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THE MARGINAL OPERATORS FOR GAMES ON CONVEX GEOMETRIES

J. M. BILBAO*, N. JIMÉNEZ, E. LEBRÓN and J. J. LÓPEZ

Department of Applied Mathematics, University of Seville
Camino de los Descubrimientos, 41092 Sevilla, Spain
**mbilbao@us.es*
http://hercules.us.es/~mbilbao

In this work we study situations in which communication among the players is not complete and it is represented by a family of subsets of the set of players. Although several models of partial cooperation have been proposed, we shall follow a model derived from the work of Faigle and Kern. We define the games on convex geometries and introduce marginal worth vectors and quasi-supermodular games. Furthermore, we analyze some properties of the marginal operators on the space of games on convex geometries.

Keywords: Marginal operators; quasi-supermodular games.

Subject Classification: 91A12

1. Introduction

In the framework of cooperative games, one often assumes that all players will cooperate. However, there are many intermediate possibilities between universal cooperation and no cooperation. In this work we discuss a class of partial cooperation structures and develop a model of cooperative games in which only certain coalitions are allowed to form.

The interest in the particular study of games defined on families of feasible coalitions which have the structure of a convex geometry is derived, initially, from the observations indicated by G. Owen (1986) when he analyzes the simplifications which are produced in the computation of the Myerson value if, in the communication situation (N, v, G), the graph which models the partial cooperation among the players is a tree. Later, the special characteristics of these singular communication situations (N, v, G), in which $G = (N, E)$ is a tree have been made clear, among others, in Borm, Nouweland, Owen and Tijs (1993) when the problems of allocation of costs are analyzed and integral formulas are found for the calculation of the Myerson value; Grafe, Mauleon and Iñarra (1995) in the search of a procedure for computing the nucleolus; and Potters and Reijnierse (1995) in the study of an equilibrium character of games restricted by communication graphs, their nucleolus and the relations between the bargaining set and the core. Faigle and Kern's paper

(1992) on cooperative games under precedence constraints bears the closest relation to our work.

In general, the specific properties of the trees, in relation with other types of graphs, lie in their connected character, in that the intersection of connected subgraphs leads to a connected subgraph and that any connected graph is the graph originated by the convex hull of its extreme points. These characteristics and the results obtained in the previously mentioned studies make it interesting to plan the study of games defined on families of feasible coalitions which have a combinatorial structure with analogous properties to the aforementioned.

In the next section, we introduce convex geometries and define the games on these set families. We introduce marginal worth vectors and quasi-supermodular games and study several relations between certain solution concepts for these games. In the third section we analyze the minimal and the maximal marginal operator for games on convex geometries. Both concepts were introduced by Curiel and Tijs (1991) for cooperative games.

2. Games on Convex Geometries

In this section, we define the concept of convex geometry (see Edelman and Jamison (1985)) and we describe some of its fundamental properties.

A *convex geometry* on a finite set N is a family $\mathcal{L} \subseteq 2^N$ which satisfies the following properties:

(C1) $\emptyset, N \in \mathcal{L}$,
(C2) If $A, B \in \mathcal{L}$, then $A \cap B \in \mathcal{L}$,
(C3) If $A \in \mathcal{L}$ with $A \neq N$ then $A \cup \{i\} \in \mathcal{L}$ for some $i \in N \backslash A$.

Unless stated otherwise throughout this paper we suppose $N = \{1, \ldots, n\}$ and $\mathcal{L}$ is always a convex geometry on N.

An element in a convex geometry $\mathcal{L}$ is called *convex set.* A convex geometry $\mathcal{L} \subseteq 2^N$ is called *atomic* if $\{i\} \in \mathcal{L}$ for all $i \in N$. For a convex set $A \in \mathcal{L}$, an element $a \in A$ is an *extreme point* of A if $A \backslash \{a\} \in \mathcal{L}$. We denote by $ex(A)$ the set of all extreme points of A.

