Lattice games

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Abstract

Lattice games are real-valued functions defined on a finite lattice L. The basic players are the nonzero join-irreducible elements of the lattice and the coalitions are its elements. If L is the Boolean algebra 2^N then we obtain a n-person game. Gilboa and Lehrer introduced the global games, which are lattice games where $L = \Pi_n$, the lattice of all partitions of N ordered by refinement. Faigle and Kern investigated a special type of cooperative game with precedence constraints. In this model L is the distributive lattice of the order ideals of a partially ordered set. We study two special classes of lattice games: topological games and convex geometry games. Finally, we define the join and the union Shapley values for decomposition of coalitions with the join and the set-union operations.

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1 Games and lattices

Let N be a finite set of players. A pair (N, v), where $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, is a game in coalitional form and v is called a worth function. The subsets $S \in 2^N$ are called coalitions. If $v(S) \leq v(T)$, for all $S \subseteq T$, then v is monotonic. The game v is a simple game if v is a monotonic $\{0, 1\}$-value worth function. The game is superadditive if $v(S) + v(T) \leq v(S \cup T)$, whenever $S \cap T = \emptyset$. If the worth function is supermodular,

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T), \quad \text{for all } S, T \in 2^N,$$

then we say that the game (N, v) is convex .

The coalitions of the game are the subsets of the set N and $\left(2^N, \cup, \cap\right)$ form a Boolean algebra. The structure of a finite Boolean algebra is completely determined once the number of atoms (basic players) is known, namely each member of Boolean algebra is the unique join of the atoms contained in it.

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An important generalization of a Boolean algebra is a distributive lattice which has the join ($\vee$) and meet ($\wedge$) operations with the same properties as in the case of Boolean algebras but without the complementation operation. It is no true that every member of the lattice is a join of atoms, but it is true that every member is a join of join-irreducible elements.

A join-irreducible is an element of the lattice which cannot be represented as a join of elements distinct from itself. For example, the family of the sets $\{\emptyset, \{1\}, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$ ordered by inclusion, form a distributive lattice L. The join-irreducible elements of L are $\emptyset$, $\{1\}$, $\{2\}$ and $\{2,3\}$. Birkhoff (1937) has proved that every element of a finite distributive lattice has a unique irredundant decomposition as a join of join-irreducible elements. It is easy to find non-distributive lattices having this property. That is, for each element $x \in L$ there exists a unique subset S of $J(L)$ (the set of all join-irreducibles) such that $x = \bigvee S$ and $x < \bigvee (S \setminus \{j\})$ for every $j \in J(L)$. For instance, the lattice $Co(3)$, whose elements are the convex of the chain $1 < 2 < 3$, is a non-distributive lattice having unique irredundant join decomposition.

Dilworth (1940) investigated representations of an element a in a finite and semimodular lattice as a meet $a = \bigwedge M$ of a set M of meet-irreducible elements. Dilworth calls a lattice L locally distributive if all the intervals $[a, a^+]$, where $a^+ := \bigvee \{x \in L : x \succ a\}$ is the join of all the covers of a, are distributive. Note that if $[a, a^+]$ is distributive then is a Boolean algebra. Dilworth showed that the following conditions are equivalent (see Jónsson (1990)) in a finite lattice:

(i) Every element of L has a unique irredundant meet-irreducible decomposition.

(ii) L is upper semimodular and locally distributive.

There are in fact a lot of characterizations of locally distributive lattices (see Monjardet (1990)). A finite lattice L is called meet-distributive if for every $a \in L$ the interval $[a^-, a]$, where $a^- = \bigwedge \{x : x \prec a\}$, is a Boolean algebra ($\prec$ is the cover relation). If L is a meet-distributive lattice then every element of L has a unique irredundant join-irreducible decomposition (Lemma 3 in Reuter (1989)).

2 Lattice Games

In this section, we define a more general framework which includes the ordinary games, the global games of Gilboa and Lehrer (1991), the games under precedence constraints and the partition games of Faigle and Kern (1992) (1997) and the arc game corresponding to communication situations of Borm, Owen and Tijs (1992).

