Math Meth Oper Res (2001) 54:359–371

Mathematical Methods
of Operations Research

The Myerson value for union stable structures

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Manuscript received: January 2001/Final version received: June 2001

Abstract. We study cooperation structures with the following property: Given any two feasible coalitions with non-empty intersection, its union is a feasible coalition again. These combinatorial structures have a direct relationship with graph communication situations and conference structures à la Myerson. Characterizations of the *Myerson value* in this context are provided using the concept of *basis* for *union stable systems*. Moreover, *TU*-games restricted by union stable systems generalizes graph-restricted games and games with permission structures.

Mathematics Subject Classification 2000: 91A12

Key words: Myerson value, communication situations, permission structures, union stable systems, Shapley value

1 Introduction

A *transferable utility game* or *TU-game* on a finite set of players N is a pair (N, v) where $v : 2^N \to \mathbb{R}$ is such that $v(\emptyset) = 0$. The elements in N are called *players*, the subsets $S \in 2^N$ *coalitions* and $v(S)$ is the *worth* of the coalition S. For every subset of players S, $v(S)$ represents the maximal monetary gains that these players can achieve when they decide to cooperate and form the coalition S. We will denote by Γ^N the set of all *TU*-games on N. Cooperative game theory usually focuses on the negotiation process within the grand coalition N of all players. So the central question is how divide the gains of N among the players in a fair and justifiable way such that cooperation between all players will persist. To better allow for a consistent approach the theory does not restrict to the local level of one particular game but more globally tries to analyze and *solve* classes of *TU*-games.

A one-point solution concept for *TU*-games is a function $f : \Gamma^N \to \mathbb{R}^N$

which assigns to every TU-game a n-dimensional real vector $f(v) \in \mathbb{R}^N$ which represents payoffs to the players in the sense that each player $i \in N$ gets a payoff of $f_i(v)$. One of the most interesting and best-studied solution concepts in cooperative game theory is the *Shapley value* as introduced by Shapley [11].

In a TU-game model it is generally assumed that there are no restrictions on cooperation and therefore each subgroup of players can form a coalition. However, in practice this seems to be inappropriate in modelling certain situations where social asymmetries among the players make certain coalitions infeasible. Several models of restricted cooperation have been proposed, among which are those derived from *communication situations* as introduced by Myerson [6]. In this model, the bilateral communication relations among the players are taken into account and modelled by means of an undirected graph. The feasible coalitions are those that induce connected subgraphs. Another type of asymmetry among the players in a TU-game is introduced in Gilles, Owen and van den Brink [5], and van den Brink [4]. In their model, the possibilities of coalition formation are determined by the positions of the players in a so-called *permission structure*.

An important aspect in communication situations is the study of the *Myerson value* [6] and the *position value* [3] that were defined with the aid of the Shapley value of two different types of communication games. This line of research in communication situations was continued by Owen [9], Borm, Owen and Tijs [3] and Potters and Reijnierse [10], among others. However, Myerson in [7] pointed out the need to generalize this model towards restricted cooperation situations which can not be modelled by a graph. This idea has been studied by van den Nouweland, Borm and Tijs [8] and Algaba, Bilbao, Borm and López [1].

In Algaba et al. [1] it is assumed that if two feasible coalitions have common elements, these ones will act as intermediaries between the two coalitions in order to establish meaningful cooperation in the union of these coalitions. These feasible coalition systems are called *union stable systems*. This mathematical feature will be essential in our study and it is satisfied for the feasible coalitions coming from graph communication situations and permission structures. This approach has already been successfully applied to the position value in [1]. The current paper in some sense complements this study by focusing on the Myerson value.

To be self-contained section 2 recalls the main definitions on restricted cooperation by means of union stable systems including the crucial driving notion of *basis*. Section 3 introduces the Myerson value for games restricted by union stable systems and studies in detail some properties of this value. The concept of basis allows to extend the axiomatic characterizations provided for the Myerson value in Myerson [6] in an elegant way. Section 4 deals with some computational aspects of the Myerson value. The paper concludes with some remarks on the relationship among the models of partial cooperation mentioned above and in particular about relations between the Myerson value and position value in union stable systems, hypergraph communication situations and permission structures.

2 Union stable systems

Definition 2.1. *Let $N = \{1, 2, \ldots, n\}$ be a finite set of players and $\mathscr{F} \subseteq 2^N$ a system of feasible coalitions. The set system $\mathscr{F}$ is called union stable if for all $A, B \in \mathscr{F}$ with $A \cap B \neq \emptyset$ it is satisfied that $A \cup B \in \mathscr{F}$.*

A communication situation is a triple (N, v, E), where (N, v) is a game and (N, E) is a simple graph, i.e., a graph without self-loops and parallel edges. It is easy to see that the set system $\mathscr{F}$, defined by

$$\mathscr{F} = \{S \subseteq N : (S, E(S)) \text{ is a connected subgraph of } (N, E)\},$$

is union stable. However, a union stable system can not always be modelled by a communication situation. If $N = \{1, 2, 3\}$ and $\mathscr{F} = \{\emptyset, \{1\}, \{2\}, N\}$ then $\mathscr{F}$ is union stable and $|S| \neq 2$ for every $S \in \mathscr{F}$. Then $\mathscr{F}$ does not coincide with the connected subgraph system of any graph.

Let $\mathscr{F}$ be a union stable system and $\mathscr{G} \subseteq \mathscr{F}$. We define inductively the families $\mathscr{G}^{(0)} = \mathscr{G}$, $\mathscr{G}^{(n)} = \{S \cup T : S, T \in \mathscr{G}^{(n-1)}, S \cap T \neq \emptyset\}$, $n = 1, 2, \ldots$ Notice that $\mathscr{G}^{(0)} \subseteq \mathscr{G}^{(n-1)} \subseteq \mathscr{G}^{(n)} \subseteq \mathscr{F}$, since $\mathscr{G} \subseteq \mathscr{F}$ and $\mathscr{F}$ is union stable.