A *maximal chain* C of $\mathcal{L}$ is an ordered collection of convex sets

$$C : \emptyset = C_0 \subset C_1 \subset \cdots \subset C_{n-1} \subset C_n = N$$

such that there does not exist a convex set M and $0 \leq j \leq n-1$ satisfying $C_j \subset M \subset C_{j+1}$. We denote by $\mathcal{C}(\mathcal{L})$ the set of all maximal chains of $\mathcal{L}$. Note that when $\mathcal{L} = 2^N$ there are $n!$ maximal chains corresponding to all orderings of players, and for any $S \in 2^N$, $ex(S) = S$.

We shall often use the following property of convex geometries (Edelman and Jamison (1985)): For each $S, T \in \mathcal{L}$ with $S \subset T$, there exists a maximal chain $C \in \mathcal{C}(\mathcal{L})$ containing both S and T.

Example 1. The family $\mathcal{L} = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$ is a convex geometry. There are three maximal chains in $\mathcal{L}$,

$$C_1 : \emptyset \subset \{1\} \subset \{1,2\} \subset \{1,2,3\},$$
$$C_2 : \emptyset \subset \{2\} \subset \{1,2\} \subset \{1,2,3\},$$
$$C_3 : \emptyset \subset \{2\} \subset \{2,3\} \subset \{1,2,3\},$$

and $ex(\{1,2\}) = \{1,2\}$, $ex(\{2,3\}) = \{3\}$ and $ex(\{1,2,3\}) = \{1,3\}$.

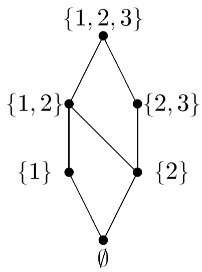

Fig. 1. Diagram of $(\mathcal{L}, \subseteq)$.

Some families of sets with convex geometry structure which have appeared in literature related to partial cooperation are the following.

Example 2. The family of convex subsets of a finite partially ordered set $(N, \leq)$ is a convex geometry (Birkhoff and Bennett (1985)). In this context, $S \subseteq N$ is a convex set if $a \in S,\ b \in S$ and $a \leq b$ imply $c \in S$ for all $c \in N$ such that $a \leq c \leq b$. The convex geometry obtained considering the natural order on N, which we name $Co(N)$, has been used by Edelman (1997) to model a class of voting games.

Example 3. A communication situation is a triple (N, G, v), where (N, v) is a cooperative game and $G = (N, E)$ is a graph. This concept was first introduced by Myerson (1977), and investigated by Borm, Owen and Tijs (1992). If $G = (N, E)$ is a connected and cycle-complete graph, then the family of all coalitions of N that induce connected subgraphs $\mathcal{L} = \{S \subseteq N : (S, E(S))$ is connected in G $\}$, is a convex geometry.

Example 4. Let $(P, \leq)$ be a finite partially ordered set. For any $X \subseteq P$,

$$X \mapsto \bar{X} = \{y \in P : y \leq x \text{ for some } x \in X\},$$

defines a closure operator on P. Its closed sets (that is, $X \subseteq P$ such that $\bar{X} = X$) form a convex geometry $J(P)$. Faigle and Kern (1992) studied games defined on distributive lattices $J(P)$.

A *game on a convex geometry* $\mathcal{L}$ is a function $v : \mathcal{L} \to \mathbb{R}$ such that $v(\emptyset) = 0$.

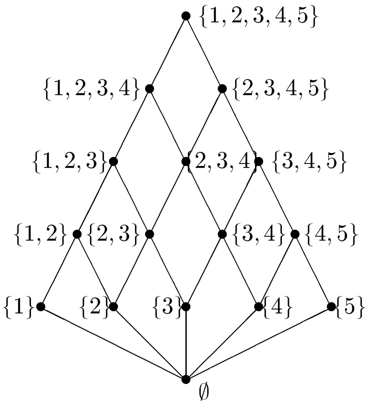

Fig. 2. The convex geometry $Co(N)$ with $N = \{1, 2, 3, 4, 5\}$.

Throughout this work, $\Gamma(\mathcal{L})$ will denote the vector space of the games on $\mathcal{L}$.

A game $v \in \Gamma(\mathcal{L})$ is said to be *quasi-superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \in \mathcal{L}$ such that $S \cap T = \emptyset$ and $S \cup T \in \mathcal{L}$. A game $v \in \Gamma(\mathcal{L})$ is called *quasi-subadditive* if the reverse inequality holds and *additive* if equality holds.