Definition 2.1. A lattice game is an ordered pair $(J(L); v)$, where $J(L)$ is the set of all nonzero join-irreducible elements of a finite lattice L and $v : L \to \mathbb{R}$; is a real value function with $v(\hat{0}) = 0$:

The basic players are the elements of $J(L)$ or J; defined by

$$J := \left\{ j \in L : j \neq \hat{0} \text{ and } (j = a \vee b) \Rightarrow (j = a \text{ or } j = b) \right\}:$$

Note that every $a \in L$ has a $\vee$-representation $a = j_1 \vee \cdots \vee j_n$; where the basic players $\{j_1; \ldots; j_n\} \subseteq J$. Then the coalitions of basic players are the elements of the lattice L. The symbols $\hat{0}$ and $\hat{1}$ are used for the least and greatest elements of a lattice. A lattice game is called monotonic, superadditive or convex when $v : L \to \mathbb{R}$ satisfies the corresponding property for the partial order, and the join and meet operations.

Remark 2.1. Whittle (1993) introduced weighted lattices $(L; f)$; where L is a finite lattice and f is a function from L into the non-negative integers $\mathbb{N}$ such that $f(\hat{0}) = 0$; and if $a; b \in L$ with $a \leq b$; then $f(a) \leq f(b)$:

Example: If L is a Boolean algebra of rank n then $J(L)$ is the set of n atoms of L; hence the lattice game is an ordinary game in coalitional form.

Example: Faigle and Kern (1992) (1997) investigated a cooperative game such that the set of players to be (partially) ordered by some precedence relation and introduced the antichain game $(A; c)$ where A is the set of antichains of a partially ordered set (poset) P; or equivalently the order ideals of P.

Example: Gilboa and Lehrer (1991) defines a global game for to analize situations where the worth is defined for all players together, as is the case with issues of environmental clean-up, medical research, and so forth. A global game is a real-valued function defined in the lattice of all partitions of the set N. In this lattice, any partition $\pi \in \Pi_n$ is the join of those atoms $\{B_1, \ldots, B_{n-1}\}$ such that $|B_1| = 2$ and B_1 is a subset of some block of π. Hence Π_n is a finite semimodular and atomic lattice, i.e. a geometric lattice, Stanley (1986). In our model, the basic players of the lattice game defined by Π_n are the join-irreducible elements, that is, the atoms $\pi = \{B_1, \ldots, B_{n-1}\}$ with $|B_1| = 2$. The cardinal of $J(\Pi_n)$ is $\binom{n}{2}$.

Example: A communication situation is a triple (N, v, G), where (N, v) is a game and $G = (N, E)$ is a graph. This concept were studied first by Myerson (1977), who introduced the graph-restricted games. Myerson's point of view focuses on the role of the vertex-connectivity of the graph. Borm, Owen and Tijs (1992) consider the role of the edges (communication link) and defined the following edge game $r^v : 2^E \to \mathbb{R}$,

$$r^v(L) := \sum \{v(T) : T \text{ is a component of the subgraph } (N, L)\},$$

for all $L \subseteq E$. We introduce a lattice game for this situation. Let L_G the geometric lattice of all partitions π of the vertex set N ordered by refinement, such that every block of π is connected (see Stanley (1986)). Thus, we define $h_v : L_G \to \mathbb{R}$, by $h_v(\pi) := \sum \{v(B) : B \in \pi\}$. Note that the element $\hat{0} \in L_G$ is $\hat{0} = \{\{1\}, \ldots, \{n\}\}$ and the element $\hat{1} = \{N_1, \ldots, N_p\}$, where N_k are the components of the graph G. The basic players of the lattice game are the atoms of L_G (partitions with $n - 1$ blocks). Then each basic player has exactly one block with two elements $\{i, j\} \in E$, and the number of basic players is equal to the number of edges $|E|$ of G.

A pair $(J, -)$ is a closure space if J is a set and $-$ is a closure operator $2^J \to 2^J : A \mapsto \overline{A}$, which satisfies $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$, $A \subseteq \overline{A}$, and $\overline{\overline{A}} = \overline{A}$, for all $A, B \subseteq J$. A set $A \subseteq J$ is called closed if $A = \overline{A}$. Denote the family of all closed sets by $L = L(J, -)$. If $(J, -)$ is a closure space then $L \subseteq 2^J$, ordered by inclusion, is a complete lattice in which meet and join operations are defined by $A \wedge B = A \cap B$ and $A \vee B = \overline{A \cup B}$ for all $A, B \in L$.

Conversely, let $(L, \vee, \wedge)$ any complete lattice, and $J \subseteq L$ a $\vee$-generating set. Then L is isomorphic to $L(J, -)$, where $(J, -)$ is the closure space defined by $\overline{A} := \{p \in J : p \leq \bigvee A\}$ for all $A \subseteq J$.