Definition 2.2. *Let $\mathscr{F}$ be a union stable system and let $\mathscr{G} \subseteq \mathscr{F}$. We define $\overline{\mathscr{G}}$ by $\overline{\mathscr{G}} = \mathscr{G}^{(k)}$, where k is the smallest integer such that $\mathscr{G}^{(k+1)} = \mathscr{G}^{(k)}$.*

For each union stable family $\mathscr{F}$, we are interested in finding a minimal subset $\mathscr{B}(\mathscr{F})$ such that $\overline{B(\mathscr{F})} = \mathscr{F}$. Let $\mathscr{F}$ be a union stable system and $\mathscr{G} \subseteq \mathscr{F}$. If $\mathscr{G}$ is union stable, there can be feasible coalitions which can be written as the union of two feasible coalitions with non-empty intersection. So, we can consider the following set:

$$D(\mathscr{G}) = \{G \in \mathscr{G} : G = A \cup B, A \neq G, B \neq G, A, B \in \mathscr{G}, A \cap B \neq \emptyset\}.$$

Definition 2.3. *Let $\mathscr{F}$ be a union stable system. The set $\mathscr{B}(\mathscr{F}) = \mathscr{F} \backslash \mathscr{D}(\mathscr{F})$, is called the basis of $\mathscr{F}$, and the elements of $\mathscr{B}(\mathscr{F})$ are called supports of $\mathscr{F}$.*

We remark that the basis $B(\mathscr{F})$ is the minimal subset of the union stable system $\mathscr{F}$ such that $\overline{B(\mathscr{F})} = \mathscr{F}$ (see Algaba et al. [1]).

Definition 2.4. *Let $\mathscr{G} \subseteq 2^N$ be a set system and let $S \subseteq N$. A set $T \subseteq S$ is called a $\mathscr{G}$-component of S if it is satisfied that $T \in \mathscr{G}$ and there exists no $T' \in \mathscr{G}$ such that $T \subset T' \subseteq S$.*

Therefore, the $\mathscr{G}$-components of S are the maximal feasible coalitions that belong to $\mathscr{G}$ and are contained in S. We denote by $C_{\mathscr{G}}(S)$ the collection of the $\mathscr{G}$-components of S.

Proposition 2.1. *The set system $\mathscr{F} \subseteq 2^N$ is union stable if and only if for any $S \subseteq N$ with $C_{\mathscr{F}}(S) \neq \emptyset$, the $\mathscr{F}$-components of S are a collection of pairwise disjoint subsets of S.*

Proof. Let $\mathscr{F}$ be union stable. Let C^1, C^2 be two different maximal feasible coalitions of S. If $C^1 \cap C^2 \neq \emptyset$, then $C^1 \cup C^2 \in \mathscr{F}$ since $\mathscr{F}$ is union stable and $C^1 \cup C^2 \subseteq S$. This contradicts the fact that C^1 and C^2 are $\mathscr{F}$-components of S.

Conversely, assume for any S such that $C_{\mathscr{F}}(S) \neq \emptyset$, that its $\mathscr{F}$-components form a collection of pairwise disjoint subsets of S. Suppose that $\mathscr{F}$ is not union stable, then there are $A, B \in \mathscr{F}$, with $A \cap B \neq \emptyset$ and $A \cup B \notin \mathscr{F}$. Hence, there must be an $\mathscr{F}$-component $C_1 \in C_{\mathscr{F}}(A \cup B)$, with $A \subseteq C_1$ and an $\mathscr{F}$-component $C_2 \in C_{\mathscr{F}}(A \cup B)$, with $B \subseteq C_2$ such that $C_1 \neq C_2$. This contradicts the fact that the $\mathscr{F}$-components of $A \cup B$ are disjoint. □

It is obvious that if $\mathscr{F}$ is a union stable system such that $\{i\} \in \mathscr{F}$, for all $i \in N$, then the $\mathscr{F}$-components of S form a partition of S. We have also the following consequence of the definitions.

Proposition 2.2. *Let $\mathscr{F}$ be a union stable system. Let $S \subseteq N$ and consider $\mathscr{F}_S = \{F \in \mathscr{F} : F \subseteq S\}$. Then, the following conditions are satisfied:*

(a) *$\mathscr{F}_S$ is union stable.*
(b) $C_{\mathscr{F}}(S) = C_{\mathscr{F}_S}(N)$.
(c) $\mathscr{B}(\mathscr{F}_S) = \{B \in \mathscr{B}(\mathscr{F}) : B \subseteq S\}$.

In order to establish a relation between conference structures à la Myerson and union stable systems, we give the following results. Moreover, the next theorem will be essential in order to prove the uniqueness in the axiomatization of the Myerson value in union stable systems.

Definition 2.5. *Let $\mathscr{F}$ be a union stable system. The players $i, j \in N$, are called connected by $\mathscr{C}(\mathscr{F}) = \{B \in \mathscr{B}(\mathscr{F}) : |B| \geq 2\}$ if there exists a sequence of non-unitary supports $(B_1, \ldots, B_k)$, such that $i \in B_1$, $j \in B_k$ and if $k \geq 2$, $B_p \cap B_{p+1} \neq \emptyset$, for all $p = 1, \ldots, k-1$.*

Theorem 2.3. *Let $\mathscr{F}$ be a union stable system. Let $S \in \mathscr{F}$ and $i, j \in N$, $i \neq j$. Then $\{i, j\} \subseteq S$ if and only if i and j are connected by $\mathscr{C}(\mathscr{F})$ with supports contained in S.*

Proof. Let $\{i, j\} \subseteq S$. If $S \in \mathscr{C}(\mathscr{F})$, it suffices to take $k = 1$ and $B_1 = S$. If $S \notin \mathscr{C}(\mathscr{F})$, then $S = A \cup B$, with $A, B \in \mathscr{F}$, and $A \cap B \neq \emptyset$. If $A, B \in \mathscr{C}(\mathscr{F})$ then we obtain the result. Otherwise, we repeat this decomposition and, proceeding in this manner, we obtain the sequence of supports. The converse is obvious. □

Corollary 2.4. *Let $\mathscr{F}$ be a union stable system and $i, j \in N$, $i \neq j$. Then i and j are in the same $\mathscr{F}$-component of N if and only if i and j are connected by $\mathscr{C}(\mathscr{F})$.*

3 The Myerson value: properties and axiomatizations

This section deals with a solution concept for games restricted by union stable structures: the *Myerson value*.