One of the main topics dealt with in cooperative game theory, is to divide the amount $v(N)$ between the players if the grand coalition N is formed. The *core* of the game $v \in \Gamma(\mathcal{L})$ is the set

$$Core(\mathcal{L}, v) = \{x \in \mathbb{R}^n : x(N) = v(N),\ x(S) \geq v(S) \ \text{ for all } S \in \mathcal{L}\},$$

where $x(S) = \sum_{i \in S} x_i$ and $x(\emptyset) = 0$. Note that the core of a game $v \in \Gamma(\mathcal{L})$ is a polyhedron that is not necessarily bounded. However, it is a polytope if and only if the family $\mathcal{L}$ is atomic (see Bilbao, Jiménez, Lebrón (1999)).

Since every maximal chain of $\mathcal{L}$ has exactly $n+1$ convex sets, there is a minimal convex set in every maximal chain that contains a player i. Thus, for every $i \in N$ and $C \in \mathcal{C}(\mathcal{L})$, we denote by $C(i)$ the minimal convex set in chain C which contains player i, that is, $C(i)$ represents the coalition of $\mathcal{L}$ formed by the player i and his predecessors in the chain C. Note that $i \in ex(C(i))$ since $C(i)\backslash\{i\} \in \mathcal{L}$.

The property (C3) of a convex geometry allows us to consider that the coalition formed by all players is reached by sequetial processes of one by one incorporation of the players in the game. A maximal chain $C \in \mathcal{C}(\mathcal{L})$ can be considered as defining a possible order in which the players form the grand coalition N. Each player i obtains the amount that he contributes to the coalition $C(i)\backslash\{i\}$ already formed. The marginal contribution of i if the grand coalition N is formed in the order given by the chain C is defined by $a_i^C(v) = v(C(i)) - v(C(i)\backslash\{i\})$. The marginal worth vector associated with the maximal chain C is $a^C(v) \in \mathbb{R}^n$ whose i-th coordinate is equal to $a_i^C(v)$. These vectors satisfy the following property.

Lemma 1. *Let $v \in \Gamma(\mathcal{L})$ be a game and $C \in \mathcal{C}(\mathcal{L})$. For all $S \in C$, we have*

$$\sum_{j \in S} a_j^C(v) = v(S).$$

Proof. Let $v \in \Gamma(\mathcal{L})$ and $C \in \mathcal{C}(\mathcal{L})$. For every $k \in N$, we denote by S_k the coalition of C such that $|S_k| = k$. Take $S_0 = \emptyset$ and $S_k = \{i_1, \ldots, i_k\}$ for all $1 \leq k \leq n$. For each $k \in N$, it follows that

$$\begin{aligned}\sum_{j \in S_k} a_j^C(v) &= \sum_{j=1}^{k} a_{i_j}^C(v) = \sum_{j=1}^{k} [v(C(i_j)) - v(C(i_j) \backslash \{i_j\})] \\ &= \sum_{j=1}^{k} [v(S_j) - v(S_{j-1})] = v(S_k).\end{aligned}$$

Note that $S_n = N$ and hence $\sum_{j \in N} a_j^C(v) = v(N)$. □

The *Weber set of* $v \in \Gamma(\mathcal{L})$ is the convex hull of marginal worth vectors associated to all maximal chains in $\mathcal{L}$, that is,

$$Weber(\mathcal{L}, v) = \text{ conv}\{a^C(v) : C \in \mathcal{C}(\mathcal{L})\}.$$

Weber (1988) and Derks (1992) proved that if $\mathcal{L} = 2^N$, then the relation $Core(\mathcal{L}, v) \subseteq Weber(\mathcal{L}, v)$ is always verified. However, this inclusion is not true in general if $\mathcal{L} \neq 2^N$, as we show in the following example.

