Mathematical Social Sciences 46 (2003) 63–80

mathematical social sciences

www.elsevier.com/locate/econbase

Computing power indices in weighted multiple majority games

E. Algaba, J.M. Bilbao*, J.R. Fernández García, J.J. López

Matematica Aplicada II, Escuela Superior de Ingenieros, Camino de los Descubrimientos s/n, Sevilla 41092, Spain

Received 1 August 2002; received in revised form 1 October 2002; accepted 1 November 2002

Abstract

The *Shapley–Shubik power index* in a voting situation depends on the number of orderings in which each player is pivotal. The *Banzhaf power index* depends on the number of ways in which each voter can effect a swing. If the input size of the problem is n, then the function which measures the worst case running time for computing these indices is in $O(n2^n)$. We present a method based on *generating functions* to compute these power indices efficiently for weighted multiple majority games and we study the *temporal complexity* of the algorithms. Finally, we apply the algorithms obtained with this method to compute the Banzhaf and the Shapley–Shubik indices under the two decision rules adopted in the Nice European Union summit.

Keywords: Majority games; Power indices; Generating functions

JEL classification: C71

1. Introduction

The analysis of power is central in political science. In general, it is difficult to define the idea of power, but for the special case of voting situations several quantitative measures for evaluating the power of a voter or coalition have been proposed. The two classical *power indices* have received the most theoretical attention as well as application to political structures. The first such power index was proposed by Shapley and Shubik (1954). The second power index was introduced by Banzhaf (1965) and has

*Corresponding author. Tel.: +34-95-448-6174; fax: +34-95-448-6165.
E-mail address: mbilbao@us.es (J.M. Bilbao).

0165-4896/02/$ – see front matter
doi:10.1016/S0165-4896(02)00086-0

been used in arguments in various legal proceedings. The computation of these power indices is complex in practice, because the algorithms have exponential complexity. However, using generating functions, Cantor (see Lucas, 1983), Mann and Shapley (1962), Brams and Affuso (1976) and Tannenbaum (1997) have obtained significant results for computing the Shapley–Shubik and the Banzhaf indices in *weighted voting* games.

In this paper we focus on computing the Banzhaf index and the Shapley–Shubik index by using *generating functions* for *weighted multiple majority* games. The interest in these games lies in that, nowadays, some international organizations which gather a diversity of countries are considering a revision of the actual voting system of qualified majority. So, the possible introduction of multiple majority systems to improve and simplify the current decision systems is being discussed. Section 2 briefly recalls the concept of weighted m-majority games. In Section 3 we compute the Banzhaf power index by generating functions for weighted m-majority games and analyze its temporal complexity. A similar study for the Shapley–Shubik index is described in Section 4. In Section 5 we apply the algorithms obtained to compute both the Banzhaf and the Shapley–Shubik indices under the two decision rules adopted in the Nice European Union summit, which will be used in the European Union enlarged to 27 countries.

2. Voting games

A *simple game* is a cooperative game (N,v) where $N=\{1,\dots,n\}$ is a finite set and $v:2^N \to \{0,1\}$, such that $v(\emptyset)=0$ and $v(S)\le v(T)$ whenever $S\subseteq T$. A coalition is *winning* if $v(S)=1$, and *losing* if $v(S)=0$. The collection of all winning coalitions is denoted by $\mathcal{W}$. We will write $S\cup i$ and $S\backslash i$ instead of $S\cup\{i\}$ and $S\backslash\{i\}$, respectively.

An important subclass of simple games is the class known as *weighted voting games* which are used in many voting schemes. A weighted voting game is represented by $[q;\, w_1,\dots,w_n]$. Here, there are n players, w_i represents the voting weight of player i with $0<w_i<q$, for all i, and q is the quota needed to win. We shall assume that $q>\frac{1}{2}\sum_{i\in N} w_i$. In such voting games the characteristic function is defined by $v(S)=1$ if $w(S)\ge q$ and $v(S)=0$ otherwise, where $w(S)=\sum_{i\in S} w_i$. We suppose that all the weights and the quota are positive integers. Given the simple games $v_1,\dots,v_m$ we consider the simple game defined by:

$$(v_1\wedge \dots \wedge v_m)(S)=\min\{v_t(S):1\le t\le m\}$$

A *weighted m-majority game* is the simple game $v_1\wedge\dots\wedge v_m$ where $v_t=[q^t;\, w_1^t,\dots,w_n^t]$, $1\le t\le m$ are weighted voting games. Then:

$$(v_1\wedge \dots \wedge v_m)(S)=\begin{cases}1, & \text{if } w^t(S)\ge q^t,\quad 1\le t\le m\\ 0, & \text{otherwise}\end{cases}$$

where $w^t(S)=\sum_{i\in S} w_i^t$.

3. The normalized Banzhaf power index

The Banzhaf index is concerned with the number of times each player could change a coalition from losing to winning and it requires to know the number of swings for every player i (see Dubey and Shapley, 1979). A *swing* for player i is a pair of coalitions $(S \cup i, S)$ such that $S \cup i$ is winning and S is not. For each $i \in N$, we denote by $\eta_i(v)$ the number of swings for i in game v, that is, the number of winning coalitions in which player i is critical. The total number of swings is $\overline{\eta}(v) = \sum_{i \in N} \eta_i(v)$ and the *normalized Banzhaf index* is the vector $\beta(v) = (\beta_1(v), \ldots, \beta_n(v))$ where:

$$\beta_i(v) = \frac{\eta_i(v)}{\overline{\eta}(v)}$$

The most useful method for counting the number of elements $f(k)$ of a finite set is to obtain its generating function. The *generating function* of $f(k)$ is the formal power series:

$$\sum_{k \geq 0} f(k) x^k$$

We can work with generating functions of several variables:

$$\sum_{k \geq 0} \sum_{j \geq 0} \sum_{l \geq 0} f(k,j,l)\, x^k y^j z^l$$

For each $n \in \mathbb{N}$, the number of subsets of k elements of the set $N = \{1,2, \ldots, n\}$ is given by the explicit formula of the binomial coefficients:

$$\binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k!}$$

A generating function approach to binomial coefficients may be obtained as follows. Let $S = \{x_1, x_2, \ldots, x_n\}$ be an n-element set. Regard the elements $x_1, x_2, \ldots, x_n$ as independent indeterminates. It is an immediate consequence of the process of multiplication that:

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) = \sum_{T \subseteq S} \prod_{x_i \in T} x_i$$

Note that if $T = \emptyset$ then we obtain 1. If we put each $x_i = x$, we obtain:

$$(1 + x)^n = \sum_{T \subseteq S} \prod_{x \in T} x = \sum_{T \subseteq S} x^{|T|} = \sum_{k \geq 0} \binom{n}{k} x^k$$

Brams and Affuso (1976) obtained generating functions for computing the normalized Banzhaf index. Let $v = [q; w_1, \ldots, w_n]$ be a weighted voting game. They noted that the number of swings for player i satisfies:

$$\eta_i(v) = |\{S \notin \mathcal{W} : S \cup i \in \mathcal{W}\}| = \sum_{k=q-w_i}^{q-1} b_k^i$$

where b_k^i is the number of coalitions that do not include i with weight k.

Proposition 1. (Brams–Affuso) *Let* $v=[q;\ w_1,\dots,w_n]$ *be a weighted voting game. Then the generating functions of numbers* $\{b_k^i\}$ *are given by:*

$$B_i(x)=\prod_{j=1,j\neq i}^{n}\left(1+x^{w_j}\right)$$

We now present generating functions for computing the Banzhaf power index in weighted m-majority games.

Proposition 2. *Let* (N,v) *be a weighted m-majority game such that* $v=v_1\wedge\dots\wedge v_m$, *with* $v_t=[q^t;\ w_1^t,\dots,w_n^t]$, $1\leq t\leq m$. *For every* $i\in N$,

1. *The number of swings of player i is given by:*

$$\eta_i(v)=\sum_{\substack{k_t=q^t-w_i^t\\1\leq t\leq m}}^{w^t(N\setminus i)} b^i_{k_1\cdots k_m}-\sum_{\substack{k_t=q^t\\1\leq t\leq m}}^{w^t(N\setminus i)} b^i_{k_1\cdots k_m}$$

   where $b^i_{k_1\cdots k_m}$ *is the number of coalitions S such that* $i\not\in S$, $w^t(S)=k_t$ *for all* $1\leq t\leq m$.
2. *The generating functions of numbers* $\{b^i_{k_1\cdots k_m}\}_{k_1,\dots,k_m\geq 0}$, *are given by:*

$$B_i(x_1,\dots,x_m)=\prod_{j=1,\ j\neq i}^{n}\left(1+x_1^{w_j^1}\cdots x_m^{w_j^m}\right)$$

Proof. 1. First of all, we consider the set of all coalitions S such that $i\not\in S$ with $w^t(S)\geq q^t-w_i^t$ for all $1\leq t\leq m$. Its cardinal is given by:

$$s_1^i=\sum_{k_1=q^1-w_i^1}^{w^1(N\setminus i)}\cdots\sum_{k_m=q^m-w_i^m}^{w^m(N\setminus i)} b^i_{k_1\cdots k_m}$$

As $w^t(S\cup i)\geq q^t$, $1\leq t\leq m$, then s_1^i coincides with the number of winning coalitions in which the player i participates.

On the other hand, inside of the set of the winning coalitions that contain player i, we consider the subset of those coalitions in which player i is not necessary to win. The cardinal of this subset coincides with the set of all coalitions S such that $i\not\in S$ with $w^t(S)\geq q$, $1\leq t\leq m$, and it is given by:

$$s_2^i=\sum_{k_1=q^1}^{w^1(N\setminus i)}\cdots\sum_{k_m=q^m}^{w^m(N\setminus i)} b^i_{k_1\cdots k_m}$$

Therefore, the number of swings of player i is $\eta_i(v)=s_1^i-s_2^i$.

2. Expanding the function:

$$\begin{aligned} B(x_1,\ldots,x_m) &= \prod_{j=1}^{n}\left(1+x_1^{w_j^1}\cdots x_m^{w_j^m}\right)=\sum_{S\subseteq N}\prod_{i\in S}x_1^{w_i^1}\cdots x_m^{w_i^m}\\ &= \sum_{S\subseteq N}x_1^{w^1(S)}\cdots x_m^{w^m(S)}\\ &= \sum_{k_1=0}^{w^1(N)}\cdots\sum_{k_m=0}^{w^m(N)} b_{k_1\cdots k_m}x_1^{k_1}\cdots x_m^{k_m}\\ &= \sum_{\substack{k_t=0\\1\le t\le m}}^{w^t(N)} b_{k_1\cdots k_m}x_1^{k_1}\cdots x_m^{k_m} \end{aligned}$$

Then $B(x_1,\ldots,x_m)$ is a generating function for the numbers $b_{k_1\ldots k_m}$ where each $b_{k_1\ldots k_m}$ is the number of coalitions S such that $w^t(S)=k_t$, $1\le t\le m$. To obtain the numbers $\{b^i_{k_1\ldots k_m}\}_{k_1,\ldots,k_m\ge 0}$, it suffices to delete the factor $\left(1+x_1^{w_i^1}\cdots x_m^{w_i^m}\right)$ in the polynomial $B(x_1,\ldots,x_m)$ giving rise to the generating function $B_i(x_1,\ldots,x_m)$. □

