 2, 4\}\}$	$\{\{1, 2, 4\}\}$	$\{\{1, 2, 4\}\}$	2
$\{\{1, 2, 3\}, \{1, 2, 4\}\}$	$\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$	$\{\{1, 2, 3, 4\}\}$	3

Thus,

$$\pi(N, v, \mathcal{F}) = \left(1, 1, \frac{1}{2}, \frac{1}{2}\right).$$

3 The Myerson Value and Superfluous Supports

In [8] the classical axiomatization of the Myerson value is given. In this section, we focus on two *superfluous support* properties.

First, we recall some standard axioms. An allocation rule $\gamma : \mathrm{US}^N \to \mathbb{R}^n$ is called *component efficient* iff for all $(N, v, \mathcal{F}) \in \mathrm{US}^N$, with $M \in C_{\mathcal{F}}(N)$, it is satisfied that $\sum_{i \in M} \gamma_i(N, v, \mathcal{F}) = v(M)$.

An allocation rule $\gamma : \mathrm{US}^N \to \mathbb{R}^n$ satisfies the *component dummy* property iff for all players $i \notin \bigcup_{M \in C_{\mathcal{F}}(N)} M$, we have $\gamma_i(N, v, \mathcal{F}) = 0$.

An allocation rule $\gamma : \mathrm{US}^N \to \mathbb{R}^n$ is *additive* iff

$$\gamma(N, v + w, \mathcal{F}) = \gamma(N, v, \mathcal{F}) + \gamma(N, w, \mathcal{F}),$$

for all $(N, v, \mathcal{F}), (N, w, \mathcal{F}) \in \mathrm{US}^N$.

The Myerson value satisfying these three properties is shown in Algaba et al. [8]. There, it is shown that the Myerson value also satisfies point anonymity. A union stable structure $(N, v, \mathcal{F})$ is called *point anonymous* iff there exists a function $f : \{0, 1, \dots, |D(\mathcal{F})|\} \to \mathbb{R}$, such that $v^{\mathcal{F}}(S) = f(|S \cap D(\mathcal{F})|)$, for all coalition $S \subseteq N$, where $D(\mathcal{F}) = \{i \in N : \mathcal{C}_i \neq \emptyset\}$. When there is no confusion, we will often write D instead of $D(\mathcal{F})$.

An allocation rule γ satisfies *point anonymity* iff for all point anonymous $(N, v, \mathcal{F})$, there exists $\alpha \in \mathbb{R}$ such that

$$\gamma_i(N, v, \mathcal{F}) = \begin{cases} \alpha, & \text{for all } i \in D, \\ 0, & \text{otherwise.} \end{cases}$$

Weakening point anonymity, in a similar way, as the corresponding property for communication structures of Borm et al. [3] is weakened by van den Brink et al. [17], yields point unanimity. A union stable structure $(N, v, \mathcal{F})$ is called *point unanimous* iff it is point anonymous with $f : \{0, 1, \dots, |D|\} \to \mathbb{R}$, such that $f(k) = 0$, for all $k \in \{0, 1, \dots, |D| - 1\}$, i.e., the restricted game is a multiple of the unanimity game on D.

An allocation rule γ satisfies *point unanimity* iff for all point unanimous $(N, v, \mathcal{F})$, there exists $\alpha \in \mathbb{R}$ such that

$$\gamma_i(N, v, \mathcal{F}) = \begin{cases} \alpha, & \text{for all } i \in D, \\ 0, & \text{otherwise.} \end{cases}$$

Since the Myerson value satisfies point anonymity, it also satisfies the weaker point unanimity.

3.1 Axiomatization of the Myerson Value on US^N Using the Strong Superfluous Support Property

We now introduce the strong superfluous support property which is based on the restricted game. An allocation rule $\gamma : \mathrm{US}^N \to \mathbb{R}^n$ satisfies the *strong superfluous* *support property* iff for all $(N, v, \mathcal{F}) \in \mathrm{US}^N$ and for all non-unitary supports $H \in \mathcal{C}$ satisfying

$$v^{\mathcal{F}}(S) = v^{\overline{\mathcal{B}\setminus\{H\}}}(S), \quad \text{for all } S \subseteq N,$$

i.e., whose absence did not influence the $\mathcal{F}$-restricted game, we have

$$\gamma(N, v, \mathcal{F}) = \gamma\bigl(N, v, \overline{\mathcal{B} \setminus \{H\}}\bigr).$$

According to this new property, the outcome of the solution is the same after removing a support, which has no influence on the communication, in the sense that it does not change the worth of coalitions in the restricted game.

Before establishing the new axiomatic characterization of the Myerson value, on the class of all union stable structures, we prove an interesting result which ensures that for all $(N, v, \mathcal{F}) \in \mathrm{US}^N$, the outcome is the same as for the restricted game, for any allocation rule satisfying component efficiency, the component dummy property, additivity and point unanimity.

Lemma 3.1 *If γ is an allocation rule that satisfies additivity, component efficiency, the component dummy property and point unanimity, then*

$$\gamma(N, v, \mathcal{F}) = \gamma\bigl(N, v^{\mathcal{F}}, \mathcal{F}\bigr),$$

for all $(N, v, \mathcal{F}) \in \mathrm{US}^N$.