Definition 3.1. *Let (N, v) be a cooperative n-person game in coalitional form and $\mathscr{F} \subseteq 2^N$ a union stable system. The $\mathscr{F}$-restricted game $v^{\mathscr{F}} : 2^N \to \mathbb{R}$, is defined by $v^{\mathscr{F}}(S) = \sum_{T \in C_{\mathscr{F}}(S)} v(T)$.*

A *union stable structure* is a triple $(N, v, \mathscr{F})$ where $N = \{1, 2, \ldots, n\}$ is the set of players, (N, v) is a game $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$, and $\mathscr{F}$ is a union stable system. The Shapley value of a game (N, v) is the vector $\Phi(N, v) \in \mathbb{R}^N$ defined by

$$\Phi_i(N, v) = \sum_{\{S \subseteq N : i \in S\}} \frac{(s-1)!(n-s)!}{n!}(v(S) - v(S \setminus \{i\})),$$

where $i \in N$, $n = |N|$ and $s = |S|$.

Definition 3.2. *The Myerson value of a union stable structure* $(N, v, \mathscr{F})$ *is the Shapley value of the* $\mathscr{F}$*-restricted game, i.e.,* $\mu(N, v, \mathscr{F}) = \Phi(N, v^{\mathscr{F}})$.

Example. Consider the player set $N = \{1, 2, 3, 4\}$ and the union stable system given by $\mathscr{F} = \{\{1\}, \{1, 2, 3\}, \{2, 3, 4\}, N\}$. Let $v : 2^N \to \mathbb{R}$ be the game defined by $v(S) = |S| - 1$, $S \neq \emptyset$, and $v(\emptyset) = 0$. Then the basis $\mathscr{B}(\mathscr{F}) = \{\{1\}, \{1, 2, 3\}, \{2, 3, 4\}\}$ and $\mathscr{C}(\mathscr{F}) = \{\{1, 2, 3\}, \{2, 3, 4\}\}$. In this case, it is clear that

$$v^{\mathscr{F}}(S) = \begin{cases} |S| - 1, & \text{if } S \in \mathscr{F}, \\ 0, & \text{otherwise,} \end{cases}$$

and the Myerson value is $\mu(N, v, \mathscr{F}) = \frac{1}{12}(5, 13, 13, 5)$.

We now consider some properties that would be desirable for *an allocation rule*, and we focus on the study of these properties for the Myerson value. The set of all union stable structures with player set N will be denoted by US^N.

Definition 3.3. *An allocation rule on* US^N *is a map* $\gamma : US^N \to \mathbb{R}^N$*, satisfying:*

(1) *Component-efficiency:* $\sum_{i \in M} \gamma_i(N, v, \mathscr{F}) = v(M)$*, for all* $(N, v, \mathscr{F}) \in US^N$ *and* $M \in C_{\mathscr{F}}(N)$.
(2) *Component-null:* $\gamma_i(N, v, \mathscr{F}) = 0$*, for all* $i \notin \bigcup_{M \in C_{\mathscr{F}}(N)} M$.

Lemma 3.1. *The Myerson value* $\mu : US^N \to \mathbb{R}^N$ *is an allocation rule.*

Proof. Let $(N, v, \mathscr{F}) \in US^N$. If $N \in \mathscr{F}$, then N is its unique $\mathscr{F}$-component, and hence $\sum_{i \in N} \mu(N, v, \mathscr{F}) = \sum_{i \in N} \Phi_i(N, v^{\mathscr{F}}) = v^{\mathscr{F}}(N) = v(N)$. Suppose that $N \notin \mathscr{F}$ and consider the set $C_{\mathscr{F}}(N)$. To each $\mathscr{F}$-component M of N is associated the game u^M which is defined in the following way

$$u^M : 2^N \to \mathbb{R}, \quad u^M(T) = v^{\mathscr{F}}(T \cap M) = \sum_{H \in C_{\mathscr{F}}(T \cap M)} v(H), \quad \text{for all } T \subseteq N.$$

Moreover, for any coalition $T \subseteq N$, $C_{\mathscr{F}}(T) = \bigcup_{R \in C_{\mathscr{F}}(N)} C_{\mathscr{F}}(T \cap R)$, and hence, it is immediate that $v^{\mathscr{F}} = \sum_{R \in C_{\mathscr{F}}(N)} u^R$. Taking it into account we get

$$\sum_{i \in M} \mu_i(N, v, \mathscr{F}) = \sum_{i \in M} \Phi_i(N, u^M) + \sum_{\{R \in C_{\mathscr{F}}(N) : R \neq M\}} \left[\sum_{i \in M} \Phi_i(N, u^R) \right].$$

Since $\sum_{i \in M} \Phi_i(N, u^M) = v^{\mathscr{F}}(M)$, and for all $R \neq M$ by Proposition 2.1 we have that $\Phi_i(N, u^R) = 0$, for all $i \in M$, the above expression implies that $\sum_{i \in M} \mu_i(N, v, \mathscr{F}) = v^{\mathscr{F}}(M) = v(M)$.