Example. We consider the weighted double majority game $v=v_1\wedge v_2$, where $v_1=[8;\ 5,3,2,2]$ and $v_2=[3;\ 1,1,1,1]$. Its characteristic function is $v_1\wedge v_2(S)=1$ if $w^1(S)\ge 8$ and $w^2(S)\ge 3$, and $v_1\wedge v_2(S)=0$ otherwise. We first calculate the functions $B_i(x,y)=\prod_{j=1,j\ne i}^{n}\left(1+x^{w_j^1}y^{w_j^2}\right)$:

$$B_1(x,y)=1+2x^2y+x^3y+x^4y^2+2x^5y^2+x^7y^3$$

$$B_2(x,y)=1+2x^2y+x^5y+x^4y^2+2x^7y^2+x^9y^3$$

$$B_3(x,y)=1+x^2y+x^3y+x^5y+x^5y^2+x^7y^2+x^8y^2+x^{10}y^3$$

$$B_4(x,y)=1+x^2y+x^3y+x^5y+x^5y^2+x^7y^2+x^8y^2+x^{10}y^3$$

To compute the number of swings for each player the following differences are calculated:

$$\eta_1(v)=\sum_{k=3}^{7}\sum_{r=2}^{3}b^1_{kr}-\sum_{k=8}^{7}\sum_{r=3}^{3}b^1_{kr}=4-0=4$$

$$\eta_2(v)=\sum_{k=5}^{9}\sum_{r=2}^{3}b^2_{kr}-\sum_{k=8}^{9}\sum_{r=3}^{3}b^2_{kr}=3-1=2$$

$$\eta_3(v)=\sum_{k=6}^{10}\sum_{r=2}^{3}b^3_{kr}-\sum_{k=8}^{10}\sum_{r=3}^{3}b^3_{kr}=3-1=2$$

$$\eta_4(v)=\sum_{k=6}^{10}\sum_{r=2}^{3}b^4_{kr}-\sum_{k=8}^{10}\sum_{r=3}^{3}b^4_{kr}=3-1=2$$

As the total number of swings is 10, we obtain $\beta(v)=(2/5,1/5,1/5,1/5)$.

With the aim of making easier the computation of the coefficients $b^i_{k_1\cdots k_m}$, we can use a table m-dimensional, to store the coefficients of $B_i(x_1,\ldots,x_m)$. If we arrange the coefficients by increasing powers of x_t with $t\in\{1,\ldots,m\}$, the element $b^i_{k_1\cdots k_m}$ is placed in the position $(k_1+1,\ldots,k_m+1)$. For instance, the matrix that contains the coefficients of $B_3(x,y)$ is

$$\begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \\ x^7 \\ x^8 \\ x^9 \\ x^{10} \end{array} \begin{array}{c} \begin{array}{cccc} 1 & y & y^2 & y^3 \end{array} \\ \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \end{array}$$

Proposition 3. *Let (N,v) be a weighted m-majority game such that $v=v_1\wedge\ldots\wedge v_m$, where $v_t=[q^t;\, w^t_1,\ldots,w^t_n]$, $1\le t\le m$. Then*

1. *The number c of terms of:*

$$B(x_1,\ldots,x_m)=\prod_{j=1}^{n}\left(1+x_1^{w_j^1}\cdots x_m^{w_j^m}\right)$$

   satisfies that $n+1\le c\le \min\left(2^n,\prod_{t=1}^{m} w^t(N)+1\right)$.
2. *The number of terms of $B_i(x_1,\ldots,x_m)$, for every $i\in N$, is bounded by c.*

Proof.

1. A lower bound of c is obtained in the case in which the weights of all players are equal, that is $w^t_i=w^t$, $1\le t\le m$, $1\le i\le n$. The number of terms of $(1+x_1^{w^1}\cdots x_m^{w^m})^n$ is always less or equal than the number of terms of $B(x_1,\ldots,x_m)$. On the other hand, we have that:

$$B(x_1,\ldots,x_m)=\sum_{k_1=0}^{w^1(N)}\cdots\sum_{k_m=0}^{w^m(N)} b_{k_1\cdots k_m}x_1^{k_1}\cdots x_m^{k_m}$$

   is a polynomial of degree $w^t(N)$ in x_t, $1\le t\le m$, and in which there are no terms such as $x_t^{k_t}$, $1\le t\le m$. Therefore, $c\le\prod_{t=1}^{m} w^t(N)+1$. Moreover, at worst, $c\le 2^n$ because all exponents of the terms of $B(x_1,\ldots,x_m)$ are different and then the number c coincides with the number of subsets of N.
2. It is straightforward by part 1. □

The *time complexity* function $f:\mathbb{N}\to\mathbb{N}$ of an algorithm give us the maximum time $f(n)$ needed to solve any problem instance of encoding length at most $n\in\mathbb{N}$. A function $f(n)$ is $O(g(n))$ if there is a constant k such that $|f(n)|\leq k|g(n)|$ for all integers $n\in\mathbb{N}$. We analyze our algorithms in the *arithmetic model*, that is, we count elementary arithmetic operations and assignments. For instance, the algorithm for computing the product of two $n\times n$ matrices is $O(n^3)$. We obtain *pseudo polynomial* algorithms, i.e. polynomial in n and c, for computing the Banzhaf and the Shapley–Shubik indices.

Proposition 4. *Let* (N,v) *be a weighted m-majority game such that* $v=v_1\wedge\ldots\wedge v_m$, *where* $v_t=[q^t;\ w_1^t,\ldots,w_n^t]$, $1\leq t\leq m$. *Then*

1. *To expand the polynomial* $B(x_1,\ldots,x_m)$, *a time* $O(nC)$ *is required where* $C=\min\left(2^n,\prod_{t=1}^{m} w^t(N)+1\right)$.
2. *For each* $i\in N$, *to expand the polynomial* $B_i(x_1,\ldots,x_m)$, *a time* $O(nc)$ *is required.*

Proof.

