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Values for Interior Operator Games

J.M. BILBAO mbilbao@us.es (http://www.esi2.us.es/˜mbilbao/)
A. JIMÉNEZ-LOSADA
E. LEBRÓN
C. CHACÓN
Matemática Aplicada II, Escuela Superior de Ingenieros, Camino de los Descubrimientos s/n, 41092 Sevilla, Spain

Abstract. The aim of this paper is to study a new class of cooperative games called interior operator games. These games are additive games restricted by antimatroids. We consider several types of cooperative games as peer group games, big boss games, clan games and information market games and show that all of them are interior operator games. Next, we analyze the properties of these games and compute the Shapley, Banzhaf and Tijs values.

Keywords: cooperative games, antimatroids, Shapley value, Banzhaf value, Tijs value

1. Introduction

A cooperative game describes a situation in which a finite set of n players can generate certain payoffs by cooperation. A value for cooperative games is a function which assigns to every cooperative game a n-dimensional real vector which represents a payoff distribution over the players. Two well-known values are the *Shapley value* as proposed by Shapley (1953), and the *Banzhaf value*, initially introduced in the context of voting games by Banzhaf (1965), and later on extended to arbitrary games by Dubey and Shapley (1979). Compromise values are a special type of values which assign to each game a payment based on two vectors. These vectors, so-called *upper* and *lower*, are the maximum and the minimum payment that the players can expect to get. The *Tijs value* (Tijs, 1981) was the first compromise value introduced for cooperative games.

In a cooperative game the players are assumed to be identical in the sense that every player can cooperate with every other player. However, in practice there exist asymmetries among the players. For this reason, the game theoretic analysis of decision processes in which one imposes asymmetric constraints on the behavior of the players has been and continues to be an important subject to study. Some models which analyze social asymmetries among players in a cooperative game are the *coalition structures* by Aumann and Maschler (1964), and the *communication situations* studied by Myerson (1977), Owen (1986), and Borm, Owen, and Tijs (1992).

Another type of asymmetry among the players in a cooperative game is introduced in Gilles, Owen, and van den Brink (1992), and van den Brink (1997). In these models, the possibilities of coalition formation are determined by the positions of the players in a hierarchical *permission structure*. Games on antimatroids were introduced in Jiménez-Losada (1998) and Algaba et al. (2004) who showed that the feasible coalition systems derived from both the conjunctive and disjunctive approach to games with a permission structure were identified to certain families of antimatroids: *poset antimatroids* and *antimatroids* with the *path property*, respectively. On the other hand, Brânzei, Fragnelli, and Tijs (2002) introduced *peer group games* that were described by a rooted tree. These games are restricted games on poset antimatroids with the path property. This class of antimatroids are the *permission forest* and *permission tree structures* which are often encountered in the economic literature. Another model in which cooperation possibilities in a game are limited by some hierarchical structure on the set of players can be found in Faigle and Kern (1993) who consider feasible rankings of the players.

The paper is organized as follows. In Section 2 we recall some preliminaries on antimatroids. Section 3 is devoted to introduce cooperative games and the games called *interior operator games*. Throughout Sections 4 and 5 we deals with formulas for the Shapley, Banzhaf and Tijs values for interior operator games.

2. Antimatroids

Antimatroids were introduced by Dilworth (1940) as particular examples of semimodular lattices. Several authors have obtained the same concept by abstracting combinatorial properties. Edelman (1980) showed a crucial property of closures induced by convex geometries, a dual concept of antimatroids. A systematic study of these structures was started by Edelman and Jamison (1985), emphasizing the combinatorial abstraction of convexity. Jiménez-Losada (1998) introduced cooperative games on feasible coalitions given by antimatroids. In this section we introduce some basic concepts of antimatroid theory. The reader can use (Korte, Lóvasz, and Schrader, 1991) for more details.

Definition 1. An antimatroid $\mathcal{A}$ on N is a family of subsets of 2^N, satisfying

A0. $\emptyset \in \mathcal{A}$.
A1. (Accessibility) If $S \in \mathcal{A}$, $S \neq \emptyset$, there exists $e \in S$ such that $S \setminus \{e\} \in \mathcal{A}$.
A2. (Closed under union) If $S, T \in \mathcal{A}$ then $S \cup T \in \mathcal{A}$.