Proof It suffices to show that $\gamma(N, v - v^{\mathcal{F}}, \mathcal{F}) = 0$, for all $(N, v, \mathcal{F}) \in \mathrm{US}^N$, since γ is additive. First $(N, v - v^{\mathcal{F}}, \mathcal{F})$ is point unanimous, $(v - v^{\mathcal{F}})^{\mathcal{F}}(S) = 0$, for all $S \subseteq N$. (Indeed, let $f : \{0, 1, \ldots, |D|\} \to \mathbb{R}$, such that $f(k) = 0$, with $k = 0, 1, \ldots, |D|$. Then, $(v - v^{\mathcal{F}})^{\mathcal{F}}(S) = f(|S \cap D|) = 0$, for all $S \subseteq N$.) Therefore, applying point unanimity, there exists $\alpha \in \mathbb{R}$ such that

$$\gamma_i\bigl(N, v - v^{\mathcal{F}}, \mathcal{F}\bigr) = \begin{cases} \alpha, & \text{if } i \in D, \\ 0, & \text{otherwise.} \end{cases}$$

Let $M \in C_{\mathcal{F}}(N)$. As γ satisfies component efficiency, we have

$$\sum_{i \in M} \gamma_i\bigl(N, v - v^{\mathcal{F}}, \mathcal{F}\bigr) = \alpha|M| = \bigl(v - v^{\mathcal{F}}\bigr)(M) = 0,$$

and thus, $\alpha = 0$. Hence, $\gamma_i(N, v - v^{\mathcal{F}}, \mathcal{F}) = 0$, for all $i \in \bigcup_{M \in C_{\mathcal{F}}(N)} M$. Finally, the component dummy property implies that $\gamma_i(N, v - v^{\mathcal{F}}, \mathcal{F}) = 0$, for all players i, such that $i \notin \bigcup_{M \in C_{\mathcal{F}}(N)} M$. □

Obviously, the above lemma is satisfied when substituting point unanimity by point anonymity.

In the proof of the characterization below, we use a characterization of the Myerson value given in [8] using the so-called superfluous player property. Let $(N, v, \mathcal{F}) \in \mathrm{US}^N$. An allocation rule γ satisfies the *superfluous player property* iff for all $(N, v, \mathcal{F})$, and every player $i \in N$, that is a null player in $v^{\mathcal{F}}$ (i.e., $v^{\mathcal{F}}(S) = v^{\mathcal{F}}(S \setminus \{i\})$, for all $S \subseteq N$) it holds $\gamma(N, v, \mathcal{F}) = \gamma(N, v, \mathcal{F}_{N\setminus\{i\}})$, where $\mathcal{F}_{N\setminus\{i\}} = \{F \in \mathcal{F} : F \subseteq N \setminus \{i\}\}$.

Theorem 3.1 *The Myerson value* $\mu : \mathrm{US}^N \to \mathbb{R}^n$ *is the unique allocation rule on* US^N *that satisfies component efficiency, the component dummy property, additivity, the strong superfluous support property and point unanimity.*

Proof By definition, it is straightforward that the Myerson value satisfies the strong superfluous support property. The Myerson value satisfying component efficiency, the component dummy property, additivity and point anonymity (and thus also point unanimity) is already shown in Algaba et al. [8]. Therefore, it only remains to prove uniqueness. Let γ be an allocation rule on US^N that satisfies component efficiency, the component dummy property, additivity, the strong superfluous support property and point unanimity. To prove that γ is uniquely determined, we first prove that if an allocation rule on US^N satisfies the strong superfluous support property, then it satisfies the superfluous player property. Hence, as the Myerson value is the unique allocation rule on US^N that satisfies additivity, the superfluous player property and point unanimity [8, Theorem 3.7], we deduce that $\mu = \gamma$. (In particular, this characterization was shown with point anonymity, but it is easy to check that the result holds with point unanimity.)

Let $i \in N$ and $(N, v, \mathcal{F}) \in \mathrm{US}^N$ such that

$$v^{\mathcal{F}}(S) = v^{\mathcal{F}}\big(S \setminus \{i\}\big), \quad \text{for all } S \subseteq N. \tag{2}$$

We have to prove that

$$\gamma(N, v, \mathcal{F}) = \gamma(N, v, \mathcal{F}_{N\setminus\{i\}}),$$

where $\mathcal{F}_{N\setminus\{i\}} = \{F \in \mathcal{F} : F \subseteq N \setminus \{i\}\} = \{F \in \mathcal{F} : F \subseteq N, i \notin F\} = \overline{\mathcal{B} \setminus \mathcal{B}_i}$ with $\mathcal{B}_i = \{B \in \mathcal{B} : i \in B\}$.

As γ is an additive allocation rule that satisfies component efficiency, the component dummy property and point unanimity, by Lemma 3.1 we have

$$\gamma(N, v, \mathcal{F}) = \gamma\big(N, v^{\mathcal{F}}, \mathcal{F}\big). \tag{3}$$

Note that $C_{\mathcal{F}}(S \setminus \{i\}) = C_{\mathcal{F}_{N\setminus\{i\}}}(S)$, for all $S \subseteq N$. Hence, if $i \in N$ satisfies (2), then for all $S \subseteq N$

$$v^{\mathcal{F}}(S) = v^{\mathcal{F}}\big(S \setminus \{i\}\big) = \sum_{T \in C_{\mathcal{F}}(S\setminus\{i\})} v(T) = \sum_{T \in C_{\mathcal{F}_{N\setminus\{i\}}}(S)} v(T) = v^{\mathcal{F}_{N\setminus\{i\}}}(S).$$

Thus, applying (3),

$$\gamma(N, v, \mathcal{F}) = \gamma\big(N, v^{\mathcal{F}}, \mathcal{F}\big) = \gamma\big(N, v^{\mathcal{F}_{N\setminus\{i\}}}, \mathcal{F}\big).$$

As $\mathcal{F}_{N\setminus\{i\}} = \overline{\mathcal{B} \setminus \mathcal{B}_i}$ and $v^{\mathcal{F}} = v^{\mathcal{F}_{N\setminus\{i\}}}$, then any support $B \in \mathcal{B}_i$ has no influence in the $\mathcal{F}$-restricted game, since for all $S \subseteq N$,

$$v^{\mathcal{F}}(S) = v^{\mathcal{F}_{N\setminus\{i\}}}(S) = v^{\overline{\overline{\mathcal{B}\setminus\mathcal{B}_i}\setminus\{B\}}}(S),$$

because $\overline{\mathcal{B} \setminus \mathcal{B}_i} = \overline{\overline{\mathcal{B} \setminus \mathcal{B}_i} \setminus \{B\}}$.

