LOCALLY CONVEX GAMES

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Abstract. We introduce a new class of cooperative games by using the notion of local supermodularity. Therefore locally convex games can be considered as relaxations of convex games. Cooperative games with the unit marginal worth property are defined and characterizations of locally concave (convex) games with this property are proved. Moreover, we obtain a characterization of a subclass of locally concave games in terms of the rank function of a matroid.

1. Introduction

A game (N, v) is *convex* if it is *supermodular*, i.e.,

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T),$$

for all $S, T \subseteq N$. This concept was introduced by Shapley [6]. A game (N, v) is convex if and only if for each player $i \in N$, its marginal contribution function is nondecreasing, that is,

$$v(S \cup i) - v(S) \leq v(T \cup i) - v(T)$$

whenever $S \subseteq T \subseteq N \setminus i$ (see Peleg and Sudhölter [5]).

Amenta [1], Gärtner [2], Matousěk, Sharir and Welzl [4] have formulated a theory of LP-type problems, also known as *Generalized Linear Programming.*

Definition 1.1. *A generalized linear program (GLP) is a pair* (H, w), *where* H *is a finite set of constraints, and* $w : 2^H \to (P, \leq)$ *is an objective function valued on a totally ordered set, with the following properties:*

(1) $F \subseteq G \subseteq H$ *implies* $w(F) \leq w(G)$.
(2) *For all* $F \subset H$ *and all* $p, q \in H$,

$$w(F) = w(F \cup p) = w(F \cup q) \quad \text{implies} \quad w(F) = w(F \cup \{p, q\}).$$

Property 2 is known as *local submodularity.* Another example is the rank function of a greedoid. This function is nondecreasing and locally submodular as the following theorem states.

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Theorem 1.1. *A function $r : 2^N \to \mathbb{Z}_+$ is the rank function of a greedoid if and only if for all $X, Y \subseteq N$ and all $x, y \in N$ the following conditions hold:*

1. $r(\emptyset) = 0$,
2. $r(X) \leq |X|$,
3. $X \subseteq Y$ *implies* $r(X) \leq r(Y)$,
4. $r(X) = r(X \cup x) = r(X \cup y)$ *implies* $r(X) = r(X \cup \{x, y\})$.

Proof. See Korte, Lovász and Schrader [3, Chapter V, Theorem 1.1]. □

Let us briefly outline the contents of our work. In the next section, we introduce locally concave games and locally convex games by using locally submodular and locally supermodular functions respectively. We also obtain some properties about the relationship between locally concave (convex) games and concave (convex) games. The aim of the third section is to analyze cooperative games integer valued. Games with the unit marginal worth property are defined and characterizations of locally concave (convex) games with this property are proved. Finally, we obtain a characterization of a subclass of locally concave games in terms of the rank function of a matroid.

2. Locally convex games

We will now define a new class of cooperative games and investigate its properties.

Definition 2.1. *A game (N, c) is locally concave if the function c is locally submodular, i.e., for all coalitions $S \subset N$ and all players $i, j \in N$,*

$$c(S) = c(S \cup i) = c(S \cup j) \quad \textit{implies} \quad c(S) = c(S \cup \{i, j\}).$$

A game (N, v) is locally convex if the function v is locally supermodular, i.e., for all nonempty coalitions $S \subseteq N$ and all players $i, j \in N$,

$$v(S) = v(S \setminus i) = v(S \setminus j) \quad \textit{implies} \quad v(S) = v(S \setminus \{i, j\}).$$

Definition 2.2. *If (N, c) is a game then its dual game (N, c^*) is defined by $c^*(S) = c(N) - c(N \setminus S)$ for all $S \subseteq N$.*

The following three results are immediate:

(i) c is nondecreasing if and only if c^* is nondecreasing.
(ii) (N, c) is concave if and only if (N, c^*) is convex.
(iii) (N, v) is convex if and only if (N, v^*) is concave.

Proposition 2.1. *(N, c) is locally concave (convex) if and only if its dual game (N, c^*) is locally convex (concave).*

Proof. Suppose first that the function c is locally submodular, and that

$$c^*(S) = c^*(S \setminus i) = c^*(S \setminus j)$$

for all nonempty $S \subseteq N$ and all $i, j \in S$. By definition, we have

$$\begin{aligned} c(N) - c(N \setminus S) &= c(N) - c(N \setminus (S \setminus i)) \\ &= c(N) - c(N \setminus (S \setminus j)). \end{aligned}$$