The definition of antimatroid implies the following *augmentation property*: if $S, T \in \mathcal{A}$ with $|T| > |S|$ then there exists $e \in T \setminus S$ such that $S \cup \{e\} \in \mathcal{A}$. We name *feasibles* the sets in $\mathcal{A}$ and we will consider only *normal* antimatroids, i.e., for all $e \in N$ there exists $S \in \mathcal{A}$ such that $e \in S$. In particular, this implies that $N \in \mathcal{A}$. Elements $e \in N$ such that $\{e\} \in \mathcal{A}$ are called *atoms* and the set of atoms in $\mathcal{A}$ is $a(\mathcal{A})$. For those $e \in N$ that satisfy $N \setminus \{e\} \in \mathcal{A}$ we will use *coatoms* (in fact, $N \setminus \{e\}$ is the coatom), and the coatoms form the set $ca(\mathcal{A})$. An antimatroid $(N, \mathcal{A})$ is *coatomic* if $ca(\mathcal{A}) = N$.

Let $(N, \mathcal{A})$ be an antimatroid. This set family allows us to define the interior operator $int : 2^N \to \mathcal{A}$, given by

$$int(S) = \bigcup_{\{T \in \mathcal{A}: T \subseteq S\}} T.$$

By A2, the interior of a set S is the unique maximal feasible set that the elements of S can form among them (the basis of the set S). The following properties are verified for the interior operator:

I1. $int(S) \subseteq S$, and $int(S) = S$ if and only if S is feasible,

I2. If $S \subseteq T$ then $int(S) \subseteq int(T)$.

Let $(N, \mathcal{A})$ be an antimatroid. An element e of a subset $S \in \mathcal{A}$ is an *endpoint* of S if $S \setminus \{e\} \in \mathcal{A}$. A feasible set S is a *path* if it has only one endpoint, if e is this unique endpoint we use *e-path*. Equivalently, an e-path is a minimal feasible set containing e. It is known any set is feasible if and only if is the union of paths. The set of e-paths in $(N, \mathcal{A})$ is denoted by $A(e)$.

A *poset antimatroid* is an antimatroid which is closed under intersection. An ideal of a poset $P = (N, \leq)$ is a subset $I \subseteq N$ such that $e \in I, \quad e' \leq e$ implies $e' \in I$. A poset antimatroid coincides with the set of the ideals of a poset. Moreover, Goecke, Korte, and Lovász [1986, Theorem 2.1] obtain the following characterization:

PA. An antimatroid $(N, \mathcal{A})$ is a poset antimatroid if and only if $|A(e)| = 1$ for all $e \in N$.

Example 1. Let N be a finite set, and $C \subseteq N$, $C \neq \emptyset$. The family

$$\mathcal{A} = \{S \subseteq N : S \subseteq C \text{ or } C \subseteq S\}$$

is a poset antimatroid. The only e-path for every $e \in N$ is

$$\begin{cases} \{e\} & \text{if } e \in C, \\ C \cup \{e\} & \text{if } e \notin C. \end{cases}$$

Example 2. Let N be a finite set, and $I \subset N$, $|I| \geq 2$. The family

$$\mathcal{A} = \{S \subseteq N : S \cap I \neq \emptyset\} \cup \{\emptyset\}$$

is a coatomic antimatroid. Since $\{e, e'\}$ is an e-path for all $e' \in I$ if $e \notin I$, it is not a poset antimatroid.

A *convex geometry* is a set system $(N, \mathcal{L})$ where $\emptyset \in \mathcal{L} \subseteq 2^N$ and the following holds: for any $S \in \mathcal{L}$, $S \neq \emptyset$ there exists an element $e \in S$ with $S \setminus \{e\} \in \mathcal{L}$, and for all $S, T \in \mathcal{L}$ we have $S \cap T \in \mathcal{L}$. Since the feasible sets of an antimatroid $(N, \mathcal{A})$ are closed under union, the family $\mathcal{L} = \{N \setminus S : S \in \mathcal{A}\}$ is a convex geometry. Notice that the poset antimatroids are the antimatroids which are also convex geometries. On a convex geometry can be defined the *closure* of a set $S \subseteq N$,

$$\overline{S} = \bigcap_{\{T \in \mathcal{L} : T \supseteq S\}} T.$$

By the intersection, the closure of a set S is the only smallest feasible set in $\mathcal{L}$ containing S. If $(N, \mathcal{A})$ is an antimatroid and $(N, \mathcal{L})$ is the associated convex geometry, there is a duality relationship between their interior and closure operators. This property is that $\overline{N \setminus S} = N \setminus int(S)$ for all $S \in \mathcal{A}$.

3. Interior operator games

A cooperative game is a pair (N, v), where $N \subseteq \mathbb{N}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ is a *characteristic function* on N satisfying $v(\emptyset) = 0$. The elements of $N = \{1, \ldots, n\}$ are called *players*, the subsets $S \subseteq N$ are *coalitions* and $v(S)$ is the maximal profit for the players in the coalition S. We denote the set of cooperative games with set of players N by $\Gamma(N)$. Now, we consider *restricted games*. To be exact, we analyze additive games restricted by the sets of an antimatroid $\mathcal{A}$. Our study can be applicable to improve the analysis of several types of classical games. We introduce some concepts of cooperative game theory which will be used in this work. A cooperative game (N, v) is called

- *monotonic* if $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$,
- *superadditive* if $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$,
- *convex* if $v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$ for all $S, T \subseteq N$.