Applying the strong superfluous support property repeatedly, for all supports $B \in \mathcal{B}_i$, we have

$$\gamma(N, v, \mathcal{F}) = \gamma\big(N, v^{\mathcal{F}_{N\setminus\{i\}}}, \mathcal{F}\big) = \gamma\big(N, v^{\mathcal{F}_{N\setminus\{i\}}}, \mathcal{F}_{N\setminus\{i\}}\big) = \gamma(N, v, \mathcal{F}_{N\setminus\{i\}}).$$

Therefore, γ is an allocation rule that satisfies the superfluous player property, and we conclude

$$\gamma(N, v, \mathcal{F}) = \mu(N, v, \mathcal{F}), \quad \text{for all } (N, v, \mathcal{F}) \in \mathrm{US}^N. \qquad \square$$

Taking into account that point anonymity implies point unanimity, the Myerson value $\mu : \mathrm{US}^N \to \mathbb{R}^n$ is the unique allocation rule on US^N that satisfies component efficiency, the component dummy property, additivity, the strong superfluous support property and point anonymity.

By satisfying the strong superfluous support property and point anonymity, the Myerson value distinguishes itself from other *Harsanyi power solutions* (introduced for communication situations in van den Brink et al. [17], and generalized to games on union stable systems by Algaba et al. [18]) which do not satisfy these properties, in general.

3.2 Axiomatization of the Myerson Value on USI^N Using the Superfluous Support Property

Next, we characterize the Myerson value on the subclass USI^N of union stable cooperation structures, where the position value was characterized (see [7]) using a weaker superfluous support property.

The support $H \in \mathcal{C}$ is *superfluous*, for $(N, v, \mathcal{F}) \in \mathrm{US}^N$ iff it is satisfied $v^{\mathcal{C}}(\mathcal{A}) = v^{\mathcal{C}}(\mathcal{A} \setminus \{H\})$, for all $\mathcal{A} \subseteq \mathcal{C}$, i.e., if support H is a null player in the conference game. An allocation rule $\gamma : \mathrm{US}^N \to \mathbb{R}^N$ satisfies the *superfluous support property* iff $\gamma(N, v, \mathcal{F}) = \gamma(N, v, \overline{\mathcal{B} \setminus \{H\}})$, for all $(N, v, \mathcal{F}) \in \mathrm{US}^N$ and for every superfluous support $H \in \mathcal{C}$, for $(N, v, \mathcal{F})$.

Firstly, before showing that the Myerson value verifies the superfluous support property, we establish the link between the *conference game* and the *restricted game*, which will be applied to state the superfluous support property for the Myerson value.

Proposition 3.1 *Let $(N, v, \mathcal{F})$ be a union stable cooperation structure and $(\mathcal{C}, v^{\mathcal{C}})$ the associated conference game. Then, for each $S \subseteq N$ we have*

$$v^{\mathcal{F}}(S) = v^{\mathcal{C}}(\mathcal{C}_S),$$

where $\mathcal{C}_S = \{C \in \mathcal{C} : C \subseteq S\}$.

Proof Let $S \subseteq N$. If $C_{\mathcal{F}}(S) = \emptyset$, it is straightforward that $\mathcal{C}_S = \emptyset$, and thus, $v^{\mathcal{F}}(S) = 0$. Otherwise, $v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T)$. Consider the collection $\mathcal{F}_S = \{F \in \mathcal{F} : F \subseteq S\}$. Then, $\mathcal{F}_S$ is a union stable system such that $C_{\mathcal{F}_S}(N) = C_{\mathcal{F}}(S)$. Its basis is $\mathcal{B}_S = \{B \in \mathcal{B} : B \subseteq S\}$. Let $\mathcal{C}_S$ be the set formed by the supports of $\mathcal{B}_S$ which have cardinality at least two. It holds that $\mathcal{C}_S \subseteq \mathcal{C}$. Therefore, for any coalition $S \subseteq N$, we have

$$v^{\mathcal{C}}(\mathcal{C}_S) = v^{\bar{\mathcal{C}}_S}(N) = v^{\bar{\mathcal{B}}_S}(N) = \sum_{M \in C_{\bar{\mathcal{B}}_S}(N)} v(M) = \sum_{T \in C_{\mathcal{F}}(S)} v(T),$$

since, $C_{\mathcal{F}_S}(N) = C_{\mathcal{F}}(S)$, and the game (N, v) is zero-normalized. □

Proposition 3.2 *Let $H \in \mathcal{C}$ be a superfluous support for the union stable structure $(N, v, \mathcal{F})$. Then $v^{\mathcal{F}}(S) = v^{\overline{\mathcal{B}\setminus\{H\}}}(S)$, for all $S \subseteq N$.*

Proof Let $H \in \mathcal{C}$ be a superfluous support for $(N, v, \mathcal{F}) \in \mathrm{US}^N$.

Proposition 3.1 implies that $v^{\mathcal{C}}(\mathcal{C}_S) = v^{\mathcal{F}}(S)$.