We only consider finite lattices and we suppose that $\emptyset \in L$ and this not lead us to loss of generality (see Libkin (1993)). The general theory of closure spaces can be found in Wild (1994) and games on closure spaces are introduced by Bilbao (1998b). In what follows, we use a simplified notation and write $A \cup b$ to mean $A \cup \{b\}$, and $A \setminus a$ to mean $A \setminus \{a\}$.

If L is a finite lattice, there is a canonical construction of a closure space such that the lattice of its closed sets L and L are isomorphic. Let $J = J(L)$ be the set of its nonzero join-irreducibles and we consider the representation of the lattice L given by $a \mapsto J(a) := \{j \in J : j \leq a\}$. Then for all $X \subseteq J$ we have

$$\overline{X} = \left\{ j \in J : j \leq \bigvee X \right\} = J\left(\bigvee X \right);$$

and therefore (J, Γ) is a closure space such that the family of its closed sets $L = \{J(a) : a \in L\} \subseteq 2^J$; is a lattice of subsets isomorphic to L.

Remark 2.2. We may identify the lattice games $(J(L), v)$ and $(J(L), v)$ where $J(L) = \left\{ \overline{\{j\}} : j \in J(L) \right\}$ and $v(J(a)) = v(a)$, for all $a \in L$. Obviously $v(\emptyset) = v\left(\hat{0}\right)$ and $v(J(L)) = v\left(\hat{1}\right)$.

We recall that L is a lattice of closed sets containing $\emptyset, J$ and closed under set-intersection. If L is also closed under set-union, we say that the closure space (J, Γ) is a topological space. Then, the following conditions are equivalent (see Libkin (1993)) in a closure space (J, Γ):

(i) $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$ for all $X, Y \in 2^J$.

(ii) $A \vee B = A \cup B$ for all $A, B \in L$.

(iii) L is a distributive lattice (sublattice of the Boolean algebra 2^J).

Definition 2.2. A topological game is a lattice game $(J(L), v)$ such that its lattice L of closed sets is distributive.

Given a closure space (J, Γ); we obtain the partitioning of 2^J induced by the equivalence relation $A \cong B \Leftrightarrow \overline{A} = \overline{B}$. For a closed set $S \in L$ an element of the equivalence class $[S]$ is a generating set of S. A minimal element of $[S]$ is a basis of S. Hence, $A \subseteq S$ is a basis of S if $\overline{A} = \overline{S}$ and $\overline{A \setminus a} \neq \overline{S}$ for all $a \in A$.

For $A \subseteq J$ an element $a \in A$ is an extreme point of A if $a \notin \overline{A \setminus a}$. Let $Ex(A)$ be the set of all extreme points of A. We have that $Ex(A)$ (possibly empty) is contained in each generating set of $\overline{A}$. The convex geometries are the abstract convex spaces satisfying the finite Minkowski-Krein-Milman property: Every closed set is the closure of its extreme points (see Edelman and Jamison (1985)). Indeed convex geometries are exactly those closure spaces where all equivalence classes $[A]$, $A \in L$, have smallest elements. Then $Ex(A)$ is the unique basis of every $\overline{A} \in L$. Edelman (1980) obtained: A finite lattice is meet-distributive if and only if it is isomorphic to some convex geometry. We call such a closure structure, as well as its lattice, convex geometry.

Definition 2.3. A convex geometry game is a lattice game $(J(L), v)$ whose lattice L of closed sets is a convex geometry.

Example: A subset S of a poset $(P, \leq)$ is convex whenever $a \in S$, $b \in S$ and $a \leq b$ implies $[a, b] \subseteq S$. The convex subsets of any poset P form a closure system $Co(P)$. If P (or, equivalently $Co(P)$) is finite, then each element is between a maximal and a minimal one. If $C \in Co(P)$ then $Ex(C)$ is the union of the maximal and minimal elements of C. Moreover, $Co(P)$ is a convex geometry (Birkhoff and Bennett (1985)). If the poset P is a chain then the sequencing games (see Curiel, Pederzoli and Tijs (1989)) are convex geometry games.

Example: Let $(P, \leq)$ be a partially ordered set. For any $X \subseteq P$,

$$X \mapsto \overline{X} := \{y \in P : y \leq x \text{ for some } x \in X\},$$

defines a closure operator on P. Its closed sets are the order ideals (down sets) of P, and we denote this lattice $I(P)$. Since the union and intersection of order ideals is again an order ideal, it follows that $I(P)$ is a sublattice of 2^P. Then $I(P)$ is a distributive lattice, so that $I(P)$ is a convex geometry closed under set-union and $Ex(S)$ is the set of all maximal points of the subposet $S \in I(P)$. When P is finite, there is a 1-1 correspondence between antichains of P and order ideals. Then the games $(\mathcal{C}, v)$ and $(\mathcal{A}, c)$, where $\mathcal{C}$ is the family of down sets of P defined by Faigle and Kern (1992) (1997) and $\mathcal{A}$ is the set of antichains of a rooted tree are topological games.