1. If $f(n)$ is the number of necessary operations to expand the polynomial $B^{(n)}(x_1,\ldots,x_m)=\prod_{j=1}^{n}\left(1+x_1^{w_j^1}\cdots x_m^{w_j^m}\right)$, we can establish the following recurrence relation:

$$O(f(n))=\begin{cases}O(1), & \text{if } n=1\\ O(f(n-1)+3C), & \text{if } n\geq 2\end{cases}$$

since $f(1)=1$ and for $n\geq 2$, at worst, the computation of:

$$B^{(n)}(x_1,\ldots,x_m)=B^{(n-1)}(x_1,\ldots,x_m)\left(1+x_1^{w_n^1}\cdots x_m^{w_n^m}\right)$$

requires a number of products and sums with upper bounds of $2C$ and C, respectively, because C is a upper bound of the number of nonzero coefficients of $B(x_1,\ldots,x_m)$. If we leave out the notation $O(\cdot)$ and expand the above recurrence, we have:

$$f(n)=f(n-1)+3C=f(n-2)+2(3C)=\cdots=f(n-k)+k(3C).$$

For $k=n-1$, it holds $f(n)=f(1)+(n-1)(3C)$. That is, $O(f(n))=O(nC)$.
2. It is straightforward by 1. □

Next, we describe the function *m-banzhafPower* which will be used to compute the normalized Banzhaf index of all players in a weighted m-majority game and we study its time complexity.

Function m-banzhafPower (*weights_1*, . . . ,*weights_m*,$q_1,\ldots,q_m$)

{*weights_t*: list of n integers; q_t: integer; $1\leq t\leq m$}

```
for i from 1 to n
    for t from 1 to m
        list_aux_t ← delete({w_1^t, ..., w_n^t}, w_i^t)
    end for
    (1) B_i(x_1,...,x_m) ← ∏_{w_j^t ∈ list_aux_t, 1≤t≤m} (1 + x_1^{w_j^1} ··· x_m^{w_j^m})
    (2) coef f ← Coefficients of B_i(x_1,...,x_m)
    (3) For k_t from q^t − w_i^t + 1 to w^t(N\i) + 1, 1 ≤ t ≤ m
        s_1^i ← Sum[coef f[k_1,...,k_m]]
    (4) For k_t from q^t + 1 to w^t(N\i) + 1, 1 ≤ t ≤ m
        s_2^i ← Sum[coef f[k_1,...,k_m]]
    (5) η_i ← s_1^i − s_2^i
end for

η̄ ← Σ_{i=1}^{n} η_i

for i from 1 to n
    β_i ← η_i / η̄
end for

output {β_1,...,β_n}
```

Proposition 5. *Let (N,v) be a weighted m-majority game such that $v = v_1 \wedge \cdots \wedge v_m$, where $v_t = [q^t;\ w_1^t, \ldots, w_n^t]$, $1 \le t \le m$. To compute the normalized Banzhaf index of all players, with the function m-BanzhafPower a time $O(\max(m, n^2 c))$ is required where c is the number of terms of $B(x_1, \ldots, x_m)$.*

Proof. In this situation, the complexity order of the function $f(n)$ which determines the execution time of the function m-BanzhafPower, is given by:

$$\begin{aligned} O(f(n)) &= O(t(loop1) + t(assignment) + t(loop2)) \\ &= O(\max(t(loop1), n, 2n)) = O(t(loop1)) \end{aligned}$$

As $O(t(loop1)) = O(n(t(loop\_int) + t(assignment1) + \cdots + t(assignment5)))$ and taking into account that:

$$O(t(loop\_int)) = O(m\, t(list\_aux\_t)) = O(m)$$

$$O(t(assignment1)) = O(t(polynomial)) = O(nc)$$

$$O(t(assignment2)) = O(t(coeff)) = O(c)$$

$$O(t(assignment3)) = O(t(sum1)) = O(c)$$

$$O(t(assignment4)) = O(t(sum2)) = O(c)$$

$$O(t(assignment5)) = O(t(difference)) = O(1)$$

it holds $O(f(n)) = O(t(loop1)) = O(nt(assignment1)) = O(n^2c)$. □

4. The Shapley–Shubik power index

The Shapley–Shubik index for a simple game $v:2^N \to \{0,1\}$ is the vector $\Phi(v) = (\Phi_1(v), \ldots, \Phi_n(v))$ defined by:

$$\Phi_i(v) = \sum_{\{S \notin \mathcal{W}: S \cup i \in \mathcal{W}\}} \frac{s!(n-s-1)!}{n!} = \sum_{j=0}^{n-1} \frac{j!(n-j-1)!}{n!} d_j^i$$

where each d_j^i is the number of swings of player i in coalitions of size j.

David G. Cantor used generating functions for computing exactly the Shapley–Shubik index for large voting games. As related by Mann and Shapley (1962), Cantor's contribution was the following result. If $v = [q; w_1, \ldots, w_n]$ is a weighted voting game, then the number d_j^i satisfies:

$$d_j^i = \sum_{k=q-w_i}^{q-1} a_{kj}^i$$

where a_{kj}^i is the number of ways in which j players, other than i, can have a sum of weights equal to k.

Proposition 6. (*Cantor*) *Let* $v = [q; w_1, \ldots, w_n]$ *be a weighted voting game. Then the generating functions of numbers* $\{a_{kj}^i\}$ *are given by*:

$$S_i(x,z) = \prod_{j=1, j\neq i}^{n} (1 + z x^{w_j})$$

Similar to what happens with the Banzhaf index in weighted m-majority games, it is also possible to obtain, using generating functions, an analogous result for the calculation of the Shapley–Shubik power index. When the game (N,v) is given by $v = v_1 \wedge \ldots \wedge v_m$, where $v_t = [q^t; w_1^t, \ldots, w_n^t]$, $1 \leq t \leq m$, then we have that:

$$d_j^i(v) = \sum_{\substack{k_t = q^t - w_i^t \\ 1 \leq t \leq m}}^{w^t(N \setminus i)} a_{k_1 \cdots k_m j}^i - \sum_{\substack{k_t = q^t \\ 1 \leq t \leq m}}^{w^t(N \setminus i)} a_{k_1 \cdots k_m j}^i$$

where $a_{k_1 \cdots k_m j}^i$ is the number of coalitions S, with cardinal j, such that $i \notin S$, and, $w^t(S) = k_t$ for all $1 \leq t \leq m$.