Other solution concept that we will use is the core introduced by Gillies (1953) as

$$C(N, v) = \{x \in \mathbb{R}^N : x(N) = v(N),\ x(S) \geq v(S) \text{ for all } S \subseteq N\}.$$

Balanced games are games which nonempty core, and it is said that the game $v \in \Gamma(N)$ is *totally balanced* if the induced subgames $v_S \in \Gamma(S)$ are balanced for all $S \subseteq N$, $S \neq \emptyset$. Recall that the induced subgame v_S is defined as $v_S(T) = v(T)$ for all $T \subseteq S$.

Let $(N, \mathcal{A})$ be an antimatroid, $w \in \mathbb{R}^N_+$ a vector such that $w \geq 0$, and (N, w) the additive game defined by $w(S) = \sum_{e \in S} w_e$ for all nonempty $S \subseteq N$, $w(\emptyset) = 0$. In these conditions, we introduce a new class of cooperative games.

Definition 2. The interior operator game $(N, w_{\mathcal{A}})$ is the cooperative game $w_{\mathcal{A}} : 2^N \to \mathbb{R}$ defined by $w_{\mathcal{A}}(S) = w(int(S))$ for all $S \subseteq N$.

Note that if $S \in \mathcal{A}$ then $w_{\mathcal{A}}(S) = w(S)$. For any player $e \in N$ an e-path means that e depends on the players of the path. We introduce the sets

$$P^e = \bigcup_{S \in A(e)} S, \quad P_e = \bigcap_{S \in A(e)} S.$$

The feasible coalition P^e is the set of the players for which e has some dependency. The set P_e are the players that control player e. From now on, we use the notation $\overline{\{e\}}$ to indicate the closure of the individual coalition $\{e\}$ in the convex geometry $(N, \mathcal{L})$ dual to the antimatroid $(N, \mathcal{A})$, with $e \in N$. This closure set makes possible to represent the players who are controlled by him.

Proposition 1. Let $(N, \mathcal{A})$ be an antimatroid and $e \in N$. Then,

$$\overline{\{e\}} = \{e' \in N : e \in P_{e'}\}.$$

That is, $\overline{\{e\}}$ is the set of players controlled by e.

Proof. Recall that $\overline{\{e\}} = N \setminus int\,(N \setminus \{e\})$. Let $e' \in N$ such that $e \in P_{e'}$. Suppose that $e' \notin \overline{\{e\}}$, that is, $e' \in int\,(N \setminus \{e\})$. Then there exists $T \in A(e')$ such that $T \subseteq int\,(N \setminus \{e\})$. Hence $e \notin T$ and it is not possible because $e \in P_{e'}$.

To prove the another inclusion, we take $e' \in \overline{\{e\}}$ and $T \in A(e')$. If we suppose that $e \notin T$, so we have $T \subseteq N \setminus \{e\}$ and, also, $e' \in T \subseteq int\,(N \setminus \{e\})$, but it is a contradiction. □

Now, we can establish that the following classical games are interior operator games.

Example 3. Games on permission structures (van den Brink, 1997; Gilles, Owen, and van den Brink 1992). In a hierarchical situation among the players of a finite set N, a directed tree, we can consider the vector of the profits obtained from each player, $w \in \mathbb{R}^N_+$. The disjunctive case proposes that a coalition S is feasible if and only if for every $e \in S$ there exists some of its predecessors in S. The set of these feasible coalitions and the empty set define an antimatroid, which is characterized in Algaba et al. (2004) as the antimatroid with the denominated path property. The conjunctive case supposes that a coalition S is feasible if and only if for every $e \in S$ all its predecessors belong to S. In Algaba et al. (2004) the authors also characterized these set systems as poset antimatroids. Then, in both cases, the games considered in Gilles, Owen, and van den Brink (1992) and van den Brink (1997), when we take the vector w, are interior operator games.

Example 4. Peer group games (Brânzei, Fragnelli, and Tijs, 2002). Given a finite set of players N and a rooted tree T among them (the root is denoted by 1), the peer group situation is denoted by (N, P, w) where P is the family of the peer groups $[1, e]$ in T for all $e \in N$ and $w \in \mathbb{R}^N_+$. The peer group game is the cooperative game (N, v)

described by

$$v(S) = \sum_{\{e \in S : [1,e] \subseteq S\}} w_e,$$

which coincides with the interior operator game $(N, w_{\mathcal{A}})$ for the poset antimatroid $\mathcal{A}$ such that its paths are the elements of P. We are considering that $[1, e]$ is the set of predecessor players of e and successor of 1 in T.