Let $\mathcal{F}'_S = \{F \in \overline{\mathcal{B} \setminus \{H\}} : F \subseteq S\}$, and let $\mathcal{B}'_S$ be the basis of $\mathcal{F}'_S$. Then, we have $\mathcal{B}'_S = \mathcal{B}_S \setminus \{H\}, \mathcal{C}'_S \subseteq \mathcal{C}$, and

$$v^{\mathcal{C}}(\mathcal{C}'_S) = v^{\overline{\mathcal{C}'_S}}(N) = v^{\mathcal{F}'_S}(N) = v^{\overline{\mathcal{B}\setminus\{H\}}}(S).$$

The fact that $\mathcal{B}'_S = \mathcal{B}_S \setminus \{H\}$, and $H \in \mathcal{C}$ is a superfluous support imply

$$v^{\mathcal{F}}(S) = v^{\mathcal{C}}(\mathcal{C}_S) = v^{\mathcal{C}}(\mathcal{C}_S \setminus \{H\}) = v^{\mathcal{C}}(\mathcal{C}'_S) = v^{\overline{\mathcal{B}\setminus\{H\}}}(S).$$ □

Note that the superfluous support property is defined using the conference game (focussing on the role of the supports), while the strong superfluous support property was defined using the restricted game (centering on the role of the players).

Although the strong superfluous support property implies the superfluous support property, the reverse is not true. Indeed, by Proposition 3.2, if an allocation rule satisfies the strong superfluous support property then

$$\gamma(N, v, \mathcal{F}) = \gamma(N, v, \overline{\mathcal{B} \setminus \{H\}}).$$

However, if $H \in \mathcal{C}$ satisfies that $v^{\mathcal{F}}(S) = v^{\overline{\mathcal{B}\setminus\{H\}}}(S)$, for all $S \subseteq N$ then the support H is not necessarily a superfluous support.

Next, we show that the axioms of Theorem 3.1 also characterize the Myerson value on the class USI^N, where the axioms on USI^N are defined, in a similar way, as before on the full class US^N. Moreover, in this case, we can even replace the strong superfluous support property by the weaker superfluous support property. Note that this is rather surprising since axiomatizations of a solution on one class usually do not characterize this solution on a subclass (because uniqueness is not guaranteed). In our case, we even can do it with weaker axioms, in particular, with a weaker superfluous support property.

Theorem 3.2 *The Myerson value $\mu : \mathrm{USI}^N \to \mathbb{R}^n$ is the unique allocation rule on USI^N that satisfies component efficiency, the component dummy property, additivity, the superfluous support property and point unanimity.*

Proof By Theorem 3.1, the Myerson value satisfies component efficiency, the component dummy property, additivity, point unanimity and the strong superfluous support property, and thus, also the superfluous support property.

To show uniqueness, let $(N, v, \mathcal{F}) \in \mathrm{USI}^N$ and let $\gamma : \mathrm{USI}^N \to \mathbb{R}^n$ satisfy the above properties. Since the game v is zero-normalized, it can be expressed as

$$v = \sum_{\{T:|T|\geq 2\}} \beta_T u_T, \quad \text{with } u_T \text{ the unanimity game on } T.$$

As γ is additive, it is sufficient to show that $\gamma(N, \beta u_T, \mathcal{F})$, $\beta \in \mathbb{R}$, is unique for all $T \subseteq N$, with $|T| \geq 2$. To prove this, fix $T \subseteq N$, with $|T| \geq 2$. We distinguish two cases

Case 1. There exists no coalition $S \in \mathcal{F}$ such that $T \subseteq S$.

Since $(\beta u_T)^{\mathcal{F}}$ is the *null game*, i.e., $(\beta u_T)^{\mathcal{F}}(S) = 0$, for all $S \subseteq N$, the triple $(N, \beta u_T, \mathcal{F})$ is point unanimous (there exists $f : \{0, 1, \ldots, |D|\} \to \mathbb{R}$, with $f(0) = \cdots = f(|D|) = 0$, such that $(\beta u_T)^{\mathcal{F}}(H) = f(|H \cap D|) = 0$, for all $H \subseteq N$). By point unanimity of γ, there exists $\alpha \in \mathbb{R}$ such that

$$\gamma_i(N, \beta u_T, \mathcal{F}) = \begin{cases} \alpha, & \text{if } i \in D, \\ 0, & \text{otherwise.} \end{cases}$$

Applying component efficiency and adding over components, we have

$$\sum_{i \in N} \gamma_i(N, \beta u_T, \mathcal{F}) = |D|\alpha = (\beta u_T)^{\mathcal{F}}(N) = 0.$$

If $D \neq \emptyset$, we have $\alpha = 0$, and, therefore

$$\gamma_i(N, \beta u_T, \mathcal{F}) = 0, \quad \text{for all } i \in N.$$

If $D = \emptyset$, then for each $i \in N$, we have two possibilities.

If $\{i\} \notin \mathcal{F}$ then $i \notin \bigcup_{M \in C_{\mathcal{F}}(N)} M$ and $\gamma_i(N, \beta u_T, \mathcal{F}) = 0$, by the component dummy property.

If $\{i\} \in \mathcal{F}$ then $\{i\} \in C_{\mathcal{F}}(N)$ and, applying component efficiency

$$\gamma_i(N, \beta u_T, \mathcal{F}) = (\beta u_T)^{\mathcal{F}}\big(\{i\}\big) = 0,$$

since $|T| \geq 2$. In any case, $\gamma_i(N, \beta u_T, \mathcal{F}) = 0$, for all $i \in N$.

Case 2. There exists a coalition $S \in \mathcal{F}$ such that $T \subseteq S$.