The fundamental theorem for finite distributive lattices states that if L is a finite distributive lattice, then $L \cong I(P)$, where P is the subposet of nonzero join-irreducibles of L. If $(J(L), v)$ is a topological game, then L is a finite distributive lattice and $J = J(L)$ is the subposet of basic players. Thus the lattice of closed sets satisfies $L = \{J(a) : a \in L\} = I(J)$.

3 The dividends in a lattice game

Let L be a finite lattice and let $L = \{J(a) : a \in L\}$ be the lattice of its closed sets. We consider the following simple lattice games. For any $T \in L$ the unanimity game $u_T : L \to \mathbb{R}$ is defined by

$$u_T(S) = \begin{cases} 1; & \text{if } T \subseteq S; \\ 0; & \text{otherwise:} \end{cases}$$

Gilboa and Lehrer (1991) obtained the following property for lattice functions $f : L \to \mathbb{R}$: The family $\{g_a : L \to \mathbb{R} : a \in L\}$ defined by $g_a(b) = 1$ if $b \geq a$, and $g_a(b) = 0$, otherwise, form a basis of the vector space of all functions $f : L \to \mathbb{R}$: This result implies the next representation of a lattice game in terms of the family $\{u_T : T \in L\}$.

Proposition 3.1. Let $v : L \to \mathbb{R}$ be a lattice game. Then there exists the unique set of coefficients $\{\Delta_v(T) : T \in L\}$ such that

$$v = \sum_{T \in L} \Delta_v(T) u_T; \quad \text{where} \quad \Delta_v(S) = \sum_{\{T \in L : T \subseteq S\}} \mu_L(T; S) v(T); \tag{3.1}$$

and μ_L is the Möbius function of the lattice.

Proof. The family $\{u_T : T \in L\}$ is a basis of the lattice games on L. Moreover, for all $S \in L$ we have

$$v(S) = \sum_{T \in L} \Delta_v(T) u_T(S) = \sum_{\{T \in L : T \subseteq S\}} \Delta_v(T):$$

The Möbius inversion formula (Stanley (1986)) for the lattice L may be applied to obtain $\Delta_v(S) = \sum_{\{T \in L : T \subseteq S\}} \mu_L(T; S) v(T)$ for all $S \in L$: ∎

Following Harsanyi (1963) we shall call $\Delta_v(T)$ the dividends of T, in the lattice game $(J(L); v)$. Let $(N; v)$ be a game on the lattice 2^N. Since the Möbius function for the Boolean algebra 2^N is $\mu(T; S) = (-1)^{|S| - |T|}$ we deduce Lemma 3 given by Shapley (1953), that is

$$v = \sum_{T \in 2^N} \Delta_v(T) u_T; \quad \text{where} \quad \Delta_v(S) = \sum_{T \subseteq S} (-1)^{|S| - |T|} v(T): \tag{3.2}$$

Theorem 3.1. Let $(\mathcal{J}(L), v)$ be a convex geometry game and let L be the lattice of its closed sets. Then the dividends of every $S \in L$ satisfies

$$\Delta_v(S) = \sum_{T \in [S^-, S]} (-1)^{|S|-|T|} v(T), \qquad (3.3)$$

where $S^- = S \setminus Ex(S)$.

Proof. The family L is a convex geometry and its Möbius function satisfies Theorem. 4.3 of Edelman and Jamison (1985), that is

$$\mu_L(T, S) = \begin{cases} (-1)^{|S|-|T|}, & \text{if } S \setminus T \subseteq Ex(S); \\ 0, & \text{otherwise.} \end{cases}$$

Thus the index set in the formula for the dividends is

$$\{T \in L : T \subseteq S, \; S \setminus T \subseteq Ex(S)\} = [S \setminus Ex(S), S].$$

$\square$

Corollary 3.1. Let $(\mathcal{J}(L), v)$ be a topological game and let $I(J)$ be the lattice of its closed sets. Then for all $S \in I(J)$

$$\Delta_v(S) = \sum_{\{T \subseteq S : S \setminus T \subseteq Max(S)\}} (-1)^{|S|-|T|} v(T),$$

where $Max(S)$ is the set of all maximal points of the subposet S.