Component-null is immediate since for $i \notin \bigcup_{M \in C_{\mathscr{F}}(N)} M$, we have $C_{\mathscr{F}}(S) = C_{\mathscr{F}}(S \setminus \{i\})$, for all $S \in \mathscr{F}$. Hence, the marginal contributions are $v^{\mathscr{F}}(S) - v^{\mathscr{F}}(S \setminus \{i\}) = 0$, and $\mu_i(N, v, \mathscr{F}) = 0$. □

Definition 3.4. *An allocation rule* γ *is fair if for all* $(N, v, \mathscr{F}) \in US^N$ *and* $B \in \mathscr{B}(\mathscr{F})$, *there exists* $c \in \mathbb{R}$ *such that* $\gamma_j(N, v, \mathscr{F}) - \gamma_j(N, v, \mathscr{F}') = c$, *for all* $j \in B$, *where* $\mathscr{F}' = \overline{\mathscr{B}(\mathscr{F}) \setminus \{B\}}$.

So, according to a fair allocation rule all players in a support B lose or gain the same amount if the support B and all coalitions that are obtained by union stability using support B are deleted. We now extend the axiomatization of the Myerson value to union stable structures.

Theorem 3.2. *The Myerson value is the unique fair allocation rule on* US^N.

Proof. (a) Uniqueness: Let $(N, v, \mathscr{F}) \in US^N$. Suppose γ^1 and γ^2 are two fair allocation rules on US^N. We prove by induction on the number $|\mathscr{C}(\mathscr{F})|$ of non-unitary supports in the basis of $\mathscr{F}$, that $\gamma^1(N, v, \mathscr{F}) = \gamma^2(N, v, \mathscr{F})$.

If $|\mathscr{C}(\mathscr{F})| = 0$, then $C_{\mathscr{F}}(N) = \{\{i\} : \{i\} \in \mathscr{F}\}$. Applying component-efficiency and component-null we obtain that $\gamma^1(N, v, \mathscr{F}) = \gamma^2(N, v, \mathscr{F})$.

Now, assume that $\gamma^1(N, v, \mathscr{G}) = \gamma^2(N, v, \mathscr{G})$ for all $\mathscr{G}$ with $|\mathscr{C}(\mathscr{G})| \leq k - 1$, and let $|\mathscr{C}(\mathscr{F})| = k$. Consider $C \in \mathscr{C}(\mathscr{F})$. Fairness implies that there exist numbers $c \in \mathbb{R}$ and $d \in \mathbb{R}$ such that

$$\gamma_j^1(N, v, \mathscr{F}) - \gamma_j^1(N, v, \overline{\mathscr{B}(\mathscr{F}) \setminus \{C\}}) = c,$$

$$\gamma_j^2(N, v, \mathscr{F}) - \gamma_j^2(N, v, \overline{\mathscr{B}(\mathscr{F}) \setminus \{C\}}) = d,$$

for all $j \in C$. Note that by the induction hypothesis

$$\gamma_j^1(N, v, \overline{\mathscr{B}(\mathscr{F}) \setminus \{C\}}) = \gamma_j^2(N, v, \overline{\mathscr{B}(\mathscr{F}) \setminus \{C\}}).$$

So there is a constant $\alpha = c - d$ such that

$$\gamma_j^1(N, v, \mathscr{F}) - \gamma_j^2(N, v, \mathscr{F}) = \alpha, \quad \text{for all } j \in C. \tag{1}$$

Given $M \in C_{\mathscr{F}}(N)$, by component-efficiency for γ^1 and γ^2, we obtain

$$\sum_{i \in M} [\gamma_i^1(N, v, \mathscr{F}) - \gamma_i^2(N, v, \mathscr{F})] = 0.$$

On the other hand, Theorem 2.3 indicates that for every $j, k \in M$, there exists a non-unitary supports sequence $(B_1, B_2, \ldots, B_p)$, contained in M such that $j \in B_1$, $k \in B_p$ and $B_i \cap B_{i+1} \neq \emptyset$, $i = 1, \ldots, p - 1$ and using equality (1) we have $\gamma_j^1(N, v, \mathscr{F}) - \gamma_j^2(N, v, \mathscr{F}) = \alpha$, for $j \in B_1$, and as $B_1 \cap B_2 \neq \emptyset$, there exists $h \in B_1 \cap B_2$ such that

$$\gamma_h^1(N, v, \mathscr{F}) - \gamma_h^2(N, v, \mathscr{F}) = \alpha.$$

Thus, applying Theorem 2.3 and equality (1) recursively for all elements of the sequence $(B_1, B_2, \ldots, B_p)$ we get

$$\gamma_j^1(N, v, \mathscr{F}) - \gamma_j^2(N, v, \mathscr{F}) = \gamma_k^1(N, v, \mathscr{F}) - \gamma_k^2(N, v, \mathscr{F}).$$

and hence $\gamma_i^1(N, v, \mathscr{F}) - \gamma_i^2(N, v, \mathscr{F}) = \alpha$, for all $i \in M$, $M \in C_{\mathscr{F}}(N)$. This implies

$$\sum_{i \in M} [\gamma_i^1(N, v, \mathscr{F}) - \gamma_i^2(N, v, \mathscr{F})] = |M|\alpha.$$

Therefore $|M|\alpha = 0$, and hence $\gamma^1(N, v, \mathscr{F}) = \gamma^2(N, v, \mathscr{F})$.