Consider the collection $\{F \in \mathcal{F} : T \subseteq F\}$, which is non-empty, since there exists $S \in \mathcal{F}$ such that $T \subseteq S$, and let

$$\bar{T} = \bigcap \{F \in \mathcal{F} : T \subseteq F\}.$$

Since $\mathcal{F} \in \mathrm{USI}^N$, the set $\bar{T}$ is feasible. Moreover, $\bar{T}$ is non-empty and it is the minimal feasible set that contains T. Hence,

$$(\beta u_T)^{\mathcal{F}}(H) = \beta u_{\bar{T}}(H) = \begin{cases} \beta, & \text{if } \bar{T} \subseteq H, \\ 0, & \text{otherwise.} \end{cases}$$

Since

(i)

$$(\beta u_T)^{\mathcal{C}}(\mathcal{A}) = \sum_{M \in C_{\bar{\mathcal{A}}}(N)} (\beta u_T)(M) = \sum_{M \in C_{\bar{\mathcal{A}}}(N)} \beta u_T(M),$$

(ii) $\sum_{M \in C_{\bar{\mathcal{A}}}(N)} \beta u_T(M) = \beta$ if and only if $u_T(M) = 1$, for some component $M \in C_{\bar{\mathcal{A}}}(N)$,

(iii)

$$u_T(M) = 1 \iff T \subseteq M, \quad \text{with } M \in \bar{\mathcal{A}} \subseteq \mathcal{F},$$

(iv) $\bar{T} \in \mathcal{F}$ is the smallest feasible set that contains T, we have

$$u_T(M) = 1 \iff T \subseteq \bar{T} \subseteq M, \quad \text{with } M \in \bar{\mathcal{A}} \subseteq \mathcal{F},$$

it follows that the conference game, associated to βu_T, is $(\beta u_T)^{\mathcal{C}} : 2^{\mathcal{C}} \to \mathbb{R}$, given by

$$(\beta u_T)^{\mathcal{C}}(\mathcal{A}) = \begin{cases} \beta, & \text{if } \bar{T} \subseteq M,\ M \in \bar{\mathcal{A}}, \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, because the expression of each non-unitary feasible coalition, as a union of non-unitary supports, is unique, there exist coalitions $B_k \in \mathcal{C},\ k \in K$, $M_i \in \mathcal{C},\ i \in I$, such that $\bar{T} = \bigcup_{k \in K} B_k$ and $M = \bigcup_{i \in I} M_i$. But then

$$\bar{T} \subseteq M \iff \{B_k\}_{k \in K} \subseteq \{M_i\}_{i \in I}.$$

As $M \in \bar{\mathcal{A}}$, $\bar{\mathcal{A}} \subseteq \mathcal{F}$, we have

$$\bar{T} \subseteq M, M \in \bar{\mathcal{A}} \iff \{B_k\}_{k \in K} \subseteq \mathcal{A}.$$

Hence,

$$(\beta u_T)^{\mathcal{C}}(\mathcal{A}) = \begin{cases} \beta, & \text{if } \{B_k\}_{k \in K} \subseteq \mathcal{A}, \\ 0, & \text{otherwise.} \end{cases}$$

All supports $B \in \mathcal{C}$, such that $B \notin \{B_k\}_{k \in K}$, are superfluous for the conference game. Therefore, repeatedly applying the superfluous support property to the allocation rule γ, we find

$$\gamma(N, \beta u_T, \mathcal{F}) = \cdots = \gamma\big(N, \beta u_T, \mathcal{F}^{'}\big),$$

where $\mathcal{F}' = (\{\overline{B_k}\}_{k \in K}) \cup (\{\{j\} : \{j\} \in \mathcal{F}\})$.

As $(\beta u_T)^{\mathcal{F}'} = \beta u_{\bar{T}}$, we have $\beta u_{\bar{T}}(S) = \beta u_{\bar{T}}(S \cap \bar{T})$, for all coalitions $S \subseteq N$, and since it holds $\bar{T} \cap S \subseteq \bar{T}$, then

$$\beta u_{\bar{T}}(S) = \beta u_{\bar{T}}(S \cap \bar{T}) = \begin{cases} \beta, & \text{if } S \cap \bar{T} = \bar{T}, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the game $(\beta u_T)^{\mathcal{F}'}$ is point unanimous with

$$D = \{i \in N : |\mathcal{C}_i| > 0\} = \bar{T},$$

and $f(0) = \cdots = f(|D| - 1) = 0$, $f(|D|) = \beta$.

As γ satisfies point unanimity, we find that, for all $i \in N$, there exists $\delta \in \mathbb{R}$, such that

$$\gamma_i(N, \beta u_T, \mathcal{F}') = \begin{cases} \delta, & \text{if } i \in \bar{T}, \\ 0, & \text{otherwise.} \end{cases}$$

Taking into account component-efficiency and the component dummy property,

$$\sum_{i \in N} \gamma_i(N, \beta u_T, \mathcal{F}') = |\bar{T}|\delta = \beta,$$

and, hence, $\delta = \frac{\beta}{|\bar{T}|}$. Therefore, for all $i \in N$,

$$\gamma_i(N, \beta u_T, \mathcal{F}') = \begin{cases} \frac{\beta}{|\bar{T}|}, & \text{if } i \in \bar{T}, \\ 0, & \text{otherwise.} \end{cases}$$ □

Since point anonymity implies point unanimity, we see that the Myerson value $\mu : \mathrm{USI}^N \to \mathbb{R}^n$ is the unique allocation rule on USI^N that satisfies component efficiency, the component dummy property, additivity, the superfluous support property and point anonymity.