Example 5. Big boss games (Muto et al. 1987). A big boss game is a cooperative game (N, v) where a player $1 \in N$, called the big boss, and v satisfy

- $v \geq 0$, and $v(N) \geq v(N \setminus \{e\})$ for all $e \in N$,
- $v(S) = 0$ if $1 \notin S$,
- $v(N) - v(N \setminus T) \geq \sum_{e \in T} [v(N) - v(N \setminus \{e\})]$ if $1 \notin T$.

Thus, every interior operator game $(N, w_{\mathcal{A}})$ where $\mathcal{A}$ is the poset antimatroid

$$\mathcal{A} = \{S \subseteq N : 1 \in S\} \cup \{\emptyset\},$$

is a big boss game (see Example 1 taking $C = \{1\}$).

Example 6. Clan games (Muto, Poos, Potters and Tijs, 1989). Clan games are a generalization of big boss games. In this case, the cooperative game (N, v) has a coalition $\emptyset \neq C \subseteq N$, called the clan, with the same properties as the big boss (replacing to $\{1\}$ by C). Then, if v is a vector $w \in \mathbb{R}_+^N$ we can consider C like a player and resolve the big boss interior operator game. Finally, we distribute the payment of C among its players, giving them the same worth. We propose here modified clan games. In this case we admit as feasibles the coalitions contained in the clan set C. Thus, the interior operator games $(N, w_{\mathcal{A}})$ where $\mathcal{A}$ is the poset antimatroid

$$\mathcal{A} = \{S \subseteq N : S \subseteq C \text{ or } C \subseteq S\},$$

(see Example 1) allow to study these games when we take a vector as worths.

Example 7. Information market games (Muto, Potters, and Tijs, 1986). Let N a finite set of firms and $w \in \mathbb{R}_+^N$ the vector of the possible profits for each firm if they sell their products in a market. An information market game assumes the existence of a subset of firms $I \subseteq N$, $|I| \geq 2$, with certain necessary information about the market. Furthermore, if a coalition does not contain some informed firm then it can not sell in the market. This game is the interior operator game $(N, w_{\mathcal{A}})$ with the antimatroid (see Example 2)

$$\mathcal{A} = \{S \subseteq N : S \cap I \neq \emptyset\} \cup \{\emptyset\}.$$

Example 8. We consider a network defined by a source r, several customers $\{1, \ldots, n\}$ and a digraph G rooted in r. In this case, the set of players is $N = \{r, 1, \ldots, n\}$. The connection with the source produces a profit for each player, w_e, and now we want to allocate the total benefit among the players. We take as feasible coalitions all the directed path since the source in G and their unions, this family $\mathcal{A}$ of sets, form the line-search antimatroid in the graph G (see Example 2.10 in Korte, Lóvasz, and Schrader (1991)). Then, this situation can be studied with the interior operator game $(N, w_{\mathcal{A}})$.

Proposition 2. An interior operator game $(N, w_{\mathcal{A}})$ is monotonic and superadditive.

Proof. Let $(N, w_{\mathcal{A}})$ an interior operator game. Let us suppose that $S \subseteq T \subseteq 2^N$. Because $int(S) \subseteq int(T)$ we have $w(int(S)) \leq w(int(T))$ and so $w_{\mathcal{A}}$ is a monotonic game.

In order to prove that $w_{\mathcal{A}}$ is superadditive, we consider now $S, T \subseteq 2^N$ satisfying $S \cap T = \emptyset$. Then $int(S) \cup int(T) \subseteq int(S \cup T)$ and

$$\begin{aligned} w_{\mathcal{A}}(S \cup T) &= w\,(int\,(S \cup T)) \geq w\,(int\,(S) \cup int(T)) \\ &= w\,(int(S)) + w\,(int\,(T)) \\ &= w_{\mathcal{A}}(S) + w_{\mathcal{A}}(T)\,, \end{aligned}$$

since $int(S) \cap int(T) = \emptyset$. □

Remark 1. In the previous section, we said that the dual structure of an antimatroid is a convex geometry. Then, if $(N, \mathcal{L})$ is a convex geometry and $w \in \mathbb{R}_+^N$ we can define a closure game as the cooperative game $(N, w_{\mathcal{L}})$ with

$$w_{\mathcal{L}}(S) = w(\overline{S}),$$

for all $S \subseteq N$. We can observe that the dual game of this closure game is the interior operator game $(N, w_{\mathcal{A}})$, where $\mathcal{A}$ is the dual antimatroid of this convex geometry,

$$\begin{aligned} (w_{\mathcal{L}})^*(S) &= w_{\mathcal{L}}(N) - w_{\mathcal{L}}(N \setminus S) = w(N) - w(\overline{N \setminus S}) \\ &= w(N \setminus \overline{N \setminus S}) = w(int(S)) = w_{\mathcal{A}}(S). \end{aligned}$$