Proposition 7. *Let* (N,v) *be a weighted m-majority game such that* $v = v_1 \wedge \cdots \wedge v_m$, *where* $v_t = [q^t; w_1^t, \ldots, w_n^t]$, $1 \le t \le m$. *For every* $i \in N$, *the generating functions of numbers* $\{a^i_{k_1 \cdots k_m\, j}\}_{k_1, \ldots, k_m,\; j \ge 0}$, *are given by*

$$S_i(x_1, \ldots, x_m, z) = \prod_{j=1, j \ne i}^{n} \left(1 + z\, x_1^{w_j^1} \cdots x_m^{w_j^m}\right).$$

Proof. Expanding the function

$$\begin{aligned} S(x_1, \ldots, x_m, z) &= \prod_{j=1}^{n} \left(1 + z\, x_1^{w_j^1} \cdots x_m^{w_j^m}\right) = \sum_{S \subseteq N} z^{|S|} \prod_{i \in S} x_1^{w_i^1} \cdots x_m^{w_i^m} \\ &= \sum_{S \subseteq N} x_1^{w^1(S)} \cdots x_m^{w^m(S)} z^{|S|} \\ &= \sum_{k_1=0}^{w^1(N)} \cdots \sum_{k_m=0}^{w^m(N)} \sum_{j=0}^{n} a_{k_1 \cdots k_m j}\, x_1^{k_1} \cdots x_m^{k_m} z^j \\ &= \sum_{\substack{(k_t=0) \\ 1 \le t \le m}}^{w^t(N)} \sum_{j=0}^{n} a_{k_1 \cdots k_m j}\, x_1^{k_1} \cdots x_m^{k_m}\, z^j \end{aligned}$$

where $a_{k_1 \cdots k_m j}$ is the number of coalitions $S \subseteq N$, such that $|S| = j$, $w^t(S) = k_t$ for all $1 \le t \le m$. To obtain the numbers $a^i_{k_1 \cdots k_m j}$, it suffices to delete the factor $\left(1 + z\, x_1^{w_i^1} \cdots x_m^{w_i^m}\right)$ in the polynomial $S(x_1, \ldots, x_m, z)$, giving rise to their generating function $S_i(x_1, \ldots, x_m, z)$. □

Note that if we know the coefficients $\{a^i_{k_1 \cdots k_m j}\}_{k_1, \ldots, k_m,\; j \ge 0}$, using the polynomial $S_i(x_1, \ldots, x_m, z)$, then the numbers $\{d^i_j\}_{j \ge 0}$ can be determined. These numbers can be identified with the coefficients of a polynomial $g_i(z) = \sum_{j=0}^{n-1} d^i_j z^j$ and taking into account that:

$$d^i_j(v) = \sum_{\substack{k_t = q^t - w_i^t \\ 1 \le t \le m}}^{w^t(N \backslash i)} a^i_{k_1 \cdots k_m\, j} - \sum_{\substack{k_t = q^t \\ 1 \le t \le m}}^{w^t(N \backslash i)} a^i_{k_1 \cdots k_m\, j}$$

it holds:

$$g_i(z) = \sum_{j=0}^{n-1} d^i_j z^j = \sum_{j=0}^{n-1} \left[\sum_{\substack{k_t = q^t - w_i^t \\ 1 \le t \le m}}^{w^t(N \backslash i)} a^i_{k_1 \cdots k_m\, j} - \sum_{\substack{k_t = q^t \\ 1 \le t \le m}}^{w^t(N \backslash i)} a^i_{k_1 \cdots k_m\, j} \right] z^j$$

Hence, we obtain that:

$$g_i(z) = \sum_{\substack{k_t = q^t - w_i^t \\ 1 \le t \le m}}^{w^t(N \backslash i)} \left[\sum_{j=0}^{n-1} a^i_{k_1 \cdots k_m\, j} z^j \right] - \sum_{\substack{k_t = q^t \\ 1 \le t \le m}}^{w^t(N \backslash i)} \left[\sum_{j=0}^{n-1} a^i_{k_1 \cdots k_m\, j} z^j \right]$$

and by Proposition 7 we have that:

$$S_i(x_1, \ldots ,x_m,z) = \sum_{\substack{k_t = q^t - w_i^t \\ 1 \le t \le m}}^{w^t(N \setminus i)} \left[\sum_{j=0}^{n-1} a^i_{k_1 \cdots k_m\, j}\, z^j \right] x_1^{k_1} \cdots x_m^{k_m}.$$

The elements of $S_i(x_1, \ldots ,x_m,z)$ can be stored in a table m-dimensional, where the element $(k_1 + 1, \ldots ,k_m + 1)$ is $\sum_{j=0}^{n-1} a^i_{k_1 \cdots k_m\, j}\, z^j$. For example, for $n = 2$, we have

$$\begin{array}{c} \\ 1 \\ x_1 \\ x_1^2 \\ \vdots \\ x^{k_1} \\ \vdots \\ x^{w^1(N\setminus i)} \end{array} \begin{array}{c} 1 \quad x_2 \quad x_2^2 \quad \ldots \quad x_2^{k_2} \quad \ldots \quad x_2^{w^2(N\setminus i)} \\ \left(\begin{array}{ccc} & \Big| & \\ & \Big| & \\ \text{———————} & \sum_{j=0}^{n-1} a^i_{k_1 \ldots k_m\, j} z^j & \\ & & \\ & & \end{array} \right) \end{array}$$

We determine a polynomial in z whose coefficients represent the winning coalitions that contain player i:

$$s_1^i(z) = \sum_{\substack{k_t = q^t - w_i^t \\ 1 \le t \le m}}^{w^t(N \setminus i)} \sum_{j=0}^{n-1} a^i_{k_1 \cdots k_m\, j} z^j$$

On the other hand, we consider a polynomial in z whose coefficients represent the number of winning coalitions that contain player i but his presence is not necessary to win:

$$s_2^i(z) = \sum_{\substack{k_t = q^t \\ 1 \le t \le m}}^{w^t(N \setminus i),} \sum_{j=0}^{n-1} a^i_{k_1 \cdots k_m\, j} z^j$$