4 The Conference Game and the Restricted Game

This paper focusses on the role of the strong superfluous support and superfluous support property, in axiomatizing the Myerson value, for games on union stable systems, one defined using the restricted game $(N, v^{\mathcal{F}})$ and the other one defined using the conference game $(\mathcal{C}, v^{\mathcal{C}})$. As a result, in this section, we want to discuss several relations between these two games. In the same way, there exist connections between the *arc game* and the *point game* for communication graph games as shown in van den Nouweland and Borm [19]. In particular, we want to discuss the inheritance of properties as balancedness, superadditivity and convexity.

First, we consider balancedness. Bondareva [20] and Shapley [21] state that a game (N, v) is *balanced* if and only if it has a non-empty *core*. The *core* of a game (N, v) is the set

$$C(v) = \{x \in \mathbb{R}^n : x(N) = v(N), x(S) \geq v(S), \text{ for all } S \subset N\},$$

where $x(S) = \sum_{i \in S} x_i$ and $x(\emptyset) = 0$. A game is called *totally balanced* iff each subgame is balanced.

Theorem 4.1 *Let $(N, v, \mathcal{F})$ be a union stable cooperation structure. If $(\mathcal{C}, v^{\mathcal{C}})$ is nonnegative and balanced, then $(N, v^{\mathcal{F}})$ is balanced.*

Proof As $(\mathcal{C}, v^{\mathcal{C}})$ is non-negative and balanced,

$$C\big(v^{\mathcal{C}}\big) = \Big\{ y \in \mathbb{R}_+^{|\mathcal{C}|} : \sum_{C \in \mathcal{C}} y_C = v^{\mathcal{C}}(\mathcal{C}), \sum_{A \in \mathcal{A}} y_A \geq v^{\mathcal{C}}(\mathcal{A}), \forall \mathcal{A} \subseteq \mathcal{C} \Big\} \neq \emptyset.$$

Let $y \in C(v^{\mathcal{C}})$. From y, we construct the vector $x \in \mathbb{R}^n$ in the following way:

$$x_i = \begin{cases} \sum_{C \in \mathcal{C}_i} \frac{1}{|C|} y_C, & \text{if } \mathcal{C}_i \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

for each $i \in N$. We have to prove that $x \in C(v^{\mathcal{F}})$, i.e.,

$$\sum_{i \in N} x_i = v^{\mathcal{F}}(N) \quad \text{and} \quad \sum_{i \in S} x_i \geq v^{\mathcal{F}}(S), \quad \forall S \subset N.$$

Indeed,

$$\begin{aligned} \sum_{i \in N} x_i &= \sum_{i \in N} \Big[\sum_{C \in \mathcal{C}_i} \frac{1}{|C|} y_C \Big] = \sum_{C \in \mathcal{C}} \Big[\sum_{i \in C} \frac{1}{|C|} y_C \Big] \\ &= \sum_{C \in \mathcal{C}} \Big[|C| \frac{1}{|C|} y_C \Big] = \sum_{C \in \mathcal{C}} y_C = v^{\mathcal{C}}(\mathcal{C}) \\ &= v^{\mathcal{F}}(N). \end{aligned}$$

On the other hand,

$$\sum_{i \in S} x_i = \sum_{i \in S} \Big[\sum_{C \in \mathcal{C}_i} \frac{1}{|C|} y_C \Big].$$

By Proposition 3.1, we find that $v^{\mathcal{C}}(\mathcal{C}_S) = v^{\mathcal{F}}(S)$, for each $S \subset N$, and as $y_C \geq 0$ and $\{C \in \mathcal{C}_S : i \in C\} \subseteq \mathcal{C}_i$, we have

$$\begin{aligned} \sum_{i \in S} x_i &= \sum_{i \in S} \Big[\sum_{C \in \mathcal{C}_i} \frac{1}{|C|} y_C \Big] \geq \sum_{i \in S} \Big[\sum_{\{C \in \mathcal{C}_S : i \in C\}} \frac{1}{|C|} y_C \Big] \\ &= \sum_{C \in \mathcal{C}_S} \Big[\frac{1}{|C|} |C| \Big] y_C = \sum_{C \in \mathcal{C}_S} y_C \geq v^{\mathcal{C}}(\mathcal{C}_S) = v^{\mathcal{F}}(S), \end{aligned}$$

where the last inequality holds because $y \in C(v^{\mathcal{C}})$. Hence, we conclude that $(N, v^{\mathcal{F}})$ is balanced. □

The inheritance does not go the other way around, as shown in the following example.

Example 4.1 Let $(N, v, \mathcal{F})$ be a union stable cooperation structure, where $(N, \mathcal{F})$ is the union stable system considered in (1), i.e., $N = \{1, 2, 3, 4\}$, the feasible coalition system $\mathcal{F} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$, and let v be the game given by

$$v(S) = \begin{cases} |S|, & \text{if } |S| \geq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the game (N, v) is totally balanced and therefore, so is the game $(N, v^{\mathcal{F}})$ (see [22]). The corresponding conference game, $(\mathcal{C}, v^{\mathcal{C}})$ is given by $v^{\mathcal{C}}(\{\{1, 2, 3\}\}) = v^{\mathcal{C}}(\{\{1, 2, 4\}\}) = 3$ and $v^{\mathcal{C}}(\{\{1, 2, 3\}, \{1, 2, 4\}\}) = 4$. Thus, it is not balanced, since $\mathcal{C} = \mathcal{B} = \{\{1, 2, 3\}, \{1, 2, 4\}\}$ and hence,

$$C\bigl(v^{\mathcal{C}}\bigr) = \bigl\{y \in \mathbb{R}^{|\mathcal{C}|} : y_{\{1,2,3\}} + y_{\{1,2,4\}} = 4, y_{\{1,2,3\}} \geq 3, y_{\{1,2,4\}} \geq 3\bigr\} = \emptyset.$$

Next, we discuss the inheritance of superadditivity.