If we take $T = N \setminus S$, then $T \cup i = N \setminus (S \setminus i)$ and $T \cup j = N \setminus (S \setminus j)$. Hence $c(T) = c(T \cup i) = c(T \cup j)$ holds and the local submodularity of c implies $c(T) = c(T \cup \{i, j\})$. Further, since $c(N \setminus S) = c(N \setminus (S \setminus \{i, j\}))$ we obtain $c^*(S) = c^*(S \setminus \{i, j\})$. A similar argument shows the converse. The result for convexity follows from $(c^*)^* = c$. □

Proposition 2.2. *Let (N, c) be a nondecreasing concave (convex) game. Then (N, c) is locally concave (convex).*

Proof. Let $S \subset N$ and $i, j \in N \setminus S$ such that $c(S) = c(S \cup i) = c(S \cup j)$. Since c is concave it follows that

$$c(S \cup \{i, j\}) \leq c(S \cup i) + c(S \cup j) - c(S) = c(S).$$

The game c is nondecreasing, so that $c(S) = c(S \cup \{i, j\})$.

Suppose that $v : 2^N \to \mathbb{R}$ is a nondecreasing convex game. Let $S \subseteq N$ and $i, j \in S$ such that $v(S) = v(S \setminus i) = v(S \setminus j)$. The convexity of v implies

$$v(S \setminus \{i, j\}) \geq v(S \setminus i) + v(S \setminus j) - v(S) = v(S). \tag{1}$$

Since v is nondecreasing, we obtain $v(S) = v(S \setminus \{i, j\})$. □

There is a stronger local condition which we describe in the next result.

Definition 2.3. *A game (N, v) is said to be monotone if $v : 2^N \to \mathbb{R}$ is nondecreasing or nonincreasing.*

Theorem 2.3. *Let (N, c) and (N, v) be monotone games. Then*

(1) *(N, c) is locally concave if and only if for all $S, T \subseteq N$*

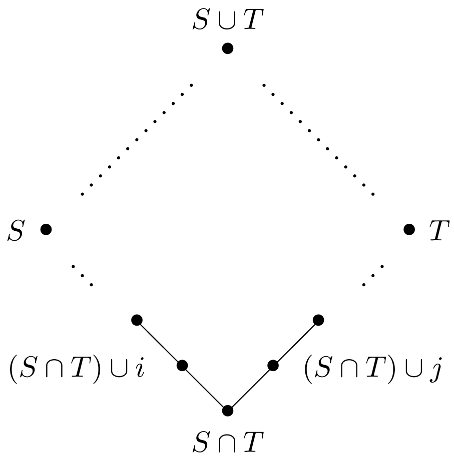
$$c(S) = c(T) = c(S \cap T) \;\; \textit{implies} \;\; c(S) = c(T) = c(S \cup T).$$

(2) *(N, v) is locally convex if and only if for all $S, T \subseteq N$,*

$$v(S) = v(T) = v(S \cup T) \;\; \textit{implies} \;\; v(S) = v(T) = v(S \cap T).$$

Proof. 1. Clearly the local condition implies local submodularity. Conversely, let $S, T \subseteq N$ such that $c(S) = c(T) = c(S \cap T)$. Since c is nondecreasing or nonincreasing, the above condition gives

$$\begin{aligned} c(S \cap T) &= c((S \cap T) \cup i) = c(S) \;\; \text{for all } i \in S \setminus T, \\ c(S \cap T) &= c((S \cap T) \cup j) = c(T) \;\; \text{for all } j \in T \setminus S. \end{aligned}$$



By repeatedly applying the local submodularity of c we deduce

$$c(S \cap T) = c((S \cap T) \cup \{i, j\}) = \cdots = c(S \cup T).$$

2. The proof is the same as 1. □

Definition 2.4. *A game (N, v) is zero-monotone if $v(S) - \sum_{i \in S} v(i)$ is nondecreasing, i.e., for all $T \subseteq S \subseteq N$,*

$$v(T) - \sum_{i \in T} v(i) \leq v(S) - \sum_{i \in S} v(i).$$

The zero-monotonicity is equivalent to

$$v(T) + \sum_{i \in S \setminus T} v(i) \leq v(S)$$

for all $T \subseteq S \subseteq N$. Note that superadditivity implies zero-monotonicity.