Example 5. Let $N = \{1, 2, 3\}$ and $\mathcal{L} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$. This convex geometry has the following maximal chains

$$\begin{aligned}C_1 &: \ \emptyset \subset \{1\} \subset \{1, 2\} \subset \{1, 2, 3\}, \\ C_2 &: \ \emptyset \subset \{2\} \subset \{1, 2\} \subset \{1, 2, 3\}, \\ C_3 &: \ \emptyset \subset \{2\} \subset \{2, 3\} \subset \{1, 2, 3\}, \\ C_4 &: \ \emptyset \subset \{3\} \subset \{2, 3\} \subset \{1, 2, 3\}.\end{aligned}$$

Let $v \in \Gamma(\mathcal{L})$ be the game given by $v(S) = |S|$ if $|S| \geq 2$, and $v(S) = 0$ otherwise. The marginal worth vectors are given by

$$\begin{aligned}a^{C_1}(v) &= (v(\{1\}) - v(\emptyset),\, v(\{1, 2\}) - v(\{1\}),\, v(N) - v(\{1, 2\})) = (0, 2, 1), \\ a^{C_2}(v) &= (v(\{1, 2\}) - v(\{2\}),\, v(\{2\}) - v(\emptyset),\, v(N) - v(\{1, 2\})) = (2, 0, 1), \\ a^{C_3}(v) &= (v(N) - v(\{2, 3\}),\, v(\{2\}) - v(\emptyset),\, v(\{2, 3\}) - v(\{2\})) = (1, 0, 2), \\ a^{C_4}(v) &= (v(N) - v(\{2, 3\}),\, v(\{2, 3\}) - v(\{3\}),\, v(\{3\}) - v(\emptyset)) = (1, 2, 0),\end{aligned}$$

and hence, $Weber(\mathcal{L}, v) = \text{conv}\{(0, 2, 1), (2, 0, 1), (1, 0, 2), (1, 2, 0)\}$.

On the other hand, $Core(\mathcal{L}, v) = \text{conv}\{(0, 2, 1), (1, 2, 0), (0, 3, 0), (1, 1, 1)\}$ and it is obvious that $Core(\mathcal{L}, v) \nsubseteq Weber(\mathcal{L}, v)$ *and* $Weber(\mathcal{L}, v) \nsubseteq Core(\mathcal{L}, v)$.

Shapley (1971) stated that if $v : 2^N \to \mathbb{R}$ is a supermodular game, then $Core(2^N, v) = Weber(2^N, v)$. Ichiishi (1981) proved that if $Weber(2^N, v) \subseteq$

$Core(2^N, v)$ then v is supermodular. Therefore, $Core(2^N, v) = Weber(2^N, v)$ if and only if v is a supermodular game.

In order to analyze the relationship between these solution concepts for $v \in \Gamma(\mathcal{L})$, we introduce the concept of *quasi-supermodular* game.

A game $v \in \Gamma(\mathcal{L})$ is *quasi-supermodular* if for all $S, T \in \mathcal{L}$ with $S \cup T \in \mathcal{L}$ it holds

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T).$$

A game $v \in \Gamma(\mathcal{L})$ is *quasi-submodular* if the reverse inequality holds.

The following results provide us with several characterizations of the class of quasi-supermodular games.

Proposition 1. *Let $v \in \Gamma(\mathcal{L})$ be a game on a convex geometry. The game v is quasi-supermodular if and only if for all $S, T \in \mathcal{L}$ such that $T \subset S$ and for all $i \in ex(S) \cap T$, it holds $v(S) - v(S \backslash \{i\}) \geq v(T) - v(T \backslash \{i\})$.*

Proof. Let $S, T \in \mathcal{L}$ with $T \subset S$ and $i \in ex(S) \cap T$. If $S' = S \backslash \{i\}$ and $T' = T$, then $S' \cap T' = (S \backslash \{i\}) \cap T = T \backslash \{i\} \in \mathcal{L}$, and $S' \cup T' = (S \backslash \{i\}) \cup T = S \in \mathcal{L}$. Applying the definition of quasi-supermodularity to S' and T', it follows that

$$v(S) + v(T \backslash \{i\}) \geq v(S \backslash \{i\}) + v(T).$$

Conversely, let $S, T \in \mathcal{L}$ such that $S \cup T \in \mathcal{L}$. If $T \subseteq S$ or $S \subseteq T$, then the equality is clear. Consider $S \cap T \neq S$ and $S \cap T \neq T$ and let $C \in \mathcal{C}(\mathcal{L})$ be a maximal chain such that $T \in C$ and $S \cup T \in C$. As $S \backslash T \neq \emptyset$, put $|S \backslash T| = k$ and write $S \backslash T = \{i_1, \ldots, i_k\}$ with $C(i_1) \subset \cdots \subset C(i_k)$, i.e., the chain C is given by

$$\cdots \subset T \subset T \cup \{i_1\} \subset \cdots \subset T \cup \{i_1, i_2, \ldots, i_k\} = T \cup S \subset \cdots$$

