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Myerson values for games with fuzzy communication structure

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Abstract

In 1977, Myerson considered cooperative games with communication structure. A communication structure is an undirected graph describing the bilateral relationships among the players. He introduced the concept of allocation rule for a game as a function obtaining an outcome for each communication structure among the players of the game. The Myerson value is a specific allocation rule extending the Shapley value of the game. More recently, the authors studied games with fuzzy communication structures using fuzzy graph-theoretic ideas. Now we propose a general framework in order to define fuzzy Myerson values. Players in a coalition need to measure their profit using their real individual and communication capacities at every moment because these attributes are fuzzy when the game is proposed. So, they look for forming connected coalitions working at the same level. The different ways to obtain these partitions by levels determine different Myerson values for the game. Several interesting examples of these ways are studied in the paper, following known models in games with fuzzy coalitions: the proportional model and the Choquet model.

Keywords: Game theory; Fuzzy sets; Fuzzy graphs; Shapley value

1. Introduction

A cooperative game over a finite set of players, see Driessen [3], is defined as a function for establishing the worth of each coalition (subset of players). The outcome of a game is a payoff vector, that is a vector with the payment for each player owing to their cooperation possibilities. The Shapley value is one of most known payoff vector for cooperative games. But the usual payoff vectors consider that all the communications among the players are feasible. Although the game is thought in a total cooperation situation, Myerson [6] considered that the communication among the players can be different at every moment. He describes the communication situation by a graph where the vertices are the players and the links are the feasible bilateral communications among them. This graph is named the communication structure of the game. Hence we will use both, graph or communication structure, alike. So, Myerson proposed as a more realistic outcome for the game, an allocation rule, that is a function for obtaining a payoff vector for each communication structure. The Myerson value is an allocation rule which coincides on the complete graph (the total cooperation situation) with the Shapley value. Given a communication structure, the Myerson value calculates the Shapley value of the game modified by the graph. So, the worth of a coalition taking into account the graph is the sum of the worths of the maximal connected coalitions in the subgraph restricted by the corresponding set of vertices.

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This allocation rule is characterized by two axioms: efficiency by components (the payoff vectors are efficient in each maximal connected coalition for each graph) and fairness (the loss of one bilateral communication implies the same loss of payment for the players involved in this link).

Aubin [1] supposed uncertainty about the membership of the players in the coalitions studying games with fuzzy coalitions. To calculate the worth of a fuzzy coalition from a game it is necessary to consider a specific partition by levels of this fuzzy set. Following this way, the uncertainty about the existence or not of the communications among the players can be extended to the uncertainty about the capacities of these communications. Recently, Jiménez-Losada et al. [4] introduced fuzzy graphs to analyze communication among players. Fuzzy graphs allow leveling the links between being feasible or not, and they also allow considering membership levels for the players. The authors defined fuzzy allocation rules for cooperative games in order to get a payoff vector for each fuzzy graph. In that paper they studied an extension of the Myerson value in a particular way.

This paper deals with a general framework to look for fuzzy allocation rules and particularly fuzzy Myerson values. The way to determine payoff vectors over a fuzzy graph depends on the partition by levels of the structure that we use. Following Myerson and Aubin we define specific fuzzy allocation rules, one for every partition selection over the fuzzy communication structures. But we consider that these fuzzy allocation rules must be named fuzzy Myerson values if and only if they satisfy similar axioms to the crisp version: efficiency by components (the measure of the profit in each component depends directly on the partition selection) and fuzzy fairness (the drop of the capacity of a link implies the same loss in the payment of both the players involved). We introduce the concept of admissible selection of partitions by conditions which guarantee fuzzy Myerson values. Two specific partitions by levels of fuzzy coalitions studied in the literature are the proportional model [2] and the Choquet model [7]. We study several options of partition selections for fuzzy graph based in both models, finding out if they are or not admissible.

In Section 2 we introduce the preliminaries reviewing the concept of fuzzy communication structure. Section 3 is dedicated to define what a fuzzy allocation rule extending the Myerson concept is. Finally, in Sections 4 and 5 we study three interesting options to get fuzzy Myerson values for different players' behaviors based on the proportional and the Choquet models.

2. Preliminaries

This section is a brief review of the necessary topics to follow this paper. The next three subsections deal with notions about games with communication structure, games with fuzzy coalitions and fuzzy graphs respectively.

2.1. Cooperative games and graphs

A superadditive *cooperative game* is a pair (N, v) where N is a finite set and $v : 2^N \to \mathbb{R}$ is a function with $v(\emptyset) = 0$ and satisfying $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subset N$ with $S \cap T = \emptyset$. The elements of $N = \{1, 2, \ldots, n\}$ are called *players*, the subsets $S \subseteq N$ *coalitions* and $v(S)$ is the *worth* (benefit) of S. The set of players N is fixed throughout this paper, hence we identify a game (N, v) with the function v. The *restriction* of a game v to the coalition $S \subseteq N$ is another game v_S defined by $v_S(T) = v(S \cap T)$ for all $T \subseteq N$. Assuming the grand coalition N will be formed in a game v then a *solution* concept defines how to distribute the profit $v(N)$ among the players. Therefore, a solution is an efficient vector $x \in \mathbb{R}^N$, this is $\sum_{i \in N} x_i = v(N)$. We use one specific solution for cooperative games. The *Shapley value* of a game v is defined for each player $i \in N$ as

$$\phi_i(v) = \sum_{\{S \subseteq N : i \in S\}} \frac{(n - |S|)!(|S| - 1)!}{n!}[v(S) - v(S \setminus \{i\})]. \tag{1}$$

This value is a linear function over the set of games, $\phi(\alpha v_1 + \beta v_2) = \alpha\phi(v_1) + \beta\phi(v_2)$ for all $\alpha, \beta \in \mathbb{R}$ and games v_1, v_2. The reader can use [3] to get more information about cooperative games. From this moment we consider a fixed game v over N throughout this paper.

Myerson [6] considered that the bilateral communication among players can be different when the game v is solved, and therefore it is necessary to change the solution. He established the players' communication possibilities with a graph. A (undirected) *graph* $g = (S, A)$ is defined by a finite set S and a set A of unordered pairs of different members of S. The elements of S are named *vertices* and the elements of A are called *links* or edges. Let $L = \{\{i, j\} : i \neq j; i, j \in N\}$

denote the set of bilateral relations among players in our game. Myerson defined a *communication structure* over N as a graph $g = (S, A)$ where $S \subseteq N$ is the subset of players who are genuinely active in the game and $A \subseteq L$ is the set of feasible communication links among players in S. Therefore, we will use throughout the paper graph or communication structure alike. The set of all the communication structures over N is denoted by CS^N. In particular $gN = (N, L)$ is the complete graph representing the total cooperation among the players. Let $g = (S, A) \in CS^N$ be a graph. A *subgraph* $g' = (S', A')$ of g is another graph which satisfies that $S' \subseteq S$ and $A' \subseteq A$. A path in g is defined by a sequence of vertices $(i_k)_{k=1}^{k=m}$ satisfying that $\{i_k, i_{k+1}\} \in A$ is a different link in g for each $k = 1, \ldots, m-1$. Two players $i, j \in S$ are connected in g if there is a path containing both of them. When any pair of players are connected in g the graph g is called *connected*. The communication structure g has a *cycle* if there is a path $(i_k)_{k=1}^{k=m}$ with $m \geq 4$ satisfying $i_1 = i_m$. A connected graph without cycles is named a *tree*. A *spanning tree* g' in a connected communication structure g is a tree subgraph of g which uses all its vertices. The *connected components* of g are the maximal subgraphs of g which are connected. If $T \subseteq N$ is a coalition then we denote g_T as the subgraph of g using only the vertices in $T \cap S$ and the links in A among them. The coalitions of players in the connected components of g are $N/g = \{H \subseteq S : g_H \text{ is a connected component of } g\}$. Coalitions in N/g form a partition of S.