Remark 3.1. If $L = 2^N$, then $S \setminus Ex(S) = \emptyset$, hence the dividends of $S \in 2^N$ satisfies formula (3.2). Moreover, the interval $[S^-, S]$ is a Boolean algebra isomorphic to $2^{Ex(S)}$.

The dual of a game (N, v), denoted by (N, v^*), is defined by

$$v^*(S) = v(N) - v(N \setminus S) \text{ for all } S \in 2^N.$$

The set of the duals of the unanimity games $\left\{u_T^* : T \in 2^N\right\}$ is a basis of the vector space of all games. Observe that

$$u_T^*(S) = \begin{cases} 1, & \text{if } T \cap S \neq \emptyset, \\ 0, & \text{if } T \cap S = \emptyset. \end{cases}$$

The game $-u_T^*$ has a natural interpretation as a cost game, where $-u_T^*(S)$ is the cost incurred by S. The presence of any number of members of T in S incurs a unit cost (see Kalai and Samet (1988)).

Definition 3.1. The dual of a lattice game $v : \mathcal{L} \to \mathbb{R}$, where $\mathcal{L} \subseteq 2^J$, denoted by $(\mathcal{L}, v^*)$ is defined by $v^*(S) = v(J) - v\left(\overline{J \setminus S}\right)$ for every $S \in \mathcal{L}$.

A subset A of a closure space $(J, -)$ is open if $J \setminus A$ is closed. The anticlosure operator

$$\omega : 2^J \to 2^J : A \mapsto \omega(A) := J \setminus \left(\overline{J \setminus A}\right),$$

satisfies the properties which are duals of those of the closure operator and A is open if $\omega(A) = A$. If $\{u_T : T \in \mathcal{L}\}$ is the basis of the vector space of all lattice games $(J(\mathcal{L}), v)$ then its dual games satisfies

$$u_T^*(S) = \begin{cases} 1, & \text{if } T \cap \omega(S) \neq \emptyset, \\ 0, & \text{if } T \cap \omega(S) = \emptyset. \end{cases}$$

We need some new concepts for to study the relation between the dividends in v and a special dual game. Let $v : \mathcal{L} \to \mathbb{R}$ be a lattice game. The restriction of v to a coalition $S \in \mathcal{L}$, denoted by v_S, is the lattice game $v_S : [\emptyset, S] \to \mathbb{R}$, defined by $v_S(T) = v(T)$ for all $T \in \mathcal{L}$ such that $T \subseteq S$. A closed set of a convex geometry $\mathcal{L}$ is called free if $S = Ex(S)$.

Proposition 3.2. Let $v : \mathcal{L} \to \mathbb{R}$ be a convex geometry game. If $S \in \mathcal{L}$ is a free set, then $\Delta_v(S) = (-1)^{|S|+1}\Delta_{(v_S)^*}(S)$.

Proof. Let S be a free coalition. Then the interval $[S^-, S] = [\emptyset, S]$ is a Boolean algebra. Hence equation (3.3) implies

$$\Delta_v(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T) = \sum_{B \subseteq S} (-1)^{|B|} v(S \setminus B),$$

where $B = S \setminus T$. Consider the restriction $v_S : 2^S \to \mathbb{R}$, of the game v to S. Then the dual game satisfies for every $B \subseteq S$,

$$(v_S)^*(B) = v_S(S) - v_S\left(\overline{S \setminus B}\right) = v(S) - v(S \setminus B).$$

Therefore,

$$\begin{aligned} \Delta_v(S) &= \sum_{B \subseteq S} (-1)^{|B|}[v(S) - (v_S)^*(B)] \\ &= v(S) \sum_{B \subseteq S} \delta(\emptyset, B) + (-1)^{|S|+1} \sum_{B \subseteq S} (-1)^{|S|-|B|} (v_S)^*(B) \\ &= (-1)^{|S|+1}\Delta_{(v_S)^*}(S). \end{aligned}$$

∎

4 The Shapley value

The classical characterization of the Shapley value is as the only value that satisfies the symmetry, efficiency and additivity on the class of all superadditive games. If $(N; v)$ is a game then the Shapley value for the player $i \in N$ is

$$\Phi_i(N; v) = \sum_{\{S \in 2^N : i \in S\}} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S \setminus i)];$$

where $n = |N|$ and $s = |S|$ (see Shapley (1953)). Kalai and Samet (1988) proposed the following approach for the Shapley value.