(b) Next, we show that the Myerson value is fair. Consider the game (N, w) given by $w(S) = v^{\mathscr{F}}(S) - v^{\mathscr{F}'}(S)$, for all $S \subseteq N$, where $\mathscr{F}' = \overline{\mathscr{B}(\mathscr{F}) \backslash \{B\}}$. Let $k \in B$. Note that $w(S) = 0$, for all $S \subseteq N$, $B \nsubseteq S$. Also note that for any $S \subseteq N$ with $B \subseteq S$, $w(S \backslash \{k\}) = 0$ since $B \nsubseteq S \backslash \{k\}$. Thus, we can write for $k \in B$

$$\Phi_k(N, w) = \sum_{\{S: B \subseteq S\}} \frac{(s-1)!(n-s)!}{n!} w(S), \quad \text{where } s = |S|, \; n = |N|. \tag{2}$$

It follows that $\Phi_k(N, w) = \Phi_p(N, w)$, for all $p \in B$. In other words, the Myerson value is fair. □

Definition 3.5. *An allocation rule γ is called basis monotonic on the subclass $\mathscr{C}^N \subseteq US^N$ if for all $(N, v, \mathscr{F}) \in \mathscr{C}^N$, for all $B \in \mathscr{B}(\mathscr{F})$ such that $(N, v, \mathscr{F}') \in \mathscr{C}^N$, and for all $j \in B$, $\gamma_j(N, v, \mathscr{F}) \geq \gamma_j(N, v, \mathscr{F}')$, where $\mathscr{F}' = \overline{\mathscr{B}(\mathscr{F}) \backslash \{B\}}$.*

This condition asserts that all the players always benefit from reaching an agreement and cooperate.

Recall that a game v is *superadditive* if cooperation is profitable, that is, $v(S \cup T) \geq v(S) + v(T)$, for all disjoint coalitions $S, T \in 2^N$. A game v is called *zero-normalized* if $v(\{i\}) = 0$, for all $i \in N$.

Proposition 3.3. *Let $(N, v, \mathscr{F}) \in US^N$. If the game v is superadditive and zero-normalized, then $\mu(N, v, \mathscr{F})$ is basis monotonic.*

Proof. Equation (2) implies that is suffices to prove that $w(S) \geq 0$ for any $S \subseteq N$ such that $B \subseteq S$, where for all $S \subseteq N$, $w(S) = v^{\mathscr{F}}(S) - v^{\mathscr{F}'}(S)$, with $\mathscr{F}' = \overline{\mathscr{B}(\mathscr{F}) \backslash \{B\}}$. Any maximal feasible coalition of S in $\mathscr{F}'$ is either a maximal feasible coalition of S in $\mathscr{F}$ or it is contained in an $\mathscr{F}$-component of S. Then, taking the $\mathscr{F}'$-components of S and taking into consideration that the game (N, v) is superadditive and zero-normalized, we obtain

$$v^{\mathscr{F}'}(S) = \sum_{T' \in C_{\mathscr{F}'}(S)} v(T') \leq \sum_{T \in C_{\mathscr{F}}(S)} \left[v\left(\bigcup_{\{T' \in C_{\mathscr{F}'}(S): T' \subseteq T\}} T' \right) \right] \leq v^{\mathscr{F}}(S). \quad \square$$

To provide other axiomatic characterizations for the Myerson value, the next definitions are introduced. We use $\mathscr{C}_i(\mathscr{F})$ to denote the collection given by $\{C \in \mathscr{C}(\mathscr{F}) : i \in C\}$.

Definition 3.6. *A union stable structure* $(N, v, \mathscr{F})$ *is called point anonymous if there exists a function* $f : \{0, 1, \ldots, |D|\} \to \mathbb{R}$ *such that* $v^{\mathscr{F}}(S) = f(|S \cap D|)$ *for all* $S \subseteq N$*, where* $D = \{i \in N : \mathscr{C}_i(\mathscr{F}) \neq \emptyset\}$.

Definition 3.7. *An allocation rule* γ *satisfies point anonymity if for all point anonymous* $(N, v, \mathscr{F})$*, there exists* $\alpha \in \mathbb{R}$ *such that*

$$\gamma_i(N, v, \mathscr{F}) = \begin{cases} \alpha, & \text{for all } i \in D, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 3.4. *The Myerson value satisfies point anonymity.*

Proof. Let $(N, v, \mathscr{F}) \in US^N$ be point anonymous. If $D = \emptyset$, then the restricted game $v^{\mathscr{F}}(S) = f(|S \cap \emptyset|) = f(0) = 0$, for all $S \subseteq N$. Hence, $\mu_i(N, v, \mathscr{F}) = 0$ for all $i \in N$. Let $D \neq \emptyset$. If $i \notin D$, obviously $S \cap D = (S \backslash \{i\}) \cap D$ and $\mu_i(N, v, \mathscr{F}) = 0$. On the other hand, if $i, j \in D$ applying the symmetry property of the Shapley value we have $\mu_i(N, v, \mathscr{F}) = \mu_j(N, v, \mathscr{F})$, and hence $f(|D|) = \sum_{i \in D} \mu_i(N, v, \mathscr{F}) = |D| \mu_i(N, v, \mathscr{F})$. Therefore, $\mu_i(N, v, \mathscr{F}) = f(|D|)/|D| = \alpha$, for all $i \in D$ and $\mu_i(N, v, \mathscr{F}) = 0$, otherwise. □

Definition 3.8. *Let* $(N, v, \mathscr{F}) \in US^N$*. Then player* $i \in N$ *is called superfluous for* $(N, v, \mathscr{F})$ *if* $v^{\mathscr{F}}(S) = v^{\mathscr{F}}(S \backslash \{i\})$*, for all* $S \subseteq N$*. An allocation rule* γ *satisfies the superfluous player property if for all* $(N, v, \mathscr{F})$ *and every player* $i \in N$ *that is superfluous for* $(N, v, \mathscr{F})$ *it holds* $\gamma(N, v, \mathscr{F}) = \gamma(N, v, \mathscr{F}_{N \backslash \{i\}})$*, where* $\mathscr{F}_{N \backslash \{i\}} = \{F \in \mathscr{F} : F \subseteq N \backslash \{i\}\}$.