It is necessary to establish what profit must be allocated in order to obtain a solution of the game v for each graph. If the communication structure among the players is $g \in CS^N$ then we suppose that all the coalitions in N/g are formed. An *allocation rule* for the game v is any function $\psi^v : CS^N \to \mathbb{R}^N$ which is *efficient by components*, if $g \in CS^N$ and $H \in N/g$ then $\sum_{i \in H} \psi_i^v(g) = v(H)$ but if i is not a vertex in g then $\psi_i^v(g) = 0$. Myerson [6] introduced a particular allocation rule extending the Shapley value. He defined a “measure” of the potential profit obtained for all the players in a communication structure $g \in CS^N$ for the game v as

$$r(g) = \sum_{H \in N/g} v(H). \tag{2}$$

This measure r allows comparing different communication structures according to the game. It satisfies for every $g = (S, A)$ the following logical properties:

1. If g is connected then the grand coalition is formed and $r(g) = v(N)$.
2. It is monotone by links, $r(g) \geq r(g_{-\{i,j\}})$ if $\{i, j\} \in A$, taking $g_{-\{i,j\}} = (S, A \setminus \{i, j\})$.
3. It is additive by components, $r(g) = \sum_{H \in N/g} r(g_H)$.

Evidently, r is the only measure by v which satisfies these conditions. Therefore, Myerson thought that the payoff vector obtained for each graph should allocate the measure of its connected components. For each $g \in CS^N$ he took a new game (N, v^g) so that

$$v^g(S) = r(g_S) \tag{3}$$

for all $S \subseteq N$. The *Myerson value* is defined as $\mu^v(g) = \phi(v^g)$ for every $g \in CS^N$. This allocation rule is an extension of the Shapley value because we obtain $\mu^v(gN) = \phi(v)$. Myerson characterized his allocation rule as the only one satisfying the *fair axiom*: the allocation rule ψ^v is fair iff for all $g = (S, A) \in CS^N$ and for all $\{i, j\} \in A$ we have the equality $\psi_i^v(g) - \psi_i^v(g_{-\{i,j\}}) = \psi_j^v(g) - \psi_j^v(g_{-\{i,j\}})$. He also proved that the Myerson value for v is *stable* because for all $g = (S, A)$ and for all $\{i, j\} \in A$ we have $\psi_i^v(g) \geq \psi_i^v(g_{-\{i,j\}})$.

2.2. Games with fuzzy coalitions

Let K be a finite set. A *fuzzy set* in K is a function $\tau : K \to [0, 1]$. We denote $[0, 1]^K$ as the family of fuzzy sets in K. Each subset $S \subseteq K$ is associated to the fuzzy set $e^S \in [0, 1]^K$ with $e^S(i) = 1$ if $i \in S$ or else $e^S(i) = 0$. Specifically, we take $e^{\emptyset} = 0$. The *support* of τ is $\text{supp}(\tau) = \{i \in K : \tau(i) \neq 0\}$. We will use the word “crisp” to differentiate between fuzzy things and non-fuzzy things.

Aubin [1] defined a *fuzzy coalition* as a fuzzy set $\tau \in [0, 1]^N$ of players where each coordinate $\tau(i)$ is interpreted as the membership capacity of the player $i \in N$ in the coalition and $\text{supp}(\tau)$ is the set of players who are active. Now, we consider our crisp game v. Aubin proposed a way to value a fuzzy coalition by v using the following claim: players organize crisp coalitions where all the players work at the same level. So, for each fuzzy coalition he introduced the idea of a balanced family which we understand here as a partition by levels. If $\tau \in [0, 1]^N$ then a *partition by levels* of

τ is a finite set of crisp coalitions and levels $(S_k, s_k)_{k=1}^{k=m}$ such that $s_k \in (0,1]$ and $\tau = \sum_{k=1}^{m} s_k e^{S_k}$. We can see that a usual partition of a crisp coalition S is a partition by levels of the fuzzy coalition e^S. Let f be a choosing a partition by levels $f(\tau) = (S_k, s_k)_{k=1}^{k=m}$ for every $\tau \in [0,1]^N$ and satisfying $f(e^S) = (S, 1)$ for all $S \subseteq N$. The *extension* of v by f is a new characteristic function $f(v)$ over the fuzzy coalitions defined as $f(v)(\tau) = \sum_{k=1}^{m} s_k v(S_k)$.

Two particular cases of extensions of a crisp game are the proportional extension introduced by Butnariu [2] and the Choquet extension defined by Tsurumi et al. [7]. Let $\tau \in [0,1]^N$ be a fuzzy coalition. For the different levels (non-zero) in τ, $h_1 < \cdots < h_m$, we take the sets $S[k] = \{i \in N : \tau(i) = h_k\}$ and $S_k = \{i \in N : \tau(i) \geq h_k\}$ for all $k = 1, \ldots, m$. Butnariu considers the partition by levels $pr(\tau) = (S[k], h_k)_{k=1}^{k=m}$ and then the *proportional extension* of v is defined by $pr(v)(\tau) = \sum_{k=1}^{m} h_k v(S[k])$. He also introduces an extension of the Shapley value which calculates the solution for v if a specific fuzzy coalition is formed. So, the *proportional-Shapley value* of the game v for the coalition $\tau \in [0,1]^N$ is

$$\phi^{pr}(v)(\tau) = \sum_{k=1}^{m} h_k \phi(v_{S[k]}). \tag{4}$$

Tsurumi et al. consider the partition by levels $ch(\tau) = (S_k, h_k - h_{k-1})_{k=1}^{k=m}$ with $h_0 = 0$ and then the *Choquet extension* of v is defined by $ch(v)(\tau) = \sum_{k=1}^{m} [h_k - h_{k-1}] v(S_k)$. They also introduce other extension of the Shapley value, the *Choquet-Shapley value* of the game v for the coalition $\tau \in [0,1]^N$ is

$$\phi^{ch}(v)(\tau) = \sum_{k=1}^{m} [h_k - h_{k-1}] \phi(v_{S_k}). \tag{5}$$

Players in v take into account two elements in a fuzzy coalition: the size of the crisp coalitions in the partition and the levels of these coalitions. For us both extensions involve particular behaviors for the players.

Proportional behavior. Players can only play once and their capacities are not divisible.

Choquet behavior. Players can allocate their capacities and they try to get the biggest crisp coalition.

Example 1. Let $N = \{1,2,3,4,5\}$ be the set of players for the crisp game v. We consider the fuzzy coalition $\tau = (0.5, 0.2, 0.7, 0, 0.5)$. If players use proportional behavior then they form the proportional partition $pr(\tau) = \{(\{3\}, 0.7), (\{1,5\}, 0.5), (\{2\}, 0.2)\}$ and

$$pr(v)(\tau) = 0.7v(\{3\}) + 0.5v(\{1,5\}) + 0.2v(\{2\}).$$

Now we suppose that players have Choquet behavior. They form the Choquet partition $ch(\tau) = \{(\{1,2,3,5\}, 0.2), (\{1,3,5\}, 0.3), (\{3\}, 0.2)\}$, and

$$ch(v)(\tau) = 0.2v(\{1,2,3,5\}) + 0.3v(\{1,3,5\}) + 0.2v(\{3\}).$$

2.3. Fuzzy communication structures

Following Myerson [6] we want to extend his model using fuzzy communication structures. We suppose that the capacity of the players or their communication can be different when the game is solved. Jiménez-Losada et al. [4] introduced a fuzzy communication structure as a fuzzy graph. We now review some concepts about fuzzy graphs for which the reader can use [5]. In this paper we use the operators $\wedge, \vee$ as the minimum and the maximum respectively. Let L be the set of bilateral communications among the players in N.

Definition 1. A fuzzy communication structure for the game v is a (undirected) fuzzy graph over N, this is a pair $\gamma = (\tau, \rho)$ with $\tau \in [0,1]^N$ the fuzzy set of vertices and $\rho \in [0,1]^L$ the fuzzy set of links satisfying $\rho(i,j) \leq \tau(i) \wedge \tau(j)$ for all $\{i,j\} \in L$. The set of fuzzy communication structures over N is denoted by FCS^N.

Hence we will use fuzzy graph or fuzzy communication structure alike. Let $\gamma = (\tau, \rho) \in FCS^N$ be a fuzzy communication structure. The number $\tau(i)$ is interpreted as the real level of involvement of player $i \in N$ in the game v. The number $\rho(i,j)$ represents the maximal level to which the link $\{i,j\}$ can be used. We use $\gamma = 0$ as the null fuzzy graph where $\tau = 0$ and $\rho = 0$. A crisp communication structure $g = (S, A) \in CS^N$ is identified with the fuzzy graph

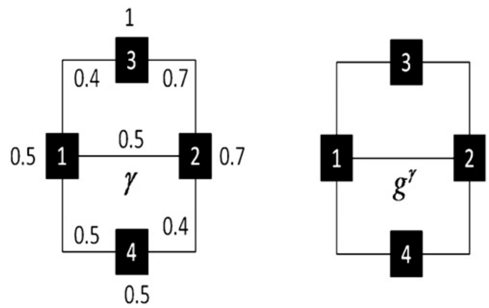

Fig. 1. Fuzzy graph and crisp version.