> "The Shapley value Φ is the linear function that assigns to each game $(N; v)$ a vector of $\mathbb{R}^{|N|}$, which for each unanimity game u_S is defined by
>
> $$\Phi_i(u_S) = \begin{cases} 1/|S|, & \text{if } i \in S; \\ 0, & \text{otherwise.} \end{cases}$$
>
> Intuitively, in the game u_S any coalition that contains S can split one unit between its members, and therefore players outside S do not contribute anything to the coalition they join. Hence, $\Phi_i(u_S) = 0$ for $i \notin S$. The members of S, on the other hand, split equally the one unit between themselves."

Thus (3.2) implies the following formula for the Shapley value

$$\Phi_i(N; v) = \sum_{\{S \in 2^N : i \in S\}} \frac{\Delta_v(S)}{|S|}. \tag{4.1}$$

Let $(J(L); v)$ be a convex geometry game and let L be the lattice of its closed (or convex) sets. Then we have that every element of L has an unique irredundant join-irreducible decomposition. That is, for all $S \in L$ we have the decomposition

$$S = \overline{Ex(S)} = J\left(\bigvee Ex(S)\right) = \bigvee \left\{ \overline{\{j\}} : j \in Ex(S) \right\};$$

and the members of S in the game $v : L \to \mathbb{R}$ are the collection of basic players $\left\{ \overline{\{j\}} : j \in Ex(S) \right\}$. The Minkowski-Krein-Milman property yields that a player has individual power to join a convex coalition when it is an extreme point of the coalition.

Let $(J(L); v)$ be a topological game; that is, $v : I(J) \to \mathbb{R}$. Theorems 1 and 2 of Faigle and Kern (1992) showed the following extension of the Shapley value to a topological game:

There is an unique function $\Phi : v \mapsto \left(\Phi_1(v), \ldots, \Phi_{|J|}(v)\right)$ satisfying the axioms of linearity, carrier and hierarchical strength. The last axiom is:

$$S \in I \text{ and } i, j \in Max(S) \text{ imply } e(S \setminus i)\Phi_j(u_S) = e(S \setminus j)\Phi_i(u_S);$$

where $e(\cdot)$ is the number of linear extensions of the corresponding subposets of J. Moreover, for every $i \in J$ we have

$$\Phi_i(I; v) = \sum_{\{T \in I : i \in Max(T)\}} \frac{e(T \setminus i)\, e(J \setminus T)}{e(J)} [v(T) - v(T \setminus i)]. \qquad (4.2)$$

Edelman and Jamison (1985) defined a compatible ordering of a convex geometry $L \subseteq 2^J$ as a total ordering $x_1 < x_2 < \cdots < x_{|J|}$ of the elements of J, such that the set $C_i := \{x_1, x_2, \ldots, x_i\} \in L$ for all $1 \leq i \leq |J|$.

Note that a compatible ordering of L corresponds exactly to a maximal chain in L. Denote by $c([S, T])$ the number of saturated chains from S to T and $c(S) := c([\emptyset, S])$ the number of saturated chains from $\emptyset$ to S. Then $c(J) = c(L)$ is the total number of maximal chains. Edelman (1997) proposed the following extension for the Shapley-Shubik index of a convex geometry game $v : L \to \mathbb{R}$, $L \subseteq 2^J$. For each $i \in J$,

$$SS(i) := \frac{1}{c(J)} \sum_{M} v\left(M^i\right) - v\left(M^i \setminus i\right);$$

where the sum is taken over all maximal chains of L, and M^i is the minimal convex in the chain M that contains i. Notice that $M^i \setminus i \in L$, hence $i \in Ex(M^i)$.

Therefore, we define the join Shapley value as the expected marginal contribution of a player i to the coalitions $S \in L$ such that $S \setminus i \in L$, when the players joining in a random compatible ordering.

Definition 4.1. Let $v : L \to \mathbb{R}$ be a convex geometry game. The join Shapley value of (L, v) is the vector $J\Phi(L, v) \in \mathbb{R}^{|J|}$, with components defined by

$$J\Phi_i(L, v) = \sum_{\{T \in L:\, i \in Ex(T)\}} \frac{c(T \setminus i)\, c([T, J])}{c(J)} [v(T) - v(T \setminus i)],$$

where $c(\cdot)$ is the number of saturated chains of the corresponding intervals of L.