These polynomials $s_1^i(z)$ and $s_2^i(z)$ are obtained by adding, respectively, the nonzero elements in the table, from the $(q^t - w_i^t)$ to the last, for all $1 \le t \le m$. Finally,

$$g_i(z) = \sum_{j=0}^{n-1} d_j^i z^j = s_1^i(z) - s_2^i(z)$$

Proposition 8. *Let (N,v) be a weighted m-majority game such that $v = v_1 \wedge \cdots \wedge v_m$, where $v_t = [q^t;\, w_1^t, \ldots ,w_n^t]$, $1 \le t \le m$. Then,*

1. *The number c of terms of*

$$S(x_1, \ldots ,x_m,z) = \prod_{j=1}^{n} \left(1 + z\, x_1^{w_j^1} \cdots x_m^{w_j^m}\right)$$

verifies that

$$n+1 \leq c \leq \min\left(2^n, n\prod_{t=1}^{m} w^t(N)+1\right).$$

2. *The number of terms of* $S_i(x_1,\ldots,x_m,z)$*, for every* $i \in N$*, is bounded by* c.

Proof.

1. A lower bound is obtained in the case in which the weights of all players are equal, that is, $w_i^t = w^t$, $1 \leq t \leq m$, $1 \leq i \leq n$. The number of terms of $(1+z\,x_1^{w^1}\cdots x_m^{w^m})^n$ is always less or equal than the number of terms of:

$$S(x_1,\ldots,x_m,z)=\prod_{j=1}^{n}\left(1+z\,x_1^{w_j^1}\cdots x_m^{w_j^m}\right)$$

Therefore, $c \geq n+1$. To determine an upper bound we note that:

$$S(x_1,\ldots,x_m,z)=\sum_{\substack{k_t=0\\1\leq t\leq m}}^{w^t(N)}\sum_{j=0}^{n} a_{k_1\cdots k_m\,j}\,x_1^{k_1}\cdots x_m^{k_m}z^j$$

is a polynomial of degree $w^t(N)$ in x_t, $1 \leq t \leq m$, degree n in z, and in which there are no terms such as z^j or x_t^k, $1 \leq t \leq m$. Therefore:

$$c \leq n\prod_{t=1}^{m} w^t(N)+1.$$

Moreover, at worst, $c \leq 2^n$ because all exponents of the terms of the polynomial $S(x_1,\ldots,x_m,z)$ are different and, then c coincides with the number of subsets of N.

2. It is straightforward by part 1. □

Proposition 9. *Let* (N,v) *be a weighted m-majority game such that* $v = v_1 \wedge \ldots \wedge v_m$*, where* $v_t = [q^t; w_1^t,\ldots,w_n^t]$, $1 \leq t \leq m$*. Then,*

1. *To expand the polynomial* $S(x_1,\ldots,x_m,z)$*, a time* $O(nC)$ *is required where*

$$C=\min\left(2^n, n\prod_{t=1}^{m} w^t(N)+1\right).$$

2. *For each* $i \in N$*, to expand the polynomial* $S_i(x_1,\ldots,x_m,z)$*, a time* $O(nc)$ *is required.*

We can compute the normalized Shapley–Shubik index of all players in a weighted *m*-majority game, using generating functions, with the function called *m-ShapleyPower*.

Function *m*-ShapleyPower(*weights_1*, . . . ,*weights_m*,$q_1,\ldots,q_m$)

{*weights_t*: list of n integers; q_t: integer; $1 \leq t \leq m$}

for i **from** 1 **to** n
  for t **from** 1 **to** m
  $list\_aux\_t \leftarrow delete(\{w_1^t, \ldots, w_n^t\}, w_i^t)$
  end for
(1) $S_i(x_1, \ldots, x_m z) \leftarrow \prod_{\substack{w_j^t \in list\_aux\_t \\ 1 \le t \le m}}^{n} \left(1 + z x_1^{w_j^1} \cdots x_m^{w_j^m}\right)$
(2) $coef\ f \leftarrow Coef\ ficients\ of\ S_i(x_1, \ldots, x_m, z)$
(3) For k_t from $q^t - w_i^t + 1$ to $w^t(N \backslash i) + 1$, $1 \le t \le m$
$s_1^i(z) \leftarrow Sum[coef\ f[k_1, \ldots, k_m]]$
(4) For k_t from $q^t + 1$ to $w^t(N \backslash i) + 1, 1 \le t \le m$
$s_2^i \leftarrow Sum[coef\ f[k_1, \ldots, k_m]]$
(5) $g_i(z) \leftarrow s_1^i(z) - s_2^i(z)$
(6) $\phi_i \leftarrow \frac{1}{n!} \sum_{j=0}^{n-1} d_j^i\, j!(n-j-1)!$
end for

output $\{\phi_1, \ldots, \phi_n\}$

We can prove to compute the normalized Shapley–Shubik index of all players in a weighted m-majority game, with the function m-ShapleyPower, a time $O(n^2 c)$ is required, where c is the number of nonzero coefficients of $S(x_1, \ldots, x_m, z)$.

5. Power indices in the European Union

The *Journal of Theoretical Politics* volume 11, number 3 (July 1999) and volume 13, number 1 (January 2001) has published several articles about the modelling of decision making process in the European Union (EU). Garrett and Tsebelis (1999a,b) criticized the classical voting power method in the context of the EU because the power indices do not take into account the preferences of players and the institutional structure of the EU. Lane and Berg (1999) assert that:

> *Cooperative theory solution concepts, on the other hand, assume that players have specified preferences and can make the types of binding commitments typically required to cement together a particular coalition in support of a particular outcome. Cooperative solutions, including power indices, therefore, are directly applicable to policy analysis only in conjunction with assumptions about preferences (spatial or otherwise) and in circumstances that suggest that binding agreements may be feasible. Therefore, criticism of cooperative game theory and power indices may be misplaced insofar as constitutional analysis is concerned, but may have more weight insofar as policy analysis is concerned, although formal power indices often help explain voting outcomes.*

Holler and Widgrén (1999) propose some ideas to combine spatial voting games and power index models. Steunenberg et al. (1999) define the strategic power index, a new approach which is based on spatial and sequential models of decision making. We agree with Lane and Berg on the need for a priori measures of power. However, we believe that the best choice is a method that takes the voting power theory, so-called *intergovernmentalism* and supplements it by the *institutional analysis* of the EU legislative process.