$g = (\tau, \rho)$ where $\tau = e^S$ and $\rho = e^A$. We can associate γ to a crisp graph. The set of vertices in γ is vert $(\gamma) = \text{supp}(\tau)$ and the set of links is link $(\gamma) = \text{supp}(\rho)$. The *crisp version* of γ is the crisp graph $g^\gamma = (\text{vert}(\gamma), \text{link}(\gamma))$. Using the crisp version several concepts from graph theory are extended to fuzzy graphs. So we say that the fuzzy graph γ is *connected* or *tree* iff its crisp version g^γ is connected or tree. The *maximal level* and the *minimal level* in γ are

$$\vee\gamma = \bigvee_{i\in N} \tau(i) \quad \text{and} \quad \wedge\gamma = \left(\bigwedge_{i\in\text{vert}(\gamma)} \tau(i)\right) \wedge \left(\bigwedge_{\{i,j\}\in\text{link}(\gamma)} \rho(i,j)\right)$$

Another fuzzy graph $\gamma' = (\tau', \rho')$ over N is a *subgraph* of γ iff $\tau' \leq \tau$ and $\rho' \leq \rho$. We use in that case $\gamma' \leq \gamma$. If $S \subseteq N$ is a crisp coalition then $\gamma_S = (\tau_S, \rho_S) \in FCS^N$ is the subgraph of γ defined as

$$\tau_S(i) = \begin{cases} \tau(i) & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \rho_S(i,j) = \begin{cases} \rho(i,j) & \text{if } i, j \in S \\ 0 & \text{otherwise} \end{cases}$$

The *connected components* of γ are its maximal connected subgraphs. We will use the set

$$N/\gamma = \{H \subseteq \text{vert}(\gamma) : \ \gamma_H \text{ is a connected component of } \gamma\}.$$

We defined three binary operations among fuzzy graphs in [4]. Let $\gamma = (\tau, \rho), \gamma' = (\tau', \rho') \in FCS^N$ be two fuzzy graphs over N:

(1) $\gamma + \gamma' = (\tau + \tau', \rho + \rho')$ $(\tau(i) + \tau'(i) \leq 1$ for all $i \in N)$,
(2) $\gamma - \gamma' = (\tau - \tau', \rho -^* \rho')$ $(\gamma' \leq \gamma)$ where for all $i, j \in N$

$$(\rho -^* \rho')(i,j) = [\rho(i,j) - \rho'(i,j)] \wedge [\tau(i) - \tau'(i)] \wedge [\tau(j) - \tau'(j)].$$

(3) $t\gamma = (t\tau, t\rho)$ $(t \in [0, 1])$.

The reader can see that the sum and subtraction of fuzzy graphs are not opposite operations. However, the following equality holds (Proposition 1, [4]): If $\gamma, \gamma', \gamma'' \in FCS^N$ are three fuzzy graphs over N such that $\gamma'' \leq \gamma - \gamma'$ and $\gamma' \leq \gamma$ then $(\gamma - \gamma') - \gamma'' = \gamma - (\gamma' + \gamma'')$. Particularly $(\gamma - \gamma') - \gamma'' = (\gamma - \gamma'') - \gamma'$.

Let $\gamma = (\tau, \rho) \in FCS^N$ be a fuzzy communication structure. If $\{i, j\} \in \text{link}(\gamma)$ and $t \in [0, \rho(i, j)]$ then we denote $\gamma^t_{-\{i,j\}}$ as γ reducing t in the capacity of the link $\{i, j\}$.

A fuzzy graph $\gamma = (\tau, \rho) \in FCS^N$ is said to be *complete by links* iff $\rho(i, j) = \tau(i) \wedge \tau(j)$ for all $i, j \in N$. If $\gamma = (\tau, \rho)$ is complete by links then ρ is just defined by τ and we can associate γ with the fuzzy coalition τ.

Example 2. We consider the fuzzy graph $\gamma = (\tau, \rho)$ over $N = \{1, 2, 3, 4\}$ given by $\rho(1, 3) = \rho(2, 4) = 0.4$, $\tau(1) = \tau(4) = \rho(1, 2) = \rho(1, 4) = 0.5$, $\tau(2) = \rho(2, 3) = 0.7$, $\tau(3) = 1$ and $\rho(3, 4) = 0$. Fig. 1 represents this fuzzy graph and its crisp version.

The maximal level is $\vee\gamma = 1$ and the minimal level is $\wedge\gamma = 0.4$. Fig. 2 calculates the difference with a tree subgraph of γ. Finally, Fig. 3 represents a fuzzy graph which is complete by links.

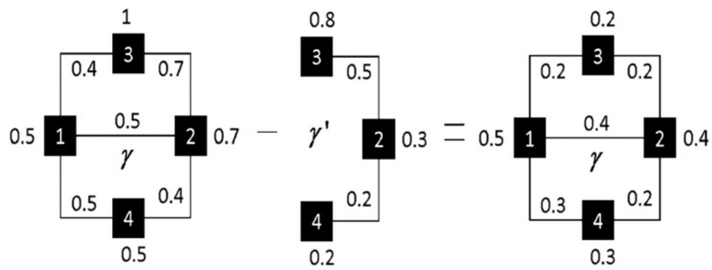

Fig. 2. The difference between fuzzy graphs.

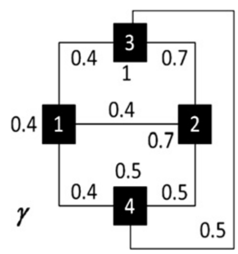

Fig. 3. Fuzzy graph which is complete by links.

3. Fuzzy Myerson values

We consider our crisp game v. A fuzzy allocation rule for v should be a function $\Psi^v : FCS^N \to \mathbb{R}^N$ obtaining a payoff vector for each fuzzy communication structure. But there is not a unique concept of efficiency. Following Myerson in (2) we need to find a "measure" for the fuzzy communication structures over N by the game v. If we consider properties of (2) good conditions for a profit measure then we introduce the next definition.3Please check the sentence 'If we consider properties...', with respect to clarity, and correct if necessary.

Definition 2. A function $\varepsilon : FCS^N \to \mathbb{R}$ is said to be a profit measure for fuzzy graphs by v if ε satisfies:

(1) $\varepsilon(g) = r(g)$ for all $g \in CS^N$,
(2) $\varepsilon(\gamma) \geq \varepsilon(\gamma^t_{-\{i,j\}})$ for every $\gamma = (\tau, \rho) \in FCS^N$ and $t \in [0, \rho(i, j)]$,
(3) $\varepsilon(\gamma) = \sum_{H \in N/\gamma} \varepsilon(\gamma_H)$ for all $\gamma \in FCS^N$.

The selection of a particular profit measure for fuzzy graphs by the game involves a particular concept of efficiency by components: the payoff vector for each fuzzy graph should be an allocation of the measure of the different connected components.

Definition 3. A fuzzy allocation rule for v is $\Psi^v : FCS^N \to \mathbb{R}^N$ satisfying that there is a profit measure ε for fuzzy graphs by v such that for all $\gamma \in FCS^N$ it holds

$$\sum_{i \in H} \Psi_i^v(\gamma) = \varepsilon(\gamma_H)$$

for each $H \in N/\gamma$ and $\Psi_i^v(\gamma) = 0$ if $i \notin \mathrm{vert}(\gamma)$.

We can do equivalence classes over the set of fuzzy allocation rules taking into consideration the rules defined by the same profit measure. So, if ε is a profit measure for the fuzzy graphs then the *ε-fuzzy allocation rules* are fuzzy allocation rules using ε.