Example: The set $\{1, 2, \ldots, n\}$ with its usual order is a chain (poset in which any two elements are comparable) denoted by n. The lattice $L = I(n)$ of its order ideals is isomorphic to $n+1$. If $S \in L$ then $Ex(S) = Max(S) = \{m_S\}$, hence $i \in Ex(S) \Leftrightarrow i = m_S$. The join Shapley value of the topological game $v : L \to \mathbb{R}$ is $J\Phi_i(v) = v(S) - v(S \setminus i)$ for all $1 \le i \le n$, where $S \in L$ is the unique closed set which satisfies $i = m_S$. The principal order ideal generated by x is $\downarrow(x) = \{y \in P : y \le x\}$. Therefore, the join Shapley value is $J\Phi_i = v\left[\downarrow(i)\right] - v\left[\downarrow(i-1)\right]$ if $2 \le i \le n$ and $J\Phi_1 = v(\{1\})$.

Proposition 4.1. Let P a poset which is a disjoint union of the chains $C_1 = \{a_1 < \cdots < a_m\}$ and $C_2 = \{b_1 < \cdots < b_n\}$. Then the join Shapley value of the topological game $v : I(P) \to \mathbb{R}$ satisfies

$$J\Phi_i(I, v) = \frac{1}{\binom{m+n}{m}} \sum_{j=0}^{n} \binom{i-1+j}{j} \binom{m-i+n-j}{n-j} v(i, j),$$

where $v(i, j) = v\left[\downarrow(i) \cup \downarrow(j)\right] - v\left[\downarrow(i-1) \cup \downarrow(j)\right]$ if $i \in C_1$ and the above formula by exchange of m and n if $i \in C_2$.

Proof. In the game $I(P)$ the players are the $m+n$ elements of P and the coalitions are the collection of all order ideal $S \in I(P)$. Note that $|Max(S)| = 1$, if $S \subseteq C_1$ or $S \subseteq C_2$, and $|Max(S)| = 2$, otherwise. If $i \in C_1$ then

$$\{T \in I : i \in Max(T)\} = \left\{\downarrow(i) \cup \downarrow(j) : j \in C_2\right\}.$$

The number of linear extensions of the poset P is $e(C_1 + C_2) = \binom{m+n}{m}$. Then the formula follows from the Faigle and Kern extension (4.2). For $i \in C_2$ the result follows by exchange of the numbers n and m. $\square$

Remark 4.1. The sum of coefficients in the join Shapley value for each i is 1: Observe that

$$\sum_{j=0}^{n}\binom{i-1+j}{j}\binom{m-i+n-j}{n-j}=\binom{n+i-1+m-i+1}{n}=\binom{n+m}{n}.$$

Example: Let $Co(n)$ be the family of the order-convex subsets of the chain $n=\{1<2<\cdots<n\}$. Note that $J=\{1,2,\dots,n\}$ and the convex sets are

$$T=[i,j]=\{i,i+1,\dots,j-1,j\},\quad \text{for } 1\le i\le j\le n.$$

For instance,

$$Co(4)=\{\emptyset,\{1\},\{2\},\{3\},\{4\},\{1,2\},\{2,3\},\{3,4\},\{1,2,3\},\{2,3,4\},J\}.$$

We obtain the join Shapley value of the game $v: Co(n)\to\mathbb{R}$ by using the following formulas proved in Lemmas 1 and 2 by Edelman (1997),

$$c(T)=2^{j-i};\quad c([T,J])=\binom{n-j+i-1}{i-1},$$

where $T=[i,j]$, $1\le i\le j\le n$, and $J=[1,n]$.

Proposition 4.2. The join Shapley value for the player $i\in[1,n]$ in a convex geometry game $v: Co(n)\to\mathbb{R}$ satisfies

$$
\begin{aligned}
J\Phi_i(v) = {} & \sum_{j=1}^{i-1}\frac{\binom{n-i+j-1}{j-1}}{2^{n-i+j}}\{v([j,i])-v([j,i-1])\}\\
& + \sum_{j=i}^{n}\frac{\binom{n-j+i-1}{i-1}}{2^{n-j+i}}\{v([i,j])-v([i+1,j])\}.
\end{aligned}
$$

If all the points (atoms) of 2^J are closed, the convex geometry $L\subseteq 2^J$ will be called atomic. In this case, every closed set $S\in L$ has a unique decomposition as a set-union of atoms contained in it. Note that a distributive lattice which is atomic is a Boolean algebra. Thus we consider the above decomposition only for atomic convex geometries.