Hence, closure games are subadditive (we can use them as cost games) and we would study this family of games through interior operator games. This idea can be applied to resolve cost allocation games. A cost allocation game is defined by Meggido (1978). He considered a tree rooted in r where the other vertices are the players and the cost of each edge. The worth of a coalition is the sum of the worths of the unique paths to the root of its players. Then, we associate to a player e the cost of the unique edge going in e and we have a vector in $\mathbb{R}_+^N$. The feasible coalitions are the same of the Example 4, in this case a poset antimatroid (convex geometry in particular) because we have a tree. The game described by Meggido is the closure game on this structure with the defined vector.

Theorem 3. An interior operator game $(N, w_{\mathcal{A}})$ is a convex game if and only if $w_e = 0$ for all $e \in N$ such that $|A(e)| \geq 2$.

Proof. Let us suppose that the interior operator game $(N, w_{\mathcal{A}})$ is convex. Let $e \in N$ such that $|A(e)| \geq 2$ and take $S, T \in A(e)$, $S \neq T$. Then, as $S \cup T \in \mathcal{A}$, we have

$$\begin{aligned} w_{\mathcal{A}}(S \cup T) + w_{\mathcal{A}}(S \cap T) &= w(S \cup T) + w(int(S \cap T)) \\ &= w(S) + w(T) - w(S \cap T) + w(int(S \cap T)) \\ &= w(S) + w(T) - w((S \cap T) \setminus int(S \cap T)) \\ &= w_{\mathcal{A}}(S) + w_{\mathcal{A}}(T) - w((S \cap T) \setminus int(S \cap T)) \\ &\geq w_{\mathcal{A}}(S) + w_{\mathcal{A}}(T) \end{aligned}$$

where the last inequality is true because $w_{\mathcal{A}}$ is convex. Therefore, we obtain $w((S \cap T) \setminus int(S \cap T)) = 0$ and then $w_e = 0$ because $e \in S \cap T$ and $e \notin int(S \cap T)$ since $S, T \in A(e)$.

Now we will prove the reverse implication. Let $S, T \subseteq N$. It is verified that $int(S) \cup int(T) \subseteq int(S \cup T)$, $int(S \cap T) \subseteq int(S) \cap int(T)$ and so

$$\begin{aligned} & w_{\mathcal{A}}(S \cup T) + w_{\mathcal{A}}(S \cap T) \\ &= w(int(S \cup T)) + w(int(S \cap T)) \\ &\geq w(int(S) \cup int(T)) + w(int(S \cap T)) \\ &= w(int(S)) + w(int(T)) - w(int(S) \cap int(T)) + w(int(S \cap T)) \\ &= w(int(S)) + w(int(T)) - w((int(S) \cap int(T)) \setminus int(S \cap T)) \\ &= w_{\mathcal{A}}(S) + w_{\mathcal{A}}(T) - w((int(S) \cap int(T)) \setminus int(S \cap T)). \end{aligned}$$

If $e' \in [(int(S) \cap int(T)) \setminus int(S \cap T)]$ there exist e-paths, R_1 and R_2, such that $R_1 \subseteq int(S)$, $R_2 \subseteq int(T)$, but it is not possible that $R_1 = R_2$ because in this case $R_1 \subseteq (int(S) \cap int(T)) \subseteq S \cap T$ and then e' must belong to $int(S \cap T)$. So, for all $e' \in [(int(S) \cap int(T)) \setminus int(S \cap T)]$ we have $\left|A(e')\right| \geq 2$, and by assumption $w_{e'} = 0$. We conclude

$$w((int(S) \cap int(T)) \setminus int(S \cap T)) = 0,$$

and thus $w_{\mathcal{A}}$ is convex. □

Corollary 4. Let $(N, \mathcal{A})$ be a fixed antimatroid. Every interior operator game $(N, w_{\mathcal{A}})$ is convex if and only if $(N, \mathcal{A})$ is a poset antimatroid.

Proof. Follows from the above theorem and property PA. □

The interior operator games defined by conjunctive approach of permission games, peer group games, big-boss games or modified clan games are convex for all $w \in \mathbb{R}^N_+$.

Balanced games play an important role in game theory because they have reasonable allocations of the profits. We will prove that the interior operator games are balanced.

Theorem 5. An interior operator game $(N, w_{\mathcal{A}})$ is totally balanced.