But now, following through the Myerson process (3), we can construct a new auxiliary game for each profit measure ε and for each fuzzy graph $\gamma \in FCS^N$. We consider the crisp game

$$v_\varepsilon^\gamma(S) = \varepsilon(\gamma_S) \tag{6}$$

for all $S \subseteq N$. If we take a crisp graph $g \in CS^N$ then, by Definition 2 and (3), $v_\varepsilon^g = v^g$ for all ε. The next proposition proves that these new crisp games are also superadditive.

Proposition 1. *If ε is a profit measure by v for the fuzzy graphs then v_ε^γ is a superadditive game for all $\gamma \in FCS^N$.*

Proof. Let $S, T \subseteq N$ be two disjoint coalitions. If $H \in N/(\gamma_S + \gamma_T)$ then either $H \in N/\gamma_S$ or $H \in N/\gamma_T$. Using the third condition in Definition 2 we obtain $v_\varepsilon^\gamma(S) + v_\varepsilon^\gamma(T) = \varepsilon(\gamma_S + \gamma_T)$. Finally, the fuzzy graph $\gamma_S + \gamma_T$ is a subgraph of $\gamma_{S\cup T}$ which can be obtained by deleting links. Applying successively the second condition in Definition 2 we get $\varepsilon(\gamma_S + \gamma_T) \leq \varepsilon(\gamma_{S\cup T})$ and so $v_\varepsilon^\gamma(S) + v_\varepsilon^\gamma(T) \leq v_\varepsilon^\gamma(S \cup T)$. □

Hence, we can define one fuzzy Myerson value for each measure.

Definition 4. For each profit measure ε for fuzzy graphs by v the ε-fuzzy Myerson value is defined as $M^{v,\varepsilon}(\gamma) = \phi(v_\varepsilon^\gamma)$ for all $\gamma \in FCS^N$.

Definition 2 implies that the ε-fuzzy Myerson value satisfies $M^{v,\varepsilon}(g) = \mu^v(g)$ for all $g \in CS^N$. Now, we are going to characterize our fuzzy Myerson values. For this, we extend the fair axiom to a fuzzy version.

Fuzzy fair axiom. Let Ψ^v be a fuzzy allocation rule for the crisp game (N, v). It is fuzzy fair if $\gamma = (\tau, \rho) \in FCS^N$ then for all $\{i, j\} \in link(\gamma)$ and $t \in [0, \rho(i, j)]$

$$\Psi_i^v(\gamma) - \Psi_i^v(\gamma_{-\{i,j\}}^t) = \Psi_j^v(\gamma) - \Psi_j^v(\gamma_{-\{i,j\}}^t).$$

Moreover, we also introduce stability for fuzzy allocation rules. A fuzzy allocation rule $\Psi^v : FCS^N \to \mathbb{R}^N$ is *fuzzy stable* iff for all $\gamma = (\tau, \rho) \in FCS^N$, $\{i, j\} \in \text{link}(\gamma)$ and $t \in (0, \rho(i, j)]$ we have $\Psi_i^v(\gamma) \geq \Psi_i^v(\gamma_{-\{i,j\}}^t)$.

Theorem 2. *Let ε be a profit measure for fuzzy graphs by v. The ε-fuzzy Myerson value is the only ε-fuzzy allocation rule for v which is fuzzy fair. Also, the ε-fuzzy Myerson value is fuzzy stable.*

Proof. The proof replicates the only theorem in [6] and Theorem 4 in [4]. In both these proofs the reader can see that the properties in Definition 2 are necessary. The fuzzy fair axiom is used at the maximum deletion, this means taking $t = \rho(i, j)$. □

Now, we define a specific method to get profit measures for fuzzy communication structures, introducing the concept of partition by levels, following Aubin [1], for a fuzzy graph.

Definition 5. A partition by levels of a fuzzy communication structure $\gamma \in FCS^N$ is the finite set $(g_k, s_k)_{k=1}^{k=m}$ with $s_k \in (0, 1]$ and $g_k \in CS^N$ such that $s_k g_k \leq \gamma - \sum_{l=1}^{k-1} s_l g_l$ for all k and $\gamma - \sum_{k=1}^m s_k g_k = 0$. A fuzzy partition selection f for N chooses a specific partition by levels for each fuzzy communication structure.

We take a fuzzy partition selection f for N. We use the following "measure" defined by the game v for all $\gamma \in FCS^N$

$$\varepsilon^f(\gamma) = \sum_{k=1}^m s_k r(g_k) \tag{7}$$

where $f(\gamma) = (g_k, s_k)_{k=1}^{k=m}$. But not all these measures are admissible because not all of them satisfy Definition 2.

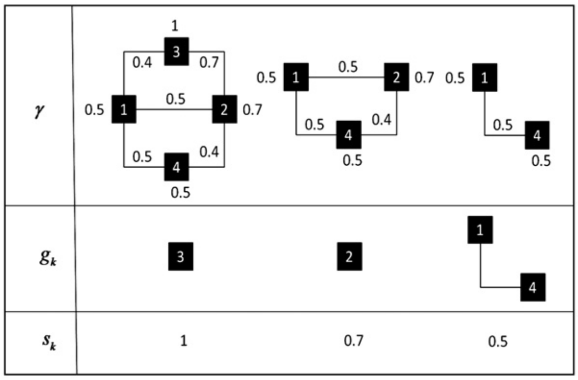

Fig. 4. pg-partition.

Definition 6. A fuzzy partition selection f for N is v-admissible if ε^f is a profit measure for fuzzy graphs by v.

For each selection f we will use v^γ_f as $v^\gamma_{\varepsilon f}$ for every fuzzy communication structure γ, and $M^{v,f}$ as M^{v,ε^f}.
The next sections study several fuzzy partition selections linked to Choquet and proportional behavior of the players.

4. The proportional model

In this section we are going to study the proportional behavior of the players looking for fuzzy Myerson values for the game v by fuzzy partition selections from different points of view.

4.1. Graph option

We consider that the object of the players' behavior is the whole fuzzy graph, the set of vertices and links. So the proportional behavior says: *Players and links only play once and their levels are not divisible*.

There is only one partition by levels representing this particular behavior of the players which is constructed using the following algorithm. Let $\gamma = (\tau, \rho) \in FCS^N$ be a fuzzy communication structure. We consider the next family of crisp versions for γ

$$g^\gamma[t] = (\{i \in N : \tau(i) = t\}, \{\{i, j\} \in L : \rho(i, j) = \tau(i) = \tau(j) = t\}).$$

We take k=0, $pg = \emptyset$ and $\gamma = \gamma$.
While $\gamma \neq 0$ do

$$k = k + 1$$

$$s_k = \vee\gamma$$

$$g_k = g^\gamma[s_k]$$

$$pg = pg \cup \{(g_k, s_k)\}$$

$$\gamma = \gamma - s_k g_k$$

The partition by levels is pg.

Example 3. Fig. 4 represents the application of the preceding algorithm to calculate $pg(\gamma)$ of a fuzzy graph γ.

Therefore, we have a fuzzy partition selection pg which is named *proportional by graphs*. In this selection, the reader can see that players only play once and they use their highest level in the graph. We will prove that this selection is admissible for the game v.

Theorem 3. *The proportional by graphs selection pg is v-admissible.*

Proof. Let $g \in CS^N$ be a non-zero crisp graph. In that case $\vee g = 1$ and $g^g[1] = g$. We get $pg(g) = \{(g, 1)\}$ and $\varepsilon^{pg}(g) = r(g)$.

We consider now $\{i, j\} \in \text{link}(\gamma)$ for any $\gamma = (\tau, \rho)$ and $t \in (0, \rho(i, j)]$. We apply the aforementioned algorithm to calculate the proportional by graphs partition of both γ and $\gamma^t_{-\{i,j\}}$. If any of the vertices i, j do not have the same level of the link, $\tau(i) > \rho(i, j)$, then for both fuzzy graphs the result of the algorithm is the same because this link is not used in any crisp graph of the partition. Hence $\varepsilon^{pg}(\gamma) = \varepsilon^{pg}(\gamma^t_{-\{i,j\}})$. If both the players satisfy $\tau(i) = \tau(j) = \rho(i, j)$ then we obtain exactly the same pairs in the algorithm until the step k where we choose as level $s_k = \rho(i, j)$. At this moment the chosen communication structures are different, if we denote g_k and g'_k as the graphs used by the algorithm for γ and $\gamma^t_{-\{i,j\}}$ respectively then $g'_k < g_k$ because both use the same set of vertices. But when any of these graphs is deleted in the structure we get the same effect because both the players i, j are deleted, therefore the other steps are exactly the same in the algorithm. As the Myerson measure (2) is monotonic (by links) we have $r(g) \geq r(g')$ and then $\varepsilon^{pg}(\gamma) \geq \varepsilon^{pg}(\gamma^t_{-\{i,j\}})$ using (7).