Let $R = S \cap T$ and denote $S_j = \{i_1, \ldots, i_j\}$ for all $1 \leq j \leq k$ with $S_0 = \emptyset$. Then $T \cup S_j \in \mathcal{L}$ for all $1 \leq j \leq k$, and $R \cup S_j = (S \cap T) \cup S_j = \cap(T \cup S_j) \in \mathcal{L}$. Since $R \cup S_j \subset T \cup S_j$ and $i_j \in ex(R \cup S_j) \cap (T \cup S_j)$, we get

$$v(R \cup S_j) - v(R \cup S_{j-1}) \leq v(T \cup S_j) - v(T \cup S_{j-1}),$$

and it follows that

$$\begin{aligned} v(S) - v(S \cap T) = v(R \cup S_k) - v(R) &= \sum_{j=1}^{k} [v(R \cup S_j) - v(R \cup S_{j-1})] \\ &\leq \sum_{j=1}^{k} [v(T \cup S_j) - v(T \cup S_{j-1})] = v(T \cup S) - v(T). \end{aligned}$$ □

A similar proof leads us to show that a game $v \in \Gamma(\mathcal{L})$ is quasi-submodular if and only if for all $S, T \in \mathcal{L}$ such that $T \subset S$ and for all $i \in ex(S) \cap T$, it holds $v(S) - v(S \backslash \{i\}) \leq v(T) - v(T \backslash \{i\})$.

Theorem 1. *A game $v \in \Gamma(\mathcal{L})$ is quasi-supermodular if and only if it satisfies $a^C(v) \in Core(\mathcal{L}, v)$, for all $C \in \mathcal{C}(\mathcal{L})$.*

Proof. Let $v \in \Gamma(\mathcal{L})$ be a quasi-supermodular game. We prove that the marginal worth vectors $a^C(v) \in Core(\mathcal{L}, v)$, for all $C \in \mathcal{C}(\mathcal{L})$. Let C be a maximal chain in $\mathcal{L}$. By Lemma 1, $\sum_{j\in S} a_j^C(v) = v(S)$ for all S in the chain C. It remains to prove that $\sum_{j\in S} a_j^C(v) \geq v(S)$ for all $S \in \mathcal{L}$ that do not belong to C. Let $S \in \mathcal{L}$ such that S is not in C with $|S| = s \geq 1$. Then $S = \{i_1, \dots, i_s\}$ where $C(i_1) \subset \cdots \subset C(i_s)$ and we denote by $S_j = \{i_1, i_2, \dots, i_j\}$ for all $1 \leq j \leq s$ and $S_0 = \emptyset$. For all $1 \leq j \leq s$, we have $S_j = S \cap C(i_j) \in \mathcal{L}$, and also $i_j \in ex(C(i_j))$. Thus, Proposition 1 implies that, for all $1 \leq j \leq s$, $v(C(i_j)) - v(C(i_j) \backslash \{i_j\}) \geq v(S_j) - v(S_{j-1})$, and hence

$$\begin{aligned}\sum_{j\in S} a_j^C(v) &= \sum_{j=1}^{s} a_{i_j}^C(v) = \sum_{j=1}^{s} [v(C(i_j)) - v(C(i_j)\backslash\{i_j\})] \\ &\geq \sum_{j=1}^{s} [v(S_j) - v(S_{j-1})] = v(S).\end{aligned}$$