Proof. We will prove that the core of the subgame $(S, (w_{\mathcal{A}})_S)$ is nonempty, for each $S \subseteq N$. Let us consider the vector $x \in \mathbb{R}^S_+$ which components are

$$x_e = \begin{cases} w_e & \text{if } e \in int(S), \\ 0 & \text{if } e \in S \setminus int(S). \end{cases}$$

So, we have

$$x(S) = w(int(S)) = (w_{\mathcal{A}})_S\,(S),$$

and for $T \subset S$,

$$x(T) = w(T \cap int(S)) \geq w(int(T)) = (w_{\mathcal{A}})_S\,(T).$$

Hence, $(N, w_{\mathcal{A}})$ is totally balanced. □

In view of the above theorem, any interior operator game $(N, w_{\mathcal{A}})$ is balanced and the vector w always is in $C\,(N, w_{\mathcal{A}})$. In particular, it is possible to determine the core using only the feasible coalitions,

$$C\,(N, w_{\mathcal{A}}) = \{x \in \mathbb{R}^N_+ : x(N) = w(N), x(S) \geq w(S) \text{ for all } S \in \mathcal{A}\}.$$

Because if $x \in \mathbb{R}^N_+$ satisfies $x(S) \geq w(S)$, $\forall S \in \mathcal{A}$, then for every non feasible $T \subset N$ we have $x(T) \geq x(int(T)) \geq w(int(T)) = w_{\mathcal{A}}(T)$.

4. The Shapley and Banzhaf values

A *value* for a cooperative game (N, v) is a solution concept that assigns to each game just one payment for each player. That is, a function $\Psi : \Gamma\,(N) \to \mathbb{R}^N$ where $\Psi\,(v)$ is the allocation vector that corresponds to the game (N, v). For a cooperative game $v \in \Gamma\,(N)$, the Shapley $Sh(N, v)$ and Banzhaf $Ba(N, v)$ values are given by the following formulas:

$$Sh_i(N, v) = \sum_{\{S \subseteq N : i \in S\}} \frac{\Delta_v(S)}{|S|}, \quad Ba_i(N, v) = \sum_{\{S \subseteq N : i \in S\}} \frac{\Delta_v(S)}{2^{|S|-1}},$$

for all $i \in N$, where $\Delta_v(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|}\, v(T)$ is the *Harsanyi dividend* (see Harsanyi and Selten, 1988) of the nonempty coalition $S \subseteq N$. In this section we calculate the Harsanyi dividends of the interior operator games and using them we obtain the Shapley and the Banzhaf values. First, we need to introduce a new concept about antimatroids.

Definition 3. Let $(N, \mathcal{A})$ be an antimatroid. A feasible set $S \in \mathcal{A}$ is a block if there exists $e \in S$ such that S is union of e-paths. In this case, S is called e-block.

By definition, any path of an antimatroid is a block. For all the poset antimatroids the only blocks are the paths, and the only antimatroids with this condition are the poset antimatroids.

Example 9. Let us consider the antimatroid on $N = \{1, 2, 3, 4, 5\}$ whose paths are

$$A(1) = \{\{1\}\}, \quad A(2) = \{\{2\}\}, \quad A(3) = \{\{1,3\}, \{2,3\}\}$$
$$A(4) = \{\{2,3,4\}, \{1,3,4,5\}\}, \quad A(5) = \{\{1,3,5\}, \{2,3,4,5\}\}.$$

Then, $\{1, 2, 3\}$ is a 3-block and N is a 4-block and a 5-block.

From the above example we observe that the element e which defines a block like union of its paths is not unique necessarily. Then we introduce the next notation for a block S in an antimatroid $(N, \mathcal{A})$,

$$Q_S = \{e \in S : S \text{ is an } e\text{-block}\}.$$

We will use the following lemma in the process of calculation of the Harsanyi dividends.

Lemma 6. Let N be a finite set and $R, S \in 2^N$ such that $R \subseteq S$. Then,

$$\sum_{T \in [R,S]} (-1)^{s-t} = \begin{cases} 0 & \text{if } R \neq S, \\ 1 & \text{if } R = S, \end{cases}$$

where $s = |S|$, $t = |T|$ and $[R, S] = \{T \subseteq N : R \subseteq T \subseteq S\}$.