Finally we see that the selection is additive by connected components. Let $\gamma = (\tau, \rho) \in FCS^N$ be. If $pg(\gamma) = (g_k, s_k)_{k=1}^{k=m}$ then for all $H \in N/\gamma$

$$pg(\gamma_H) = \{((g_k)_H, s_k) : \ k \in \{1, \ldots, m\}, \ (g_k)_H \neq 0\}$$

because each player i is used only once with level $\tau(i)$. Moreover, $\{(g_k)_H : \ H \in N/\gamma\}$ is a partition of g_k. So,

$$\varepsilon^{pg}(\gamma) = \sum_{k=1}^{m} s_k r(g_k) = \sum_{k=1}^{m} s_k \sum_{H \in N/\gamma} r((g_k)_H) = \sum_{H \in N/\gamma} \sum_{k=1}^{m} s_k r((g_k)_H) = \sum_{H \in N/\gamma} \varepsilon^{pg}(\gamma_H). \qquad \square$$

The proportional by graphs selection determines the *pg-fuzzy Myerson value* $M^{v,pg}$ from Definition 5 which is a good extension of the crisp Myerson value. This allocation rule presents nice results.

Theorem 4. *The pg-fuzzy Myerson value for v satisfies*

(1) $M^{v,pg}(\gamma) = \sum_{k=1}^{m} s_k \mu^v(g_k)$ *with* $pg(\gamma) = (g_k, s_k)_{k=1}^{k=m}$ *for each* $\gamma \in FCS^N$.

(2) $M^{v,pg}(\gamma) = \phi^{pr}(v)(\tau)$ *if* $\gamma = (\tau, \rho)$ *is complete by links.*

Proof. (1) Using (6) we have $v^{\gamma}_{pg}(S) = \varepsilon^{pg}(\gamma_S)$ for each $\gamma \in FCS^N$ and $S \subseteq N$. Let $pg(\gamma) = (g_k, s_k)_{k=1}^{k=m}$. As players only work once using their highest levels then $pg(\gamma_S) = \{((g_k)_S, s_k) : \ k \in \{1, \ldots, m\}, (g_k)_S \neq 0\}$. Hence by (7) and (3)

$$v^{\gamma}_{pg}(S) = \sum_{k=1}^{m} s_k r((g_k)_S) = \sum_{k=1}^{m} s_k v^{g_k}(S).$$

We can write that $v^{\gamma}_{pg} = \sum_{k=1}^{m} s_k v^{g_k}$. The linear property of the Shapley value (1) and the definition of the crisp Myerson value imply

$$M^{v,pg}(\gamma) = \phi(v^{\gamma}_{pg}) = \sum_{k=1}^{m} s_k \phi(v^{g_k}) = \sum_{k=1}^{m} s_k \mu^v(g_k).$$

(2) Let $\gamma = (\tau, \rho)$ be a fuzzy communication structure which is complete by links thus it is associated with the fuzzy coalition τ of its vertices. We consider its partition proportional by graphs $pg(\gamma) = (g_k, s_k)_{k=1}^{k=m}$. We prove the claim: if γ is complete by links then each g_k is a connected graph. We take two players i, j who are vertices in g_k, and therefore both of them have the same level and this coincides with $\tau(i)$. But as $\rho(i, j) = \tau(i)$ and neither of these players were used before then $\{i, j\} \in \text{link}(g_k)$. Moreover, if $S \subseteq N$ then the same reasoning implies $(g_k)_S$ is connected.

Now, we can obtain the proportional partition of τ (see Section 2.2) by $pg(\gamma)$. We get $pr(\tau) = (\text{vert}(g_k), s_k)_{k=1}^{k=m}$. The levels s_k are the different numbers in τ and we use in g_k all the vertices within the level s_k which forms a connected graph, this being $S[k] = \text{vert}(g_k)$.

If $S \subseteq N$ then $v^{g_k}(S) = r((g_k)_S) = v(\text{vert}(g_k) \cap S) = v_{S[k]}(S)$ by (3). Hence $v^{g_k} = v_{S[k]}$. Finally using (1) and (4)

$$M^{v,pg}(\gamma) = \sum_{k=1}^{m} s_k \mu^v(g_k) = \sum_{k=1}^{m} s_k \phi(v^{g_k}) = \sum_{k=1}^{m} s_k \phi(v_{S[k]}) = \phi^{pr}(v)(\tau). \qquad \square$$

The latter condition in the above theorem says that the *pg*-fuzzy Myerson value is also an extension of the proportional-Shapley value (4).

4.2. Vertex option

In this second option we consider that the behavior of the players' object is the set of vertices. So the proportional behavior in this option says: *Players only play once and their levels are not divisible*.

There is more than one partition by levels representing this specific players' behavior for each fuzzy graph because they do not have any obligation on the links and they can use a tree to connect the chosen coalition, thus keeping the unnecessary links for next cooperations. We construct any of these partitions with the following algorithm. Let $\gamma = (\tau, \rho) \in FCS^N$ be a fuzzy communication structure.

We take k=0, $pv = \emptyset$ and $\gamma = \gamma$.

While $\gamma \neq 0$ do

$k = k + 1$

$s_k = \vee\gamma$

$g_k = g^{\gamma}[s_k]$

Choose g_k^H a spanning tree for each $H \in N/g_k$

$$g_k^* = \sum_{H \in N/g_k} g_k^H$$

$pv = pv \cup \{(g_k^*, s_k)\}$

$\gamma = \gamma - s_k g_k^*$

The partition by levels is pv.

Example 4. Fig. 5 explains the preceding algorithm for a specific fuzzy graph γ. We obtain one of the feasible *pv*-partitions for γ in it.

Hence there exists different *pv*-selections depending on the spanning trees used. We denote PV as the finite set of selections *pv* obtained with the aforementioned algorithm. But all these selections get exactly the same measure over the fuzzy communication structures, moreover this measure is the same of the preceding model.

Proposition 5. *If $pv \in PV$ then $\varepsilon^{pv} = \varepsilon^{pg}$.*

Proof. Suppose that we need m steps in the algorithm to calculate the partition by levels $pv(\gamma)$ for $\gamma \in FCS^N$ and $pv \in PV$. Elements s_1 and g_1 in the first step are the same as the first step in the algorithm to get *pg*. We construct g_1^* which satisfies $N/g_1^* = N/g_1$. Therefore, $r(g_1^*) = r(g_1)$. Moreover, as s_1 is the highest level of the players in γ we get $\gamma - s_1 g_1^* = \gamma - s_1 g_1$ because all the vertices in g_1^* are deleted and all the links incidental to these vertices are also deleted. We repeat the reasoning for the next step and we have $\varepsilon^{pv}(\gamma) = \varepsilon^{pg}(\gamma)$ by (7). $\square$

Hence we always obtain the same fuzzy Myerson value using any selection *pv*, the *pg*-fuzzy Myerson value. This option is not a new model.

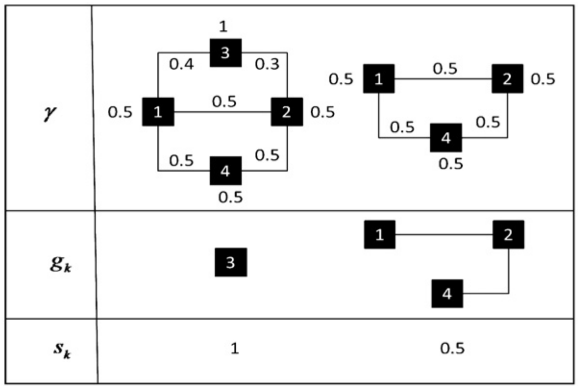

Fig. 5. *pv*-partition.

4.3. Link option

We consider now that the behavior of the players' object is a set of links. So, the proportional behavior says now: *Links only play once and their levels are not partitives.* Players can play several times.