Theorem 4.2 *Let $(N, v, \mathcal{F})$ be a union stable cooperation structure. If $(\mathcal{C}, v^{\mathcal{C}})$ is superadditive and non-negative, then $(N, v^{\mathcal{F}})$ is superadditive.*

Proof Let $S, T \subseteq N$, $S \cap T = \emptyset$. We have to prove that

$$v^{\mathcal{F}}(S \cup T) \geq v^{\mathcal{F}}(S) + v^{\mathcal{F}}(T). \tag{4}$$

By Proposition 3.1,

$$v^{\mathcal{F}}(S) = v^{\mathcal{C}}(\mathcal{C}_S), \qquad v^{\mathcal{F}}(T) = v^{\mathcal{C}}(\mathcal{C}_T) \quad \text{and} \quad v^{\mathcal{F}}(S \cup T) = v^{\mathcal{C}}(\mathcal{C}_{S \cup T}).$$

Therefore, expression (4) is equivalent to

$$v^{\mathcal{C}}(\mathcal{C}_{S \cup T}) \geq v^{\mathcal{C}}(\mathcal{C}_S) + v^{\mathcal{C}}(\mathcal{C}_T).$$

For that, it suffices to prove that $\mathcal{C}_S \cap \mathcal{C}_T = \emptyset$ and $\mathcal{C}_S \cup \mathcal{C}_T \subseteq \mathcal{C}_{S \cup T}$ since, by the superadditivity and the monotonicity of the game $(\mathcal{C}, v^{\mathcal{C}})$, we have

$$v^{\mathcal{C}}(\mathcal{C}_S) + v^{\mathcal{C}}(\mathcal{C}_T) \leq v^{\mathcal{C}}(\mathcal{C}_S \cup \mathcal{C}_T) \leq v^{\mathcal{C}}(\mathcal{C}_{S \cup T}).$$

On the one hand, since

$$\mathcal{C}_S \subseteq 2^S, \qquad \mathcal{C}_T \subseteq 2^T, \qquad S \cap T = \emptyset \quad \text{we have } \mathcal{C}_S \cap \mathcal{C}_T = \emptyset.$$

On the other hand, by construction of the sets of non-unitary supports $\mathcal{C}_S$, $\mathcal{C}_T$ and $\mathcal{C}_{S \cup T}$, we conclude that $\mathcal{C}_S \cup \mathcal{C}_T \subseteq \mathcal{C}_{S \cup T}$. □

Again, the inheritance does not go the other way around.

Example 4.2 Consider the union stable cooperation structure $(N, v, \mathcal{F})$, given in Example 4.1. The game (N, v) is superadditive, and it is easy to show that then, so is the game $(N, v^{\mathcal{F}})$. However, the conference game $(\mathcal{C}, v^{\mathcal{C}})$ is not superadditive, since

$$v^{\mathcal{C}}\bigl(\{\{1, 2, 3\}, \{1, 2, 4\}\}\bigr) \ngeq v^{\mathcal{C}}\bigl(\{\{1, 2, 3\}\}\bigr) + v^{\mathcal{C}}\bigl(\{\{1, 2, 4\}\}\bigr).$$

As mentioned in the preliminaries this example cannot be modeled by a communication situation.

Finally, we study the transmission of convexity from the conference game to the restricted game.

Theorem 4.3 *Let* $(N, v, \mathcal{F})$ *be a union stable cooperation structure. If the conference game* $(\mathcal{C}, v^{\mathcal{C}})$ *is non-negative and convex then* $(N, v^{\mathcal{F}})$ *is convex.*

Proof Let $i \in N$ and S, T such that $S \subseteq T \subseteq N \setminus \{i\}$. We have to prove that

$$v^{\mathcal{F}}\big(S \cup \{i\}\big) - v^{\mathcal{F}}(S) \leq v^{\mathcal{F}}\big(T \cup \{i\}\big) - v^{\mathcal{F}}(T).$$

By Proposition 3.1, we have

$$v^{\mathcal{F}}(S \cup \{i\}) = v^{\mathcal{C}}(\mathcal{C}_{S\cup\{i\}}), \qquad v^{\mathcal{F}}(S) = v^{\mathcal{C}}(\mathcal{C}_S),$$
$$v^{\mathcal{F}}(T \cup \{i\}) = v^{\mathcal{C}}(\mathcal{C}_{T\cup\{i\}}), \qquad v^{\mathcal{F}}(T) = v^{\mathcal{C}}(\mathcal{C}_T).$$

Therefore, the above is equivalent to prove that

$$v^{\mathcal{C}}(\mathcal{C}_{S\cup\{i\}}) - v^{\mathcal{C}}(\mathcal{C}_S) \leq v^{\mathcal{C}}(\mathcal{C}_{T\cup\{i\}}) - v^{\mathcal{C}}(\mathcal{C}_T).$$

On the one hand, as $\mathcal{C}_{S\cup\{i\}}, \mathcal{C}_{T\cup\{i\}}, \mathcal{C}_S$ and $\mathcal{C}_T$ are the sets of non-unitary supports of the union stable systems $(N, \mathcal{F}_{S\cup\{i\}})$, $(N, \mathcal{F}_{T\cup\{i\}})$, $(N, \mathcal{F}_S)$ and $(N, \mathcal{F}_T)$, respectively, the following is satisfied:

$$\mathcal{C}_{S\cup\{i\}} \cup \mathcal{C}_T \subseteq \mathcal{C}_{T\cup\{i\}} \quad \text{and} \quad \mathcal{C}_{S\cup\{i\}} \cap \mathcal{C}_T = \mathcal{C}_S.$$