Proof. Let $t \in [r, s]$ a fixed number. Because the number of sets $T \in [R, S]$ which have cardinality t is $\binom{s-r}{t-r}$, we have

$$\sum_{T \in [R,S]} (-1)^{s-t} = \sum_{t=r}^{s} \binom{s-r}{t-r} (-1)^{s-t} = \sum_{t=0}^{s-r} \binom{s-r}{t} (-1)^{s-r-t}$$
$$= (-1)^{s-r} \sum_{t=0}^{s-r} \binom{s-r}{t} (-1)^{t}.$$

Hence, if $r \neq s$,

$$\sum_{T \in [R,S]} (-1)^{s-t} = (-1)^{s-r} (1-1)^{s-r} = 0,$$

and otherwise, $\sum_{T \in [R,S]} (-1)^{s-t} = 1$. □

Although we are already prepared to justify how the dividends can be calculated in an interior operator game, we need to introduce the following notation. Given an antimatroid $(N, \mathcal{A})$ and an e-block S, we will denote by $n_i^{e,S}$ the number of different ways in those that S can express as union of i e-paths. We will denote $\Delta_w^{\mathcal{A}}(S)$ the Harsanyi dividend of the interior operator game $(N, w_{\mathcal{A}})$ for the coalition $S \subseteq N$.

Theorem 7. The Harsanyi dividends of an interior operator game $(N, w_{\mathcal{A}})$ are

$$\Delta_w^{\mathcal{A}}(S) = \begin{cases} \sum\limits_{e \in Q_S} \lambda_e^S w_e & \text{if } S \text{ is a block,} \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\lambda_e^S = \sum_{i=1}^{m} (-1)^{i-1} n_i^{e,S}$$

and m is the number of the e-paths contained in S.

Proof. By definition, the Harsanyi dividends are

$$\begin{aligned} \Delta_w^{\mathcal{A}}(S) &= \sum_{T \in [\emptyset,S]} (-1)^{|S|-|T|} w_{\mathcal{A}}(T) = \sum_{T \in [\emptyset,S]} (-1)^{|S|-|T|} w(int(T)) \\ &= \sum_{T \in [\emptyset,S]} (-1)^{|S|-|T|} \sum_{e \in int(T)} w_e \\ &= \sum_{e \in int(S)} \left(\sum_{\{T \in [\emptyset,S]: e \in int(T)\}} (-1)^{|S|-|T|} \right) w_e, \end{aligned}$$

since $e \in int(T)$ for some $T \in [\emptyset, S]$ if and only if $e \in int(S)$.

Given $T \in [\emptyset, S]$, we can observe that $e \in int(T)$ if and only if T contains some e-path. Since S is a block, let us consider all e-paths $R_1, \ldots, R_m$ contained in S. Then

$$\{T \in [\emptyset, S] : e \in int(T)\} = \{T \subseteq S : T \supseteq R_i \text{ for some } 1 \leq i \leq m\} = \bigcup_{i=1}^{m} [R_i, S].$$

If we denote λ_e^S the number

$$\lambda_e^S = \sum_{\{T \in [\emptyset,S]: e \in int(T)\}} (-1)^{|S|-|T|},$$

we claim that

$$\begin{aligned}
\lambda_e^S &= \sum_{T \in \bigcup_{i=1}^{i=m} [R_i, S]} (-1)^{|S|-|T|} \\
&= \sum_{i=1}^{m} \sum_{T \in [R_i, S]} (-1)^{|S|-|T|} - \sum_{i=1}^{m} \sum_{j=i+1}^{m} \sum_{T \in [R_i \cup R_j, S]} (-1)^{|S|-|T|} \\
&\quad + \sum_{i=1}^{m} \sum_{j=i+1}^{m} \sum_{k=j+1}^{m} \sum_{T \in [R_i \cup R_j \cup R_k, S]} (-1)^{|S|-|T|} \\
&\quad - \cdots + (-1)^{m-1} \sum_{T \in [\bigcup_{i=1}^{i=m} R_i, S]} (-1)^{|S|-|T|}.
\end{aligned}$$

We will prove the last equality as follow. Suppose that $T \in \bigcup_{i=1}^{i=m} [R_i, S]$ contains only p of the $e-$path $R_1, \dots, R_m$ contained in S. Then T appears p times in the first sum, $\binom{p}{2}$ times in second, $\binom{p}{3}$ in third and so forth until the sum p, that is the last one, in which T appears a single time, since T has p only of those e-paths. In total the number of times that T appears in the second member of the equality is one, since

$$\sum_{i=1}^{p} \binom{p}{i} (-1)^{i-1} = -\left[\left(\sum_{i=0}^{p} \binom{p}{i} (-1)^i \right) - 1 \right] = -[(1-1)^p - 1] = 1.$$