There is only one partition representing this specific players' behavior. Let $\gamma = (\tau, \rho) \in FCS^N$ be a fuzzy communication structure. We consider this other family of crisp versions

$$g_*^\gamma[t] = (\{i \in N : \exists \rho(i, j) = t\}, \{\{i, j\} \in L : \rho(i, j) = t\}).$$

Particularly $g_*^\gamma[0] = g^\gamma$.

We take k=0, $pl = \emptyset$ and $\gamma = \gamma$.

While $\gamma \neq 0$ do

$$k = k + 1$$

$$s_k = \vee\{\rho(i, j) : i, j \in N\}$$

$$g_k = g_*^{\gamma_k}[s_k]$$

$$pl = pl \cup \{(g_k, s_k)\}$$

$$\gamma = \gamma - s_k g_k$$

The partition by levels is *pl*.

Example 5. Fig. 6 contains the application of the preceding algorithm for a fuzzy graph γ.

We have a fuzzy partition selection *pl* which we name *proportional by links*. Unfortunately this selection is not admissible for all the games *v*. Therefore, we have found a "fuzzy Myerson value" which is not a good extension. The next example shows that this allocation rule is not fuzzy fair.

Example 6. We consider the fuzzy communication structure γ in the next figure with $N = \{1, 2, 3\}$. We take *v* as $v(N) = 5$, $v(\{2, 3\}) = 3$ otherwise $v(S) = 0$. The *pl*-selection is not *v*-admissible because the measure is not monotonic, in Fig. 7 we can also see the subgraph $\gamma^1_{-\{1,3\}}$ and $\varepsilon^{pl}(\gamma) = 0 < 1.5 = \varepsilon^{pl}(\gamma^1_{-\{1,3\}})$.

The modified game (6) by the fuzzy graph γ using the *pl*-selection is defined as $v^\gamma_{pl}(\{2, 3\}) = 1.5$ and $v^\gamma_{pl}(S) = 0$ otherwise. We also get the game $v_{pl}^{\gamma^1_{-\{1,3\}}}(N) = v_{pl}^{\gamma^1_{-\{1,3\}}}(\{2, 3\}) = 1.5$ and $v_{pl}^{\gamma^1_{-\{1,3\}}}(S) = 0$ otherwise. Following

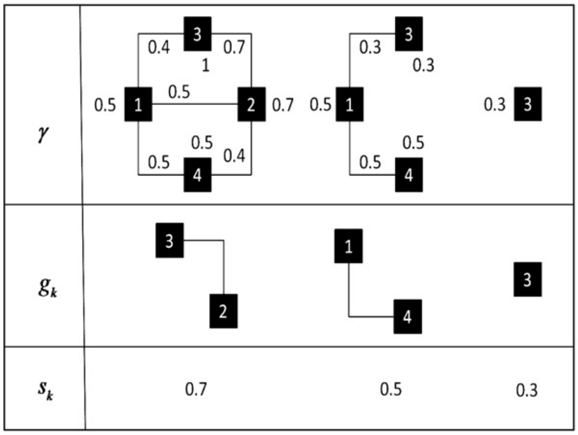

Fig. 6. *pl*-partition.

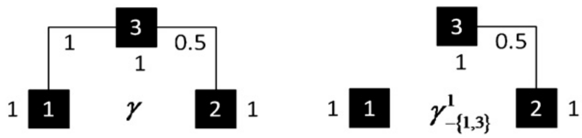

Fig. 7. Deleting the capacity of a link in a fuzzy graph.

Definition 5 and using (1) we have $M^{v,pl}(\gamma) = (-0.5, 1, 1)$ and $M^{v,pl}(\gamma^1_{-\{1,3\}}) = (0, 0.75, 0.75)$. Our value is not fuzzy fair because

$$M_3^{v,pl}(\gamma) - M_3^{v,pl}(\gamma^1_{-\{1,3\}}) = 0.25 \neq -0.5 = M_1^{v,pl}(\gamma) - M_1^{v,pl}(\gamma^1_{-\{1,3\}}).$$

5. The Choquet model

In this section we are going to introduce several fuzzy Myerson values for the game v which are interesting because they represent the Choquet players' behavior from different points of view.

5.1. Graph option

We consider that the behavior player's object of this the whole fuzzy graph, including its set of vertices and links, as in the first model. So Choquet players' behavior says: *Players and links can allocate their capacities and they try to get the biggest crisp graph.*

We can see again that there is only one partition representing this specific players' behavior. We construct this new partition with the following algorithm. Let $\gamma = (\tau, \rho) \in FCS^N$ be a fuzzy communication structure.

We take k=0, $cg = \emptyset$ and $\gamma = \gamma$.

While $\gamma \neq 0$ do

$$k = k + 1$$

$$s_k = \wedge\gamma$$

$$g_k = g^\gamma$$

$$cg = cg \cup \{(g_k, s_k)\}$$

$$\gamma = \gamma - s_k g_k$$

The partition by levels is cg.

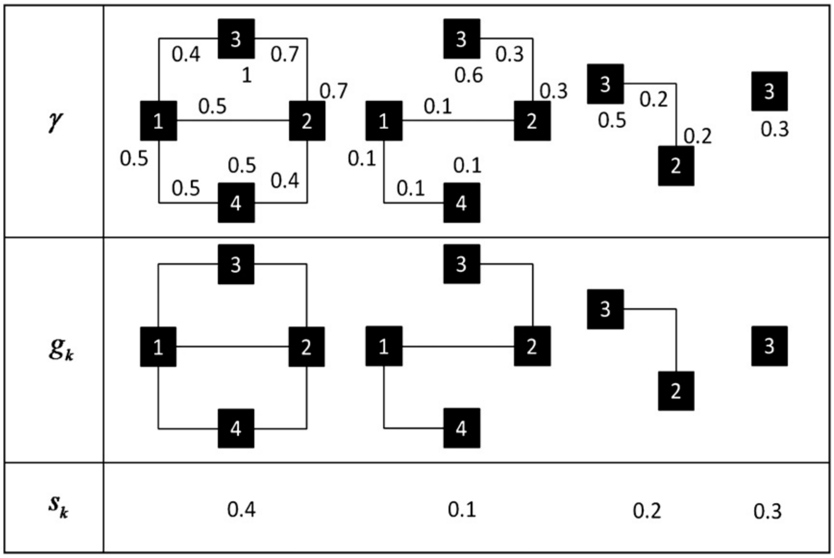

Fig. 8. *cg*-partition.

Example 7. We can see in Fig. 8 the *cg*-partition by levels of a fuzzy graph γ.

Therefore, we have a fuzzy partition selection *cg* named *Choquet by graphs*. We will prove that this selection is admissible for v.

Theorem 6. *The Choquet by graphs selection cg is v-admissible.*

Proof. Let $g \in CS^N$ be a non-zero crisp graph. In that case $s_1 = 1$ and $g_1 = g$. There are not more steps in the preceding algorithm, hence we get $\varepsilon^{cg}(g) = r(g)$ by (7).

We consider $\gamma = (\tau, \rho) \in FCS^N$ and $\{i, j\} \in \text{link}(\gamma)$. We apply the preceding algorithm with its notations to calculate Choquet by graphs partitions. Let $t \in (0, \rho(i, j)]$. The cg-partitions $(g_p, s_p)_p$, $(g'_q, s'_q)_q$ obtained through the algorithm for the fuzzy graphs γ, $\gamma^t_{-\{i,j\}}$ respectively will be compared in two different situations:

(a) First we suppose $t = \rho(i, j)$. While $\wedge\gamma < \rho(i, j)$ we get $g'_p = (g_p)_{-\{i,j\}}$ and then we obtain in the algorithm (g_p, s_p) for γ and $((g_p)_{-\{i,j\}}, s_p)$ for $\gamma^t_{-\{i,j\}}$. Let k be the step where $\wedge\gamma = \rho(i, j)$. At this moment there are two possibilities: either $\wedge\gamma^t_{-\{i,j\}} = \rho(i, j)$ or $\wedge\gamma^t_{-\{i,j\}} > \rho(i, j)$. If $\wedge\gamma^t_{-\{i,j\}} = \rho(i, j)$ then $s_k = s'_k$ and $g'_k = (g_k)_{-\{i,j\}}$. This fact implies $\gamma = \gamma^t_{-\{i,j\}}$ in the step k+1 of the algorithm. Since $r(g'_p) \leq r(g_p)$ for $p \leq k$ we have $\varepsilon^{cg}(\gamma^t_{-\{i,j\}}) \leq \varepsilon^{cg}(\gamma)$ using (7). If $\wedge\gamma^t_{-\{i,j\}} > \rho(i, j)$ then $s_k < s'_k$ and $g'_k = (g_k)_{-\{i,j\}}$. Moreover, the reader can see that s'_k is just the next minimal level to s_k in γ. Unfortunately the equality $\gamma = \gamma^t_{-\{i,j\}}$ is not true in the step k+1 of the algorithm. But in that step we get $s_{k+1} = s'_k - s_k$ and $g_{k+1} = g'_k$ because we use up the link $\{i, j\}$ in the last step. Thus we need to compare two steps of the algorithm for γ with only one for $\gamma^t_{-\{i,j\}}$. So, starting in the step k for both the fuzzy graphs, we have $\gamma = \gamma - (s_k g_k + s_{k+1} g_{k+1}) = \gamma^t_{-\{i,j\}} - s'_k g'_k$, the fuzzy graphs are the same in step $k + 2$ for γ and in step $k + 1$ for $\gamma^t_{-\{i,j\}}$. Since $s'_k r(g'_k) \leq s_k r(g_k) + s_{k+1} r(g_{k+1})$ and $r(g'_p) \leq r(g_p)$ for $p < k$ we obtain $\varepsilon^{cg}(\gamma^t_{-\{i,j\}}) \leq \varepsilon^{cg}(\gamma)$ by (7).

(b) Second we consider at the beginning $t < \rho(i, j)$. In this case the first step k which is different in the algorithm for the fuzzy graphs γ, $\gamma^t_{-\{i,j\}}$ happens when $\wedge\gamma^t_{-\{i,j\}} = \alpha - t$, where $\alpha = \rho(i, j)$ (ρ for this step). We get (g_k, s_k) in the step k following the algorithm for γ with $\alpha - t \leq s_k \leq \alpha$. In the same step k for the other fuzzy graph we obtain $s'_k = \alpha - t$ and $g'_k = g_k$ because the link $\{i, j\}$ is deleted. We begin the next step with $\gamma^t_{-\{i,j\}} = \gamma^t_{-\{i,j\}} - (\alpha - t) g_k$. The algorithm chooses $s'_{k+1} = s_k - \alpha + t$ and g'_{k+1} as g_k deleting perhaps (if $s_k < \alpha$) the link $\{i, j\}$. Since $g'_{k+1} \leq g_k$ (if we use the same vertices) we have $r(g'_{k+1}) \leq r(g_k)$ and so we obtain $s'_k r(g'_k) + s'_{k+1} r(g'_{k+1}) \leq s_k r(g_k)$. If γ is the fuzzy graph in the step $k + 1$ and, conversely, $\gamma^t_{-\{i,j\}}$ is the other fuzzy graph in the step k+2 then $t = \rho(i, j)$ in this step. Now the algorithm can go on like our first case, getting $\varepsilon^{cg}(\gamma^t_{-\{i,j\}}) \leq \varepsilon^{cg}(\gamma)$.

Finally we see that the selection is additive by connected components. If $g \in CS^N$ then $r(g) = \sum_{H \in N/g} r(g_H)$. Let $\gamma = (\tau, \rho) \in FCS^N$ and $cg(\gamma) = (g_k, s_k)_{k=1}^{k=m}$. For each $H \in N/\gamma$ we consider the indices $(k_p)_{p=1}^{p=q}$ such that $(g_{k_p+1})_H \neq (g_{k_p})_H$ (we take $k_q = m$ if $(g_m)_H \neq 0$). It is easy to test that $cg(\gamma_H) = (g'_p, s'_p)_{p=1}^{p=q}$ with

$$g'_p = (g_{k_p})_H \quad \text{and} \quad s'_p = \sum_{k=1}^{k_q} s_k. \tag{8}$$

Moreover, the measure (7) of the connected component is

$$\varepsilon^{cg}(\gamma_H) = \sum_{p=1}^{q} \left(\sum_{k=1}^{k_p} s_k \right) r((g_{k_p})_H) = \sum_{k=1}^{m} s_k r((g_k)_H),$$

where $(g_k)_H = 0$ for all $k > k_q$. So, by (2)

$$\sum_{H \in N/\gamma} \varepsilon^{cg}(\gamma_H) = \sum_{H \in N/\gamma} \sum_{k=1}^{m} s_k r((g_k)_H) = \sum_{H \in N/\gamma} \sum_{k=1}^{m} s_k \sum_{H' \in N/(g_k)_H} v(H')$$
$$= \sum_{k=1}^{m} s_k \sum_{H \in N/\gamma} \sum_{H' \in N/(g_k)_H} v(H') = \sum_{k=1}^{m} s_k r(g_k) = \varepsilon^{cg}(\gamma)$$

because $N/g_k = \bigcup_{H \in N/\gamma} N/(g_k)_H$ for every k. □

The Choquet by graphs selection determines the *cg-fuzzy Myerson value* $M^{v,cg}$ as mentioned in Definition 4. We can say then that the cg-fuzzy Myerson value is also a good extension of the crisp Myerson value. This fuzzy allocation rule satisfies nice results.

Theorem 7. *The cg-fuzzy Myerson value for v satisfies*:

(1) $M^{v,cg}(\gamma) = \sum_{k=1}^{m} s_k \mu^v(g_k)$ *where* $cg(\gamma) = (g_k, s_k)_{k=1}^{k=m}$ *for each* $\gamma \in FCS^N$.
(2) $M^{v,cg}(\gamma) = \phi^{ch}(v)(\tau)$ *if* $\gamma = (\tau, \rho) \in FCS^N$ *is complete by links.*

Proof. (1) The algorithm gets $cg(\gamma) = (g_k, s_k)_{k=1}^{k=m}$ for each $\gamma \in FCS^N$. Let $S \subseteq N$ be a crisp coalition. Following the same reasoning explained in (8) we have $cg(\gamma_S) = (g'_p, s'_p)_{p=1}^{p=q}$ with $(k_q)_{p=1}^{p=q}$ such that $(g_{k_p+1})_S \neq (g_{k_p})_S$ ($k_q = m$ if $(g_m)_S \neq 0$), $g'_p = (g_{k_p})_S$ and $s'_p = \sum_{k=1}^{k_q} s_k$. Hence we obtain by (6) and (7)

$$v^{\gamma}_{cg}(S) = \varepsilon^{cg}(\gamma_S) = \sum_{p=1}^{q} \left(\sum_{k=1}^{k_p} s_k \right) r((g_{k_p})_S) = \sum_{k=1}^{m} s_k r((g_k)_S) = \sum_{k=1}^{m} s_k v^{g_k}(S),$$

where $(g_k)_S = 0$ for all $k > k_q$. So, by Definition 4, the definition of the crisp Myerson value and the linearity of the Shapley value(1)

$$M^{v,cg}(\gamma) = \phi(v^{\gamma}_{cg}) = \sum_{k=1}^{m} s_k \phi(v^{g_k}) = \sum_{k=1}^{m} s_k \mu^v(g_k).$$

(1) Let $\gamma = (\tau, \rho)$ be a fuzzy communication structure which is complete by links thus it is associated with the fuzzy coalition τ of its vertices. We consider its Choquet by graphs partition $cg(\gamma) = (g_k, s_k)_{k=1}^{k=m}$. Since the crisp version is complete by links then $N = \text{vert}(\gamma)$ and $h_1 = \bigwedge_{i \in N} \tau(i)$. Moreover, $h_1 = \wedge\gamma = s_1$. The fuzzy graph γ is connected and this fact implies $S_1 = N = \text{vert}(g_1)$. But the new fuzzy graph in the second step of the algorithm, $\gamma = \gamma - s_1 g_1$, is also complete by links because we have reduced the level of all the elements in the graph with the same capacity h_1. We can repeat the reasoning obtaining $h_2 - h_1 = s_2$ and $S_2 = \text{vert}(g_2)$. So $ch(\tau) = (\text{vert}(g_k), s_k)_{k=1}^{k=m}$.