Proposition 3.5. *The Myerson value satisfies the superfluous player property.*

Proof. Let $i \in N$ be a superfluous player for $(N, v, \mathscr{F}) \in US^N$. We have to prove $\mu(N, v, \mathscr{F}) = \mu(N, v, \mathscr{F}_{N \backslash \{i\}})$. We observe that i is a null player in $v^{\mathscr{F}}$ and this implies that $\mu_i(N, v, \mathscr{F}) = 0$. Further, $\mu_i(N, v, \mathscr{F}_{N \backslash \{i\}}) = 0$, because $i \notin \bigcup_{M \in C_{\mathscr{F}_{N \backslash \{i\}}}(N)} M$ and μ satisfies component-null (Lemma 3.1).

For the other players, it suffices to show that $v^{\mathscr{F}}(S) = v^{\mathscr{F}_{N \backslash \{i\}}}(S)$, or equivalently that $v^{\mathscr{F}}(S \backslash \{i\}) = v^{\mathscr{F}_{N \backslash \{i\}}}(S)$, for all $S \subseteq N$, and for every i that is a superfluous player for $(N, v, \mathscr{F})$. The components satisfy $C_{\mathscr{F}}(S \backslash \{i\}) = C_{\mathscr{F}_{N \backslash \{i\}}}(S)$, and therefore

$$v^{\mathscr{F}}(S \backslash \{i\}) = \sum_{T \in C_{\mathscr{F}}(S \backslash \{i\})} v(T) = \sum_{T \in C_{\mathscr{F}_{N \backslash \{i\}}}(S)} v(T) = v^{\mathscr{F}_{N \backslash \{i\}}}(S),$$

for all $S \subseteq N$. □

Definition 3.9. *An allocation rule* γ *is called additive if for all* $(N, v, \mathscr{F})$ *and* $(N, w, \mathscr{F})$*,* $\gamma(N, v + w, \mathscr{F}) = \gamma(N, v, \mathscr{F}) + \gamma(N, w, \mathscr{F})$.

We immediately obtain that the Myerson value is additive.

Lemma 3.6. *If* γ *is an additive allocation rule that satisfies the superfluous player property, then* $\gamma(N, v, \mathscr{F}) = \gamma(N, v^{\mathscr{F}}, \mathscr{F})$*, for all* $(N, v, \mathscr{F}) \in US^N$.

Proof. By additivity of γ, it suffices to show that $\gamma(N, v - v^{\mathscr{F}}, \mathscr{F}) = 0$, for all $(N, v, \mathscr{F}) \in US^N$. Indeed, for any $S \subseteq N$,

$$(v - v^{\mathscr{F}})^{\mathscr{F}}(S) = \sum_{T \in C_{\mathscr{F}}(S)} (v - v^{\mathscr{F}})(T) = \sum_{T \in C_{\mathscr{F}}(S)} [v(T) - v^{\mathscr{F}}(T)] = 0.$$

Thus, all players are superfluous for any $(N, v - v^{\mathscr{F}}, \mathscr{F}) \in US^N$. Hence, by recursively considering all players in the same component $M \in C_{\mathscr{F}}(N)$ we have $\gamma(N, v - v^{\mathscr{F}}, \mathscr{F}) = \gamma(N, v - v^{\mathscr{F}}, \mathscr{F}_{N\setminus M})$. For all $i \in M$, we obtain $\gamma_i(N, v - v^{\mathscr{F}}, \mathscr{F}) = \gamma_i(N, v - v^{\mathscr{F}}, \mathscr{F}_{N\setminus M}) = 0$, since $i \notin \bigcup_{H \in C_{\mathscr{F}_{N\setminus M}}(N)} H$. It follows that $\gamma_i(N, v - v^{\mathscr{F}}, \mathscr{F}) = 0$, for all $i \in N$. □

Theorem 3.7. *The Myerson value is the unique allocation rule on US^N that satisfies additivity, the superfluous player property and point anonymity.*

Proof. Let γ be an allocation rule on US^N that also satisfies additivity, the superfluous player property and point anonymity. From Lemma 3.6 we deduce $\gamma(N, v, \mathscr{F}) = \gamma(N, v^{\mathscr{F}}, \mathscr{F})$. The unanimity games $\{u_S : S \in \mathscr{F}, S \neq \emptyset\}$ form a basis for the vector space of the $\mathscr{F}$-restricted games (see Bilbao [2]), that is,

$$v^{\mathscr{F}} = \sum_{\{S \in \mathscr{F} : S \neq \emptyset\}} \alpha_S u_S$$

for some coefficients α_S.

Applying additivity, it suffices to show that $\gamma(N, \alpha u_S, \mathscr{F})$, is uniquely determined for all $S \in \mathscr{F}$, $S \neq \emptyset$ and $\alpha \in \mathbb{R}$. Fix S and α. If $i \in N\setminus S$ then for all coalitions $T \subseteq N$

$$\alpha u_S(T) = \alpha \Leftrightarrow S \subseteq T \Leftrightarrow S \subseteq T\setminus\{i\} \Leftrightarrow \alpha u_S(T\setminus\{i\}) = \alpha.$$

We deduce that any player that is not in S is superfluous and hence, by the superfluous player property,

$$\gamma(N, \alpha u_S, \mathscr{F}) = \gamma(N, \alpha u_S, \mathscr{F}_{N\setminus(N\setminus S)}) = \gamma(N, \alpha u_S, \mathscr{F}_S).$$

Since $C_{\mathscr{F}_S}(N) = C_{\mathscr{F}}(S) = \{S\}$, component-null property implies that $\gamma_i(N, \alpha u_S, \mathscr{F}_S) = 0$, for all $i \in N\setminus S$. It remains only to compute $\gamma_i(N, \alpha u_S, \mathscr{F}_S)$ for all $i \in S$. First, for all $T \subseteq N$, we have

$$(\alpha u_S)^{\mathscr{F}_S}(T) = \sum_{H \in C_{\mathscr{F}_S}(T)} \alpha u_S(H) = \alpha \Leftrightarrow \exists H \in \mathscr{F}_S, \quad S \subseteq H \subseteq T.$$