Definition 4.2. Let $(\mathcal{J}(L); v)$ be a convex geometry game such that L is atomic. The union Shapley value of $(\mathcal{J}(L); v)$ is the vector $U\Phi(L; v) \in \mathbb{R}^{|J|}$, with components defined by

$$U\Phi_i(L; v) = \sum_{\{S \in L : i \in S\}} \frac{\Delta_v(S)}{|S|} \quad \text{for all } i \in \mathcal{J}(L).$$

Proposition 4.3. The union Shapley value of an atomic convex geometry game $(\mathcal{J}(L); v)$ satisfies the efficiency property:

$$\sum_{i \in J} U\Phi_i(L; v) = v(J).$$

Proof. If we denote $L_i = \{S \in L : i \in S\}$ for all $i \in J$, then we have

$$\begin{aligned}
\sum_{i \in J} U\Phi_i(L; v) &= \sum_{i \in J} \sum_{k=1}^{|J|} \frac{1}{k} \left[\sum_{\{S \in L_i : |S| = k\}} \Delta_v(S) \right] \\
&= \sum_{k=1}^{|J|} \frac{1}{k} \left[\sum_{i \in J} \sum_{\{S \in L_i : |S| = k\}} \Delta_v(S) \right].
\end{aligned}$$

Every nonempty $S \in L$ with $|S| = k$ appears exactly k times in the term in brackets. Hence by Proposition 3.1 the sum satisfies

$$\begin{aligned}
\sum_{i \in J} U\Phi_i(L; v) &= \sum_{k=1}^{|J|} \sum_{\{S \in L : |S| = k\}} \Delta_v(S) \\
&= \sum_{S \in L} \Delta_v(S) \\
&= v(J).
\end{aligned}$$

□

A convex geometry, considered as lattice, is characterized as meet-distributive. If the lattice is also upper semimodular and atomic (geometric lattice) then the lattice is Boolean. We shall study atomic convex geometries which are semimodular above its atoms (quasi-modular). Note that this condition is equivalent to intersecting property.

Definition 4.3. A convex geometry L is intersecting if for any $S, T \in L$ with $S \cap T \neq \emptyset$ we have $S \cup T \in L$.

Proposition 4.4. Let $L \subseteq 2^J$ be an intersecting convex geometry. Then the interval $[T, T^+]$ is a Boolean algebra for every nonempty $T \in L$.

Proof. If S and T are closed sets and $S \succ T$ then $S = T \cup j$ for some $j \in J \setminus T$ (see Edelman and Jamison (1985)). Let T a nonempty closed set. If $\{R, S\} \subseteq L$ with $R \succ T$ and $S \succ T$ then $R \cap S = R \wedge S = T \neq \emptyset$. Hence $R \cup S \in L$ and we have

$$T^+ = \bigcup \{S \in L : S \succ T\} = T \cup \{j \in J \setminus T : T \cup j \in L\}.$$

From this we conclude that $[T, T^+]$ is isomorphic to $2^{T^+ \setminus T}$. □

Finally, if we use the proof methods of Theorems 2 and 3 of Bilbao (1998a) then we will obtain the following results.

Theorem 4.1. Let $(J(L), v)$ be a lattice game such that L is a convex geometry which is atomic and intersecting. For all $i \in J$, we define the following sets:

$$\begin{aligned} L_i &= \{T \in L : i \in T\} \\ L_i^+ &= \{T \in L : i \in Ex(T),\ (T \setminus i)^+ = T^+\} \\ L_i^\star &= \{T \in L : i \notin T,\ T \cup i \in L,\ T^+ \neq (T \cup i)^+\}. \end{aligned}$$

Then the union Shapley value for i satisfies

$$\begin{aligned} U\Phi_i(L, v) = & \sum_{T \in L_i^+} \frac{(t-1)!(t^+ - t)!}{t^+!}[v(T) - v(T \setminus i)] \\ & + \sum_{T \in L_i \setminus L_i^+} \frac{(t-1)!(t^+ - t)!}{t^+!} v(T) \\ & - \sum_{T \in L_i^\star} \frac{(t)!(t^+ - t - 1)!}{t^+!} v(T), \end{aligned}$$

where $t = |T|$ and $t^+ = |T^+|$.

Note that if L is an intersecting convex geometry then L_i is a distributive lattice. The above formula for computing the convex values can be further simplified when the player is an extreme point of N.

Corollary 4.1. Let $(J(L); v)$ be a lattice game such that L is a convex geometry which is atomic and intersecting, and $v(\{i\}) = 0$ for all $i \in N$. Then, for every $i \in Ex(N)$, we have

$$U\Phi_i(L; v) = \sum_{T \in L} \frac{(t-1)!(t^+ - t)!}{t^+!} [v(T) - v(T \setminus i)],$$

where $t = |T|$ and $t^+ = |T^+|$.

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