Conversely, let us assume that $a^C(v) \in Core(\mathcal{L}, v)$, for all $C \in \mathcal{C}(\mathcal{L})$ and we show that v is quasi-supermodular. For any $S, T \in \mathcal{L}$ with $S \cup T \in \mathcal{L}$, let $C \in \mathcal{C}(\mathcal{L})$ be a maximal chain containing $S \cap T$ and $S \cup T$. For this chain C, we have that $\sum_{j\in S} a_j^C(v) \geq v(S)$, $\sum_{j\in T} a_j^C(v) \geq v(T)$, $\sum_{j\in S\cup T} a_j^C(v) = v(S \cup T)$ and $\sum_{j\in S\cap T} a_j^C(v) = v(S \cap T)$. Therefore,

$$\begin{aligned}v(S) + v(T) &\leq \sum_{j\in S} a_j^C(v) + \sum_{j\in T} a_j^C(v) \\ &= \sum_{j\in S\cup T} a_j^C(v) + \sum_{j\in S\cap T} a_j^C(v) = v(S\cup T) + v(S \cap T).\end{aligned}$$

□

In a likewise manner it is possible to prove that a game $v \in \Gamma(\mathcal{L})$ is quasi-submodular if and only if $\sum_{i\in S} a_i^C(v) \leq v(S)$ for all $C \in \mathcal{C}(\mathcal{L})$ and $S \in \mathcal{L}$.

As the core of $v \in \Gamma(\mathcal{L})$ is a convex set, an immediate consequence of this result is the following.

Corollary 1. *A game on a convex geometry $v \in \Gamma(\mathcal{L})$ is quasi-supermodular if and only if $Weber(\mathcal{L}, v) \subseteq Core(\mathcal{L}, v)$.*

3. The Marginal Operators

Let $v \in \Gamma(\mathcal{L})$ and $C \in \mathcal{C}(\mathcal{L})$. The additive game $a^C(v) : \mathcal{L} \to \mathbb{R}$, given by $a^C(v)(S) = \sum_{i\in S} a_i^C(v)$ for all $S \in \mathcal{L}$, is called *marginal game* associated to the chain C of the game v.

Marginal games are of interest as they lead us to the Shapley value. The Shapley-value is the average of marginal games. That is, if $v \in \Gamma(\mathcal{L})$ then the Shapley value $\Phi(v)$ is,

$$\Phi(v) = \frac{1}{|\mathcal{C}(\mathcal{L})|} \sum_{C\in\mathcal{C}(\mathcal{L})} a^C(v),$$

where it is assumed that all possible orders (maximal chains in $\mathcal{L}$) of formation of N are equally likely.

Futhermore, marginal games were used by Curiel and Tijs (1991) to introduce marginal operators on $\Gamma(2^N)$. These marginal operators (the minimarg and the maximarg operators) have nice properties. This section is devoted to define these operators on $\Gamma(\mathcal{L})$ and comprove if they verify analogous properties in this case.

The *minimal marginal operator* $Mi : \Gamma(\mathcal{L}) \to \Gamma(\mathcal{L})$ is defined by

$$Mi(v) = \min_{C \in \mathcal{C}(\mathcal{L})} a^C(v).$$

For every $v \in \Gamma(\mathcal{L})$ and $S \in \mathcal{L}$, we have $Mi(v)(S) = \min_{C \in \mathcal{C}(\mathcal{L})} a^C(v)(S)$.

The *maximal marginal operator* $Ma : \Gamma(\mathcal{L}) \to \Gamma(\mathcal{L})$ is defined by

$$Ma(v) = \max_{C \in \mathcal{C}(\mathcal{L})} a^C(v),$$

and so, for every $v \in \Gamma(\mathcal{L})$ and $S \in \mathcal{L}$, $Ma(v)(S) = \max_{C \in \mathcal{C}(\mathcal{L})} a^C(v)(S)$.

Each maximal chain of $\mathcal{L}$ represents a possible formation order of the grand coalition. Every player receives his marginal contribution in this process and the worth of a coalition is the sum of what its members receive. A feasible coalition $S \in \mathcal{L}$ may consider a worst possible order, i.e., an order for which the sum of what its members receive is minimal, that is, the coalition S takes the value $Mi(v)(S)$. On the other hand $Ma(v)(S)$ is the worth of feasible coalition S when the grand coalition is formed according to a best possible order for S. Note that if convex geometry has a unique maximal chain, it holds from Lemma 1 that $Mi(v) = Ma(v) = v$. It is also satisfied if all marginal worth vectors coincide.