On the other hand, as $(\mathcal{C}, v^{\mathcal{C}})$ is non negative and superadditive, we have

$$v^{\mathcal{C}}(\mathcal{C}_{T\cup\{i\}}) \geq v^{\mathcal{C}}(\mathcal{C}_{S\cup\{i\}} \cup \mathcal{C}_T).$$

Moreover, by convexity of the game $(\mathcal{C}, v^{\mathcal{C}})$, it holds

$$\begin{aligned} v^{\mathcal{C}}(\mathcal{C}_{S\cup\{i\}} \cup \mathcal{C}_T) - v^{\mathcal{C}}(\mathcal{C}_T) &\geq v^{\mathcal{C}}(\mathcal{C}_{S\cup\{i\}}) - v^{\mathcal{C}}(\mathcal{C}_{S\cup\{i\}} \cap \mathcal{C}_T) \\ &= v^{\mathcal{C}}(\mathcal{C}_{S\cup\{i\}}) - v^{\mathcal{C}}(\mathcal{C}_S). \end{aligned}$$

Combining the two last expressions, we conclude

$$v^{\mathcal{C}}(\mathcal{C}_{S\cup\{i\}}) - v^{\mathcal{C}}(\mathcal{C}_S) \leq v^{\mathcal{C}}(\mathcal{C}_{S\cup\{i\}} \cup \mathcal{C}_T) - v^{\mathcal{C}}(\mathcal{C}_T) \leq v^{\mathcal{C}}(\mathcal{C}_{T\cup\{i\}}) - v^{\mathcal{C}}(\mathcal{C}_T). \qquad \square$$

A study about the convexity between the original game and the conference game can be found in [7], where subclasses of union stable families for which the convexity of the original game is inherited by the restricted game, and the conference game, respectively, are given.

The literature on restricted cooperation in cooperative games, usually follows the approach of this paper, where a cooperative game is enriched with a relational structure that expresses the cooperation restrictions. However, there is literature on economic and social networks, which uses value functions that assign a worth to every graph on a set of players (instead of any coalition of players), see, for example, Jackson and Wolinsky [23]. Note that for communication graphs, the conference game is assigning a worth to every subgraph of the graph, and, thus, it is similar to such graph value functions. In this section, we considered the conference game of more general union stable systems and analyzed the inheritance of properties from the conference game to the restricted game. Finally, note that Theorem 4.3 and the fact that the Shapley value is in the core of the game, i.e., $\Phi(N,v) \in C(v)$, if v is convex, gives us the next corollary giving conditions on the conference game, which guarantee that the Myerson value is in the core of the restricted game.

Corollary 4.1 *Let $(N,v,\mathcal{F})$ be a union stable cooperation structure. If the conference game $(\mathcal{C},v^{\mathcal{C}})$ is non-negative and convex then*

$$\mu(N,v,\mathcal{F}) \in C\bigl(v^{\mathcal{F}}\bigr).$$

In [7], it is shown that $(\mathcal{C},v^{\mathcal{C}})$ is convex if $(N,v,\mathcal{F}) \in \mathrm{USI}^N$ and (N,v) is convex. Since, moreover, $(\mathcal{C},v^{\mathcal{C}})$ is positive if (N,v) is superadditive and zero-normalized, applying the above corollary, we have $\mu(N,v,\mathcal{F}) \in C(v^{\mathcal{F}})$ if $(N,v,\mathcal{F}) \in \mathrm{USI}^N$ and (N,v) is convex.

5 Conclusions

This paper makes some contributions to cooperative game theory with restricted cooperation. It not only allows for the unification and generalization of different research lines, such as communication situations or permission structures, but also the analysis of economic applications, which arise in this context. At first sight, the Myerson value and the conference game, and therefore, the supports, seem not related. In this paper, we show the existence of a relationship between the Myerson value and the conference game. First, we provided an axiomatization of the Myerson value using the strong superfluous support property. Requiring this independence only for superfluous supports (defined through the conference game), we axiomatized the Myerson value on a special class of union stable structures that contains those coming from cycle-free graphs.

For communication situations, van den Brink et al. [17] introduce the class of *Harsanyi power solutions*, which can be obtained by distributing the Harsanyi dividends in the point game proportional to some power measure for communication graphs. This class contains the Myerson value, which is obtained by using the equal power measure. This approach is generalized to games on union stable systems by Algaba et al. [18]. Although every Harsanyi power solution satisfies the superfluous support property on the class USI^N, in general, they do not satisfy this property on US^N, and therefore, they do not verify the strong superfluous support property either. Thus, in some sense, the superfluous support property and the role of the conference game are characteristic for the Myerson value, within the class of Harsanyi power solutions. On the other hand, the position value, which has a quite different interpretation and definition than the Myerson value (introduced through the conference game instead of the restricted game), also satisfies the superfluous support property on US^N. However, this is not a Harsanyi power solution, although it is on the class USI^N.

Finally, we provided a study between the restricted game and the conference game, in particular, an analysis of the transmission of properties. As a corollary, by means of the conference game, it can be established when the Myerson value is in the core of the restricted game, without requirements about the structure or the restricted game.

Acknowledgements This research was finished while the first author was visiting Tinbergen Institute and VU University Amsterdam, under grant Ref. 24022011 of Seville University. Also, this visit was partially supported by Tinbergen Institute. Moreover, this work was presented in some conferences under financial support of the project ECO201017766.

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