The Council of Ministers of the EU represents the national governments of the member states. The Council uses a voting system of qualified majority to pass new legislation. The Nice European Council in December 2000 established two decision rules for the EU enlarged to 27 countries. These rules are contained in the *Declaration on the enlargement of the European Union* and the *Declaration on the qualified majority threshold and the number of votes for a blocking minority in an enlarged Union* (Official Journal of the European Communities 10.3.2001, C 80/80-85).

Felsenthal and Machover (2001) analyzed in terms of a priori measures of power these decision rules for the Council of Ministers of the EU. They used the Bräuninger-König IOP 1.0 program and the Lemma 3.3.12 in Felsenthal and Machover (1998) to calculate the voting power of each one of the present 15 members and the future 27 ones. The new version of the program IOP 2.0 allows us to calculate voting power indices for the post Nice institutions of the EU where Council members have two kinds of weighted and one unweighted vote. In addition, an option for reporting winning and minimal winning coalitions is implemented (see Bräuninger and König, 2001).

We next present our results concerning to the Banzhaf and Shapley–Shubik power indices using the algorithms of the previous sections. We compute these indices under the two decision rules prescribed by the Treaty of Nice. Each member state represented in the future Council is considered an individual player. The players in the Council of the EU enlarged to 27 countries are:

{Germany, United Kingdom, France, Italy, Spain, Poland, Romania, The Netherlands, Greece, Czech Republic, Belgium, Hungary, Portugal, Sweden, Bulgaria, Austria, Slovak Republic, Denmark, Finland, Ireland, Lithuania, Latvia, Slovenia, Estonia, Cyprus, Luxembourg, Malta}.

The first decision rule is the weighted triple majority game $v_1 \wedge v_2 \wedge v_3$, where the three weighted voting games corresponding to votes, countries and population, are the following:

$$v_1 = [255; 29,29,29,29,27,27,14,13,12,12,12,12,12,10,10,10,7,7,7,7,7,4,4,4,4,4,3]$$

$$v_2 = [14; 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1]$$

$$v_3 = [620; 170,123,122,120,82,80,47,33,22,21,21,21,21,18,17,17,11,11,11,8,8,5,4,3,2,1,1]$$

The game v_3 is defined assigning to every country, a number of votes equal to the rate per thousand of its population over the total population and the quota represents the 62% of the total population. So, a voting will be favorable if it counts on the support of 14 countries with at least 255 votes, and with at least the 62% of the population.

The second decision rule is the weighted triple majority game $v_1 \wedge v_2' \wedge v_3$, where the weighted voting game v_2' consists of a qualified majority of $2/3$ of the countries, that is,

$$v_2' = [18; 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1]$$

Next, we give a table which contains the Banzhaf power indices of the countries if the first decision rule is used. In the column called *Pop. I* is included the percentage of population over the total, and in the column *Vote I*, the percentage of votes of every country over the total. The Banzhaf indices of the game v_1 are included in the column *Game1*, and the Banzhaf indices of the weighted triple majority game $v_1 \wedge v_2 \wedge v_3$ in the column *Game3a*.

From Table 1, we can deduce the following conclusions:

- The first decision rule, that consists of a triple majority system, is quasi equivalent to

Table 1
The Banzhaf index under the first rule

Countries	Population	Votes	Pop. I	Vote I	Game1	Game3a
Germany	82.038	29	0.170	0.084	0.0778	0.0778
United Kingdom	59.247	29	0.123	0.084	0.0778	0.0778
France	58.966	29	0.123	0.084	0.0778	0.0778
Italy	57.612	29	0.120	0.084	0.0778	0.0778
Spain	39.394	27	0.082	0.078	0.0742	0.0742
Poland	38.667	27	0.080	0.078	0.0742	0.0742
Romania	22.489	14	0.047	0.041	0.0426	0.0426
The Netherlands	15.760	13	0.033	0.038	0.0397	0.0397
Greece	10.533	12	0.022	0.035	0.0368	0.0368
Czech Republic	10.290	12	0.021	0.035	0.0368	0.0368
Belgium	10.213	12	0.021	0.035	0.0368	0.0368
Hungary	10.092	12	0.021	0.035	0.0368	0.0368
Portugal	9.980	12	0.021	0.035	0.0368	0.0368
Sweden	8.854	10	0.018	0.029	0.0309	0.0309
Bulgaria	8.230	10	0.017	0.029	0.0309	0.0309
Austria	8.082	10	0.017	0.029	0.0309	0.0309
Slovak Republic	5.393	7	0.011	0.020	0.0218	0.0218
Denmark	5.313	7	0.011	0.020	0.0218	0.0218
Finland	5.160	7	0.011	0.020	0.0218	0.0218
Ireland	3.744	7	0.008	0.020	0.0218	0.0218
Lithuania	3.701	7	0.008	0.020	0.0218	0.0218
Latvia	2.439	4	0.005	0.012	0.0125	0.0125
Slovenia	1.978	4	0.004	0.012	0.0125	0.0125
Estonia	1.446	4	0.003	0.012	0.0125	0.0125
Cyprus	0.752	4	0.002	0.012	0.0125	0.0125
Luxembourg	0.429	4	0.001	0.012	0.0125	0.0125
Malta	0.379	3	0.001	0.009	0.0094	0.0094

the first game of qualified majority. The Banzhaf power of all countries is almost the same as the power with the simple game v_1, with the double game $v_1 \wedge v_2$, and with the triple one $v_1 \wedge v_2 \wedge v_3$.
- The second decision rule, that differs from the first one because it requires the approval at least of 2/3 of the countries, is quasi equivalent to the weighted double majority game $v_1 \wedge v_2'$. In this rule, the required population quota to adopt a decision does not change the Banzhaf power of the countries.