Proposition 2.4. *If (N, v) is a convex game and $v(i) \geq 0$ for all $i \in N$, then (N, v) is locally convex.*

Proof. Consider $S \subseteq N$ and $i, j \in S$ with $v(S) = v(S \setminus i) = v(S \setminus j)$. From (1) we deduce that $v(S) \leq v(S \setminus \{i, j\})$. Since convexity implies zero-monotonicity, we have

$$\begin{aligned} v(S \setminus i) + v(i) &\leq v(S) = v(S \setminus i), \\ v(S \setminus j) + v(j) &\leq v(S) = v(S \setminus j), \end{aligned}$$

and therefore $v(i) = v(j) = 0$. Further, zero-monotonicity implies

$$v(S \setminus \{i, j\}) = v(S \setminus \{i, j\}) + v(i) + v(j) \leq v(S),$$

and hence $v(S \setminus \{i, j\}) = v(S)$. □

Proposition 2.5. *If* (N,c) *is a concave game and* $c(N\setminus i)\le c(N)$ *for all* $i\in N$*, then* (N,c) *is locally concave.*

Proof. If (N,c) is a concave game and $c(N\setminus i)\le c(N)$ for all $i\in N$, then (N,c^*) is a convex game such that $c^*(i)=c(N)-c(N\setminus i)\ge 0$ for all $i\in N$. Therefore the above proposition may be applied to obtain that (N,c^*) is locally convex. It follows from the identity $(c^*)^*=c$ and Proposition 2.1 that (N,c) is locally concave. □

3. Games integer valued

Every rank function of a greedoid which is not a matroid is an example of locally concave game which is not concave. We introduce a class of games in order to study the equivalence of these concepts.

Definition 3.1. *An integer game* $c:2^N\to\mathbb{Z}$ *has the unit marginal worth property if for all* $S\subset N$ *and* $i\in N$*,* $c(S\cup i)-c(S)\in\{0,1\}$.

Theorem 3.1. *Let* $c:2^N\to\mathbb{Z}$ *be an integer game which satisfies the unit marginal worth property. Then the following statements are equivalent:*

1. (N,c) *is locally concave.*
2. c *is the rank function of the matroid* $\mathcal{M}=\{S\subseteq N: c(S)=|S|\}$.
3. (N,c) *is nondecreasing and concave.*

Proof. $1\Rightarrow 2$. The function $c:2^N\to\mathbb{Z}$ satisfies the following conditions:

(i) $c(\emptyset)=0$,

(ii) For all $S\subset N$ and $i\in N$, the unit marginal worth property implies

$$0\le c(S\cup i)-c(S)\le 1\Longleftrightarrow c(S)\le c(S\cup i)\le c(S)+1,$$

(iii) c is locally submodular.

Therefore, the statement 2 follows from the axiomatization given by Korte *et al.* [3, Chapter II, Theorem 1.3].

$2\Rightarrow 3$. Follows from Korte *et al.* [3, Chapter II, Theorem 1.4].

$3\Rightarrow 1$. Proposition 2.2 implies the result. □

Corollary 3.2. *Let* $v:2^N\to\mathbb{Z}$ *be an integer game which satisfies the unit marginal worth property. Then the following statements are equivalent:*

1. (N,v) *is locally convex.*
2. *The function* $r^d(S)=|S|-v(S)$ *is the rank function of the matroid* $\mathcal{M}^d=\{S\subseteq N: v(S)=0\}$.
3. (N,v) *is nondecreasing and convex.*

Proof. $1 \Rightarrow 2$. The dual game v^* is locally concave, so that v^* is the rank function of the matroid $\mathcal{M} = \{S \subseteq N : v^*(S) = |S|\}$. The rank function r^d of the dual matroid $\mathcal{M}^d$ (see Welsh [8, Section 5]) is given by

$$r^d(S) = |S| - v^*(N) + v^*(N \setminus S) = |S| - v(S).$$

Hence, the dual matroid is

$$\mathcal{M}^d = \left\{S \subseteq N : r^d(S) = |S|\right\} = \{S \subseteq N : v(S) = 0\}.$$

$2 \Rightarrow 3$ and $3 \Rightarrow 1$ are immediate consequences of the equivalences of the main theorem and the formula $(v^*)^* = v$. □

Corollary 3.3. *Let $v : 2^N \to \{0,1\}$ be a game which is nondecreasing (simple game). Then the following statements are equivalent:*

(1) *(N, v) is convex.*
(2) *(N, v) is locally convex.*
(3) *The collection of winning coalitions $\mathcal{W} = \{S \subseteq N : v(S) = 1\}$ is closed under intersection.*
(4) *There exists a unique circuit (minimal dependent set) of the matroid $\mathcal{M}^d$.*
(5) *$v = u_T$, where T is the unique circuit of the matroid $\mathcal{M}^d$.*

Note that the intersection of the winning coalitions is the minimal winning coalition of the convex simple game v, i.e., the unique circuit of the matroid $\mathcal{M}^d$. A simple game with these properties is called an *oligarchic game* (see van Deemen [7, Theorem 5.4]).

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