If $H \in \mathscr{F}_S$ then $H \subseteq S$, and hence $(\alpha u_S)^{\mathscr{F}_S}(T) = \alpha$ if and only if $S \subseteq T$. Therefore, $(\alpha u_S)^{\mathscr{F}_S} = \alpha u_S$ implies

$$(\alpha u_S)^{\mathscr{F}_S}(T) = \alpha u_S(T) = \alpha \Leftrightarrow S \subseteq T \Leftrightarrow S \cap T = S.$$

It follows that there exists a function $f : \{0, 1, \ldots, |S|\} \to \mathbb{R}$, such that $(\alpha u_S)^{\mathscr{F}_S}(T) = f(|S \cap T|)$, for all $T \subseteq N$, where $f(0) = \cdots = f(|S| - 1) = 0$, and $f(|S|) = \alpha$. Hence $(N, \alpha u_S, \mathscr{F}_S)$ is point anonymous and, applying point anonymity to the rule γ, there exists $\beta \in \mathbb{R}$ such that

$$\gamma_i(N, \alpha u_S, \mathscr{F}_S) = \begin{cases} \beta, & \text{if } i \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Further, $C_{\mathscr{F}_S}(N) = \{S\}$, and using component-efficiency we get

$$\sum_{i \in S} \gamma_i(N, \alpha u_S, \mathscr{F}_S) = \alpha = |S|\beta.$$

Then $\beta = \alpha/|S|$ and we deduce that $\gamma(N, v, \mathscr{F})$ is the Myerson value. □

4 Computational aspects

One of the main problems of the Myerson value is its computation. Some formulas to compute it more easily in this context have been given by Bilbao [2]. In this section we complete this study and show that the Myerson value satisfies the nice formula of Shapley. On the other hand and taking into account that the coalition N may not be feasible, we prove that the calculation of the Myerson value can be simplified and computed by the maximal feasible coalitions of N which each player belongs to.

Definition 4.1. *An allocation rule $\gamma : US^N \to \mathbb{R}^n$, satisfies the Shapley-formula if for every $(N, v, \mathscr{F}) \in US^N$ and every $i \in N$ it is satisfied*

$$\gamma_i(N, v, \mathscr{F}) = \sum_{\{S \subseteq N : i \in S\}} \lambda(S) \left[\sum_{j \in N} \gamma_j(N, v, \mathscr{F}_S) - \sum_{j \in N} \gamma_j(N, v, \mathscr{F}_{S \setminus \{i\}}) \right],$$

where for every $S \subseteq N$,

$$\lambda(S) = \frac{(|S| - 1)!(n - |S|)!}{n!} \quad \textit{and} \quad \mathscr{F}_S = \{F \in \mathscr{F} : F \subseteq S\}.$$

Theorem 4.1. *The Myerson value satisfies the Shapley-formula.*

Proof. Taking into account the definitions of $\mu_i(N, v, \mathscr{F})$ and the one of restricted game $(N, v^{\mathscr{F}})$ we have

$$\mu_i(N, v, \mathscr{F}) = \sum_{\{S \subseteq N : i \in S\}} \lambda(S) \left[\sum_{T \in C_{\mathscr{F}_S}(N)} v(T) - \sum_{T' \in C_{\mathscr{F}_{S \setminus \{i\}}}(N)} v(T') \right],$$

since $C_{\mathscr{F}}(S) = C_{\mathscr{F}_S}(N)$ and $C_{\mathscr{F}}(S \setminus \{i\}) = C_{\mathscr{F}_{S \setminus \{i\}}}(N)$. On the other hand, as $\mu_i(N, v, \mathscr{F})$ is an allocation rule, we conclude

$$\mu_i(N, v, \mathscr{F}) = \sum_{\{S \subseteq N : i \in S\}} \lambda(S) \left[\sum_{j \in N} \mu_j(N, v, \mathscr{F}_S) - \sum_{j \in N} \mu_j(N, v, \mathscr{F}_{S \setminus \{i\}}) \right]. \qquad \square$$

Notice that the Shapley-formula coincides with the classical one when the cooperation is total, i.e., if $\mathscr{F} = 2^N$

$$\Phi_i(N, v) = \sum_{\{S \subseteq N : i \in S\}} \lambda(S)(v(S) - v(S \backslash \{i\})).$$

Theorem 4.2. *Let* $(N, v, \mathscr{F}) \in US^N$. *If* v *is a zero-normalized game and* $M \in C_{\mathscr{F}}(N)$, *such that* $i \in M$, *then* $\mu_i(N, v, \mathscr{F}) = \mu_i(M, v|_M, \mathscr{F}_M)$, *where* $v|_M$ *is the restriction of* v *to* M *and* $\mathscr{F}_M = \{F \in \mathscr{F} : F \subseteq M\}$.

Proof. Let $M \in C_{\mathscr{F}}(N)$ such that $i \in M$ and $L \in C_{\mathscr{F}}(N)$, $L \neq M$. Taking into account that for every coalition $S \subseteq N$ it is satisfied

$$C_{\mathscr{F}}(S) = \bigcup_{H \in C_{\mathscr{F}}(N)} C_{\mathscr{F}}(S \cap H).$$

we obtain $v^{\mathscr{F}}(S \cup \{i\}) - v^{\mathscr{F}}(S) = v^{\mathscr{F}}((S \cap M) \cup \{i\}) - v^{\mathscr{F}}(S \cap M)$, since $C_{\mathscr{F}}((S \cup \{i\}) \cap L) = C_{\mathscr{F}}(S \cap L)$ because $\{i\} \cap L = \varnothing$, for all $L \neq M$. Hence,

$$\begin{aligned}\mu_i(N, v, \mathscr{F}) &= \sum_{\{S \subseteq N : i \notin S\}} \lambda(S)[v^{\mathscr{F}}((S \cap M) \cup \{i\}) - v^{\mathscr{F}}(S \cap M)] \\ &= \sum_{\{R \subseteq M : i \notin R\}} \left[\sum_{\{S \subseteq N : i \notin S, S \cap M = R\}} \lambda(S) \right] [v^{\mathscr{F}}(R \cup \{i\}) - v^{\mathscr{F}}(R)].\end{aligned}$$