Table 2 contains the Shapley–Shubik indices for the first decision rule adopted, i.e. for the game $v_1 \wedge v_2 \wedge v_3$, labeled *Game3a*. In a similar way, in both cases, the results corresponding to the games v_1, labeled *Game1*, and $v_1 \wedge v_2$, labeled *Game2a* are similar. The conclusion, such as we anticipated before, is that the results corresponding to the games $v_1 \wedge v_2 \wedge v_3$ and $v_1 \wedge v_2$ are almost the same.

Next, we include the computations corresponding to the Banzhaf and Shapley–Shubik indices for the second decision rule adopted (Table 3), i.e. for the game $v_1 \wedge v_2' \wedge v_3$, labeled *Game3b*.

Summarizing, there are two main conclusions:

Table 2
The Shapley–Shubik index under the first rule

Countries	Population	Game1	Game2a	Game3a
Germany	0.170	0.0867	0.0867	0.0871
United Kingdom	0.123	0.0867	0.0867	0.0870
France	0.123	0.0867	0.0867	0.0870
Italy	0.120	0.0867	0.0867	0.0870
Spain	0.082	0.0800	0.0800	0.0799
Poland	0.080	0.0800	0.0800	0.0799
Romania	0.047	0.0399	0.0399	0.0399
The Netherlands	0.033	0.0368	0.0368	0.0368
Greece	0.022	0.0341	0.0341	0.0340
Czech Republic	0.021	0.0341	0.0341	0.0340
Belgium	0.021	0.0341	0.0341	0.0340
Hungary	0.021	0.0341	0.0341	0.0340
Portugal	0.021	0.0341	0.0341	0.0340
Sweden	0.018	0.0282	0.0282	0.0281
Bulgaria	0.017	0.0282	0.0282	0.0281
Austria	0.017	0.0282	0.0282	0.0281
Slovak Republic	0.011	0.0196	0.0196	0.0196
Denmark	0.011	0.0196	0.0196	0.0196
Finland	0.011	0.0196	0.0196	0.0196
Ireland	0.008	0.0196	0.0196	0.0196
Lithuania	0.008	0.0196	0.0196	0.0196
Latvia	0.005	0.0110	0.0110	0.0110
Slovenia	0.004	0.0110	0.0110	0.0110
Estonia	0.003	0.0110	0.0110	0.0110
Cyprus	0.002	0.0110	0.0110	0.0110
Luxembourg	0.001	0.0110	0.0110	0.0110
Malta	0.001	0.0082	0.0082	0.0082

Table 3
The power indices under the second rule

Countries	Banzhaf index			Shapley–Shubik index		
	Game1	Game2b	Game3b	Game1	Game2b	Game3b
Germany	0.0778	0.0665	0.0665	0.0867	0.0834	0.0837
United Kingdom	0.0778	0.0665	0.0665	0.0867	0.0834	0.0836
France	0.0778	0.0665	0.0665	0.0867	0.0834	0.0836
Italy	0.0778	0.0665	0.0665	0.0867	0.0834	0.0836
Spain	0.0742	0.0631	0.0631	0.0800	0.0768	0.0767
Poland	0.0742	0.0631	0.0631	0.0800	0.0768	0.0767
Romania	0.0426	0.0407	0.0407	0.0399	0.0395	0.0394
The Netherlands	0.0397	0.0386	0.0386	0.0368	0.0366	0.0365
Greece	0.0368	0.0366	0.0366	0.0341	0.0341	0.0340
Czech Republic	0.0368	0.0366	0.0366	0.0341	0.0341	0.0340
Belgium	0.0368	0.0366	0.0366	0.0341	0.0341	0.0340
Hungary	0.0368	0.0366	0.0366	0.0341	0.0341	0.0340
Portugal	0.0368	0.0366	0.0366	0.0341	0.0341	0.0340
Sweden	0.0309	0.0325	0.0325	0.0282	0.0287	0.0286
Bulgaria	0.0309	0.0325	0.0325	0.0282	0.0287	0.0286
Austria	0.0309	0.0325	0.0325	0.0282	0.0287	0.0286
Slovak Republic	0.0218	0.0263	0.0263	0.0196	0.0209	0.0208
Denmark	0.0218	0.0263	0.0263	0.0196	0.0209	0.0208
Finland	0.0218	0.0263	0.0263	0.0196	0.0209	0.0208
Ireland	0.0218	0.0263	0.0263	0.0196	0.0209	0.0208
Lithuania	0.0218	0.0263	0.0263	0.0196	0.0209	0.0208
Latvia	0.0125	0.0198	0.0198	0.0110	0.0131	0.0131
Slovenia	0.0125	0.0198	0.0198	0.0110	0.0131	0.0131
Estonia	0.0125	0.0198	0.0198	0.0110	0.0131	0.0131
Cyprus	0.0125	0.0198	0.0198	0.0110	0.0131	0.0131
Luxembourg	0.0125	0.0198	0.0198	0.0110	0.0131	0.0131
Malta	0.00942	0.0177	0.0177	0.0082	0.0106	0.0106

1. Germany has almost the same power indices that United Kingdom, France and Italy, for the weighted triple majority game $v_1 \wedge v_2 \wedge v_3$: the difference is only 4 swings with respect to 28 millions. Concerning to the weighted triple majority game $v_1 \wedge v_2' \wedge v_3$, the difference is also 4 swings in favor of Germany, over 24 millions and a half of swings. Consequently, the distinction between the Banzhaf indices of Germany and United Kingdom, France and Italy are, respectively, smaller than 1.4×10^{-7} and 1.6×10^{-7}.
2. The two rules of triple majority, adopted in the Nice summit meeting, are almost equivalent to a game of simple majority (the first) or double (the second). With both rules, the required population quota to adopt a decision does not change in practice the power of the countries.

The power indices obtained for the member states in the Council of Ministers under the Nice weighting of votes, are contained in two *notebooks* of the computer system Mathematica 4, due to Bilbao et al. (2001a,b).

Acknowledgements

This research has been partially supported by the Spanish Ministry of Science and Technology, under grant SEC2000–1243.

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