If $S \subseteq N$ then $v^{g_k}(S) = r((g_k)_S) = v(\mathrm{vert}(g_k) \cap S) = v_{S_k}(S)$ by (3), (2) and because g_k is a complete graph. Hence $v^{g_k} = v_{S[k]}$ and finally using (1)

$$M^{v,cg}(\gamma) = \sum_{k=1}^{m} s_k \mu^v(g_k) = \sum_{k=1}^{m} s_k \phi(v^{g_k}) = \sum_{k=1}^{m} [h_k - h_{k-1}] \phi(v_{S_k]}) = \phi^{ch}(v)(\tau). \qquad \square$$

The last property in the above theorem implies the cg-fuzzy Myerson value is also an extension of the Choquet–Shapley value (5).

5.2. Vertex option

The object of the Choquet behavior are the vertices. So, the Choquet behavior says: *Players can allocate their capacities and they try to get the biggest crisp coalition*. Suppose capacities of links are divisible.

This option was studied by the authors in [4] but now we introduce it in the context of the general model. Players choose one of the biggest feasible crisp coalitions in the fuzzy graph but they only need a maximal spanning tree to connect these players. There can be several maximal spanning trees in a connected graph thus we obtain several different partitions by levels of a fuzzy communication structure explaining this option. We construct one of these partitions by levels with the following algorithm. Let $\gamma = (\tau, \rho) \in FCS^N$ be a fuzzy communication structure. There is a family of crisp versions by different levels associated to a fuzzy communication structure $\gamma = (\tau, \rho) \in FCS^N$. If $t \in (0, 1]$ then the *crisp t-version (cut)* of γ is

$$g_t^{\gamma} = (\{i \in N : \tau(i) \geq t\},\ \{\{i, j\} \in L : \rho(i, j) \geq t\})$$

and $g_0^{\gamma} = g^{\gamma}$. If γ is connected then the *connection level* is the maximal level t such that g_t^{γ} is connected and $\mathrm{vert}(g_t^{\gamma}) = \mathrm{vert}(\gamma)$, which is denoted by $cl(\gamma)$. A *maximal spanning tree* in the connected fuzzy graph γ is any crisp spanning tree in $g^{\gamma}_{cl(\gamma)}$.

We take k=0, $cv = \emptyset$ and $\gamma = \gamma$.

While $\gamma \neq 0$ do

$k = k + 1$

Choose $H \in N/\gamma$

$s_k = cl(\gamma_H)$

Choose g_k a maximal spanning tree in g_H

$cv = cv \cup \{(g_k, s_k)\}$

$\gamma = \gamma - s_k g_k$

The partition by levels is cv.

Example 8. Fig. 9 shows one of the cv-partitions by levels of a fuzzy graph obtained by the preceding algorithm.

The measure from a partition by level obtained in the algorithm depends on the chosen spanning trees, as it is proved in [4]. Therefore, we have a finite family CV of fuzzy partition selections cv. In that case we supposed players seek the best possibility and then we define

$$\varepsilon^{CV} = \bigvee_{cv \in CV} \varepsilon^{cv}.$$

Theorem 8. *The measure ε^{CV} is a profit measure for fuzzy graphs by the game v.*

Proof. Let $g \in CS^N$ be a non-zero crisp graph. In that case $s_1 = 1$. For any connected component H and any maximal spanning tree g_1 of g_H we obtain $r(g_k) = v(H)$ and players in H are deleted. We can repeat the same reasoning using

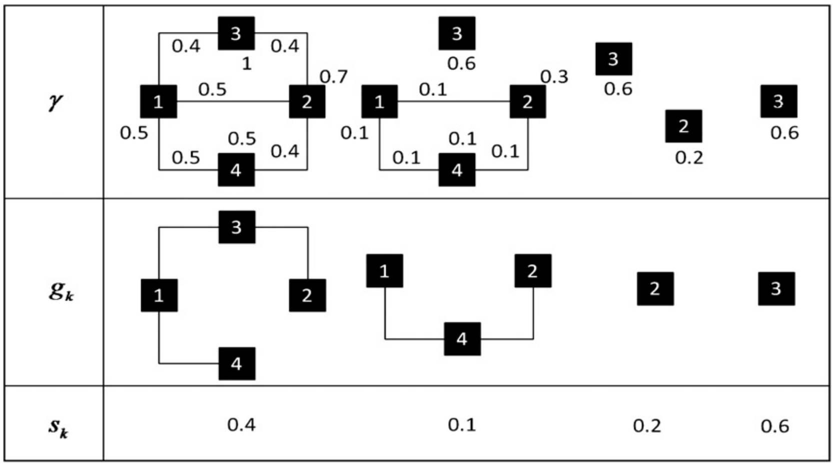

Fig. 9. *cv*-partition.

up the connected components in the other steps. So $\varepsilon^{cv}(g) = r(g)$ for all $cv \in CV$. We suppose now $\{i, j\} \in \text{link}(\gamma)$ for any $\gamma = (\tau, \rho)$ and $t \in (0, \rho(i, j)]$. Proposition 3 and Lemma 4 in [4] proved that if $\gamma' = (\tau, \rho') \leq \gamma$ (a subgraph with the same fuzzy set of vertices) then $\varepsilon^{cv}(\gamma') \leq \varepsilon^{cv}(\gamma)$ for all $cv \in CV$. So $\gamma^t_{-\{i,j\}}$ is a subgraph with the same set of vertices than γ. The additivity by connected components of ε^{CV} was seen in the proof of Theorem 6 [4]. □

Any selection $cv \in CV$ getting the maximal measure obtains the same fuzzy Myerson value $M^{v,cv}$ named the *cv-fuzzy Myerson value* from Definition 5. The *cv*-fuzzy Myerson value is also an extension of the Choquet–Shapley value (5) as it is proved in Theorem 8 [4]. But Example 7 in that paper showed that this value is not always the linear combination of crisp Myerson values. The selections using non-maximal *cv*-partitions are not *v*-admissible in general.

5.3. Link option

Finally, we consider that the object of the Choquet behavior is the set of links. On that note, the Choquet behavior says: *Links can allocate their capacities and players try to get the biggest set of links*. Suppose capacities of players are divisible.

Nonetheless, if we take all the links then we get the whole graph, thus the minimal level of the links is the minimal level in the graph (considering the isolated vertices at the end of the algorithm). Furthermore, the algorithm representing this behavior is the same one as the graph option. This is not a new model.

6. Conclusions

In this paper we have explained a general framework to extend the Myerson model for games with communication structure to games with fuzzy communication structure. Definition 2 permits defining different ways of extension depending on the profit measure in fuzzy graphs. Ultimately, we have used a specific technique to obtain profit measures using selections of partitions by levels of fuzzy graphs. Some of these selections were studied in the last two sections considering different behaviors of the players: the proportional model and the Choquet model. One of them coincides with the model introduced in [4].

Finally, we show an example of calculating the proposed fuzzy Myerson values. Suppose $N = \{1, 2, 3, 4\}$ and the game $v(S) = 2|S| - 1$ for all non-empty coalition S. The Shapley value of this game is $\phi(v) = \mu^v(gN) = (1.75, 1.75, 1.75, 1.75)$. We take γ as the fuzzy graph described in Fig. 10. If the communication structure was considered crisp then we would obtain the crisp Myerson value, $\mu^v(g^\gamma) = (1.67, 1.67, 2.17, 1.5)$.

If we consider that the elements of the graph (or the vertices only) can only be used once then we calculate the graph option (or vertex option) of the proportional model, $M^{v,pg}(\gamma) = M^{v,pv}(\gamma) = (1, 0.6, 1, 0.8)$. If the edges only can be used once then we use the link option in the proportional model, although this one does not have to be fuzzy stable

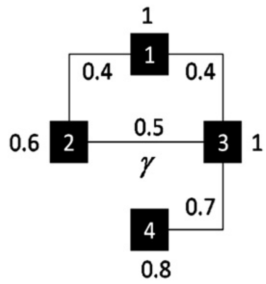

Fig. 10. Fuzzy graph γ.

nor fuzzy fair as we showed in Example 6. So, we get the solution $M^{v,pl}(\gamma) = (1.4, 0.8, 1.5, 1.1)$. Now we take into consideration the Choquet model. The graph option (or the link option) seeks the bigger graph in γ in each step, so $M^{v,cg}(\gamma) = (1.27, 0.92, 1.67, 1.15)$. The vertex option of this model [4], obtains the best profit measure for this graph and the following allocation $M^{v,cv}(\gamma) = (1.4, 1.05, 1.8, 1.15)$.

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