In the following result, we establish relations among the core of the games v, $Mi(v)$ and $Ma(v)$ and the Weber set of v.

Theorem 2. *Let $v \in \Gamma(\mathcal{L})$ be a game on a convex geometry. Then*

1. *$Mi(v)(S) \leq v(S) \leq Ma(v)(S)$, for all $S \in \mathcal{L}$ and equality holds for $S = N$. Moreover, $Core(\mathcal{L}, Ma(v)) \subseteq Core(\mathcal{L}, v) \subseteq Core(\mathcal{L}, Mi(v))$.*
2. *$Weber(\mathcal{L}, v) \subseteq Core(\mathcal{L}, Mi(v))$.*
3. *If $\mathcal{L}$ is atomic, then $Core(\mathcal{L}, Ma(v)) = \emptyset$ or $Mi(v) = v = Ma(v)$ is an additive game. Moreover, $Core(\mathcal{L}, Ma(v)) \subseteq Weber(\mathcal{L}, v)$.*

Proof. 1. For each $S \in \mathcal{L}$ there is a maximal chain $C \in \mathcal{C}(\mathcal{L})$ that contains S. For this chain, $a^C(v)(S) = v(S)$, and so $Mi(v)(S) = \min_{C \in \mathcal{C}(\mathcal{L})} a^C(v)(S) \leq v(S)$. A similar argument shows that $Ma(v)(S) \geq v(S)$ for all $S \in \mathcal{L}$. Taking $S = N$, $a^C(N) = v(N)$ for all $C \in \mathcal{C}(\mathcal{L})$ and so, $Mi(v)(N) = Ma(v)(N) = v(N)$. Hence $Core(\mathcal{L}, Ma(v)) \subseteq Core(\mathcal{L}, v) \subseteq Core(\mathcal{L}, Mi(v))$.

2. For every $C \in \mathcal{C}(\mathcal{L})$ we have $a^C(v)(S) \geq Mi(v)(S)$ for all $S \in \mathcal{L}$ and $a^C(v)(N) = v(N)$. Therefore, $a^C(v) \in Core(\mathcal{L}, Mi(v))$. Since the core is a convex set in $\mathbb{R}^n$, then $Weber(\mathcal{L}, v) \subseteq Core(\mathcal{L}, Mi(v))$.

3. Let $\mathcal{L}$ be an atomic convex geometry. First, we prove that if there exist $C_1, C_2 \in \mathcal{C}(\mathcal{L})$ such that $a^{C_1}(v) \neq a^{C_2}(v)$ then $Core(\mathcal{L}, Ma(v))$ is empty. Assume that $a^{C_1}(v) = (y_1, \ldots, y_n)$ and $a^{C_2}(v) = (z_1, \ldots, z_n)$ are not equal. Since these vectors satisfy $a^{C_1}(v)(N) = a^{C_2}(v)(N) = v(N)$, then there exist $i, j \in N$ such that $y_i < z_i$, and $y_j > z_j$. If $x \in Core(\mathcal{L}, Ma(v))$ then

$$\begin{aligned} x_i &\geq Ma(v)(\{i\}) \geq z_i > y_i, \\ x_j &\geq Ma(v)(\{j\}) \geq y_j, \\ x_k &\geq Ma(v)(\{k\}) \geq y_k, \quad \text{for } k \notin \{i, j\}. \end{aligned}$$

Thus,

$$v(N) = \sum_{i \in N} x_i > \sum_{i \in N} y_i = v(N),$$

and this contradiction proves that $Core(\mathcal{L}, Ma(v)) = \emptyset$.

On the other hand, if the $Core(\mathcal{L}, Ma(v))$ is nonempty then all marginal worth vectors coincide. Hence $Mi(v) = v = Ma(v)$ is an additive game, defined by $v(S) = a^C(v)(S)$, where $a_i^C(v) = v(\{i\})$ for all $1 \leq i \leq n$. □

In (3), the atomicity hypothesis on $\mathcal{L}$ is crucial, because if $\mathcal{L}$ is not atomic, $Core(\mathcal{L}, Ma(v))$ is not bounded and therefore, the inclusion $Core(\mathcal{L}, Ma(v)) \subseteq Weber(\mathcal{L}, v)$ does not hold.