Taking into account the classical formula of the Shapley value, we have

$$\sum_{\{S \subseteq N : i \notin S, S \cap M = R\}} \lambda(S) = \sum_{\{S \subseteq N : i \notin S, S \cap M = R\}} \frac{s!(n-1-s)!}{n!}.$$

Fixing a coalition R, the coalitions $S \subseteq N$, such that $S \cap M = R$, are expressed in the following way

$$\begin{cases} R, & \\ R \cup \{i_1\} & \text{for all } i_1 \in N \backslash M, \\ R \cup \{i_1, i_2\} & \text{for all } \{i_1, i_2\} \subseteq N \backslash M, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ R \cup (N \backslash M) & \end{cases}$$

Setting $r = |R|$, $m = |M|$, it boils down to

$$\sum_{\{S \subseteq N : i \notin S, S \cap M = R\}} \lambda(S) = \sum_{p=0}^{n-m} \binom{n-m}{p} \frac{(r+p)!(n-1-r-p)!}{n!},$$

satisfying that

$$\sum_{p=0}^{n-m} \binom{n-m}{p} \frac{(r+p)!(n-1-r-p)!}{n!} = \frac{r!(m-1-r)!}{m!}.$$

Consequently,

$$\mu_i(N, v, \mathscr{F}) = \sum_{\{R \subseteq M : i \notin R\}} \lambda(R)[v^{\mathscr{F}}(R \cup \{i\}) - v^{\mathscr{F}}(R)] = \mu_i(M, v|_M, \mathscr{F}_M). \qquad \square$$

5 Concluding remarks

As we have already indicated in the introduction, one of the most important partial cooperation models was introduced by Myerson: the so-called graph communication situations. Myerson also pointed out the necessity for generalizing this model of partial cooperation. Following Myerson's suggestion hypergraph communication situations [8] and union stable cooperation structures [1] have been investigated.

A *hypergraph communication situation* is a triple $(N, v, \mathscr{H})$ such that (N, v) is a zero-normalized game and the hypergraph $\mathscr{H} \subseteq \{H \in 2^N : |H| \geq 2\}$. The set of hypergraph communication situations is denoted by HCS^N. It is supposed that communication is only possible by the conferences $H \in \mathscr{H}$. So, given a coalition $S \subseteq N$, the feasible coalitions or interaction sets within the coalition S are defined as the sets obtained in the following way:

1. For all $i \in S$, $\{i\}$ is an interaction set of S.
2. If $H \in \mathscr{H}$ and $H \subseteq S$ then H is an interaction set of S.
3. If T_1 and T_2 are interaction sets of S and satisfying that $T_1 \cap T_2 \neq \emptyset$, then $T_1 \cup T_2$ is an interaction set of S.

Taking into account the above definition, it can be assured that hypergraph communication situations give rise to union stable cooperation structures, the feasible coalitions being all interaction sets within arbitrary coalitions. Assuming zero normalization of the underlying game, we may suppose, without loss of generality, that the unitary coalitions belong to the system. So, given a union stable cooperation structure, there are several hypergraph communication situations whose interaction sets coincides with the feasible coalitions of the union stable system. Obviously, one of the hypergraph communication situations is that whose conference set $\mathscr{H}$ is formed by the non-unitary supports of the basis.

In fact, considering US^N instead of HCS^N boils down to considering equivalence classes (respecting interaction sets) which also respect to the Myerson value. Given $(N, v, \mathscr{F}) \in US^N$, where v is zero-normalized, for any player $i \in N$, $\mu_i(N, v, \mathscr{F})$ uniquely depends on the feasible coalitions where the player i is. In the same way, given $(N, v, \mathscr{H}) \in HCS^N$, the Myerson value $\mu_i(N, v, \mathscr{H})$ only depends on the interaction sets which the player i belongs to. Therefore, if it is considered $(N, v, \mathscr{F}) \in US^N$ the Myerson value $\mu(N, v, \mathscr{F})$ is the same as the Myerson value $\mu(N, v, \mathscr{H})$, for every $(N, v, \mathscr{H})$ whose interaction set system coincides with the set of feasible coalitions of $\mathscr{F}$.

Formally, the above ideas could have been used to translate the existing characterizations of the Myerson value on HCS^N to US^N. In this paper, we have instead chosen for an approach symmetric to the one for the position value [1] for which the same line of reasoning can not be applied. Moreover, in our context a non-unitary support is the precise and natural generalization of an edge of a graph. This concept allows for more transparent proofs and results.

A final remark is concerned with so-called permission structure games: games restricted by the feasible coalition system derived from a directed graph using either the conjunctive or disjunctive approach. Gilles et al. [5] and van den Brink [4] showed that the feasible coalition system $\mathscr{F}$ derived from the conjunctive or disjunctive approach is closed under taking unions, and hence constitutes a union stable system. This property implies that every coalition $S \subseteq N$ has a unique $\mathscr{F}$-component that corresponds to the *sovereign part* of S. Then, the $\mathscr{F}$-restricted game is the same as the *conjunctive* or *disjunctive* restriction and hence, the conjunctive or disjunctive permission value coincides with the Myerson value $\mu(N, v, \mathscr{F})$.

The final conclusion is that games restricted by union stable systems generalizes graph communication situations and games with permission structures. Therefore, the Shapley value for games restricted by graphs or permission structures is the Myerson value for union stable systems.

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