It is obvious that $Mi(v)$ is quasi-superadditive and $Ma(v)$ is quasi-subadditive by definition of these games. Moreover, it is easy to prove that if $n \leq 3$, $Mi(v)$ is quasi-supermodular and $Ma(v)$ is quasi-submodular. In general, $Mi(v)$ need not be quasi-supermodular and $Ma(v)$ need not be quasi-submodular. The following result provides us with a sufficient and necessary condition for the quasi-supermodularity and quasi-submodularity of a game using the marginal operators.

Theorem 3. *Let $v \in \Gamma(\mathcal{L})$. Then*

1. *The game v is quasi-supermodular if and only if $Mi(v) = v$*
2. *The game v is quasi-submodular if and only if $Ma(v) = v$.*

Proof. We only prove the first assertion because the proof of the second one is analogous.

Assume that $Mi(v) = v$. Then $v(S) = Mi(v)(S) \leq a^C(v)(S)$ for all $S \in \mathcal{L}$ and for all $C \in \mathcal{C}(\mathcal{L})$. Hence, $a^C(v) \in Core(\mathcal{L}, v)$ for all $C \in \mathcal{C}(\mathcal{L})$ and by using the characterization of Theorem 1, the game v is quasi-supermodular.

Conversely, if the game v is quasi-supermodular, then $a^C(v) \in Core(\mathcal{L}, v)$ for all $C \in \mathcal{C}(\mathcal{L})$. Therefore, $a^C(v)(N) = v(N)$ and $a^C(v)(S) \geq v(S)$, for all $S \in \mathcal{L}$. On the other hand, for each $S \in \mathcal{L}$, there exists a maximal chain $C_S \in \mathcal{C}(\mathcal{L})$ such that $Mi(v)(S) = a^{C_S}(v)(S)$. Thus, $v(S) \geq Mi(v)(S) = a^{C_S}(v)(S) \geq v(S)$, and this implies that $Mi(v)(S) = v(S)$ for all $S \in \mathcal{L}$. □

Lastly, we show in the following theorem that applying the minimal (maximal) marginal operator iteratively to a game $v \in \Gamma(\mathcal{L})$ yields a sequence of games that converges to a quasi-supermodular (quasi-submodular).

Theorem 4. *Let $v \in \Gamma(\mathcal{L})$ be a game on an atomic convex geometry $\mathcal{L}$. Then*

1. *If we define inductively the games $v^{(1)} = v, \;\; v^{(k)} = Mi(v^{(k-1)}),\; k = 2, 3, \ldots$ then $\lim_{k\to\infty} v^{(k)}$ exists and is quasi-supermodular.*
2. *If we define inductively the games $v^{(1)} = v, \;\; v^{(k)} = Ma(v^{(k-1)}),\; k = 2, 3, \ldots$ then $\lim_{k\to\infty} v^{(k)}$ exists and is quasi-submodular.*

Proof. Let $S \in \mathcal{L}$, by Theorem 2, we have that

$$v^{(1)}(S) \geq v^{(2)}(S) \geq \cdots \geq v^{(k)}(S) \geq \cdots$$

Moreover, if w is a quasi-superadditive game, then $a^C(w)(\{i\}) \geq w(\{i\})$ for each maximal chain $C \in \mathcal{C}(\mathcal{L})$ and hence $Mi(w)(S) \geq \sum_{i\in S} w(\{i\})$. By definition, $v^{(k)}$ is a quasi-superadditive game for $k = 2, \ldots$ Then, for every $S \in \mathcal{L}$,

$$v^{(k)}(S) \geq \sum_{i\in S} Mi(v)(\{i\}),$$

and therefore, the sequence $v^{(k)}(S)$ has a lower bound, and hence it converges to a limit $z(S) = \lim_{k\to\infty} v^{(k)}(S)$. Further, Mi is a continuous operator on $\Gamma(\mathcal{L})$, and therefore

$$z = \lim_{k\to\infty} v^{(k)} = \lim_{k\to\infty} Mi(v^{(k-1)}) = Mi(z).$$

Theorem 3 leads us to the quasi-supermodularity of the game z.

An analogous argument to the above shows the result for $Ma(v)$. □

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