Note that the previous lemma implies that all the terms of the last expression are zero unless the union of the paths at issue be S. So, if $S \neq \bigcup_{i=1}^{i=m} R_i$ (that is, S is not an e-block), the coefficient $\lambda_e^S = 0$. Otherwise, if S is an e-block, we obtain

$$\lambda_e^S = \sum_{i=1}^{m} (-1)^{i-1} n_i^{e,S}.$$

Lastly, including these results in the initial formula, we obtain

$$\Delta_w^{\mathcal{A}}(S) = \sum_{e \in int(S)} \lambda_e^S w_e = \sum_{e \in Q_S} \lambda_e^S w_e,$$

and prove the assertion. □

Corollary 8. Let $(N, \mathcal{A})$ be an antimatroid and S an e-block, then

$$\lambda_e^S = 1 - \sum_{\{T \in (\emptyset, S): T \text{ is an } e\text{-block}\}} \lambda_e^T.$$

Proof. We will use the recursive formula of the Harsanyi dividends (see Harsanyi and Selten, 1988), and $w \in \mathbb{R}_+^N$ such that $w_{e'} = 0$ for all $e' \neq e$. So,

$$\begin{aligned}\Delta_w^{\mathcal{A}}(S) &= w_{\mathcal{A}}(S) - \sum_{T\in[\emptyset,S)} \Delta_w^{\mathcal{A}}(T)\\ &= w\,(int(S)) - \sum_{T\in(\emptyset,S)} \sum_{e'\in Q_T} \lambda_{e'}^T w_{e'}\\ &= w_e - \sum_{\{T\in(\emptyset,S):T \text{ is an } e'\text{-block}\}} \lambda_{e'}^T w_{e'}\\ &= \left(1 - \sum_{\{T\in(\emptyset,S):T \text{ is an } e\text{-block}\}} \lambda_e^T\right) w_e,\end{aligned}$$

and, by above theorem, $\Delta_w^{\mathcal{A}}(S) = \sum_{e\in Q_S} \lambda_e^S w_e$. Thus, taking $w_e = 1$ for instance, we obtain the result. □

Note that it has also been obtained in the previous proof that if S is not an e-block but $e \in int(S)$, then

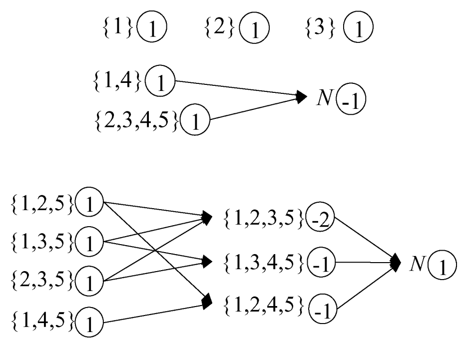
$$\sum_{\{T\in(\emptyset,S):T \text{is an } e\text{-block}\}} \lambda_e^T = 1.$$

This corollary gives a recursive formula to obtain the dividends by an easier method.

Example 10. Let $w \in \mathbb{R}_+^N$. We consider the interior operator game $(N, w_{\mathcal{A}})$ where $N = \{1, 2, 3, 4, 5\}$ and $(N, \mathcal{A})$ is the antimatroid with the following paths:

$$A(1) = \{\{1\}\},\quad A(2) = \{\{2\}\},\quad A(3) = \{\{3\}\},\quad A(4) = \{\{1,4\}, \{2,3,4,5\}\},$$
$$A(5) = \{\{1,2,5\}, \{1,3,5\}, \{2,3,5\}, \{1,4,5\}\}.$$

To obtain the dividends we determine the coefficients λ_e^S. This coefficient is 1 if the block is a path. We mark the coefficient of each block with a circle.



S	$\Delta_w^{\mathcal{A}}(S)$	S	$\Delta_w^{\mathcal{A}}(S)$	S	$\Delta_w^{\mathcal{A}}(S)$
$\{1\}$	w_1	$\{2, 3, 4, 5\}$	w_4	$\{1, 4, 5\}$	w_5
$\{2\}$	w_2	$\{1, 2, 5\}$	w_5	$\{1, 2, 3, 5\}$	$-2w_5$
$\{3\}$	w_3	$\{1, 3, 5\}$	w_5	$\{1, 3, 4, 5\}$	$-w_5$
$\{1, 4\}$	w_4	$\{2, 3, 5\}$	w_5	$\{1, 2, 4, 5\}$	$-w_5$
		N	$-w_4 + w_5$		

Corollary 9. If $(N, w_{\mathcal{A}})$ is an interior operator game with $(N, \mathcal{A})$ a poset antimatroid then

$$\Delta_w^{\mathcal{A}}(S) = \begin{cases} w_e & \text{if } S \text{ is } e\text{-path,} \\ 0 & \text{otherwise.} \end{cases}$$

Now, we are going to formulate the Shapley and the Banzhaf values.

Theorem 10. If $(N, w_{\mathcal{A}})$ is an interior operator game, the Shapley and the Banzhaf values are, for every $e \in N$,

$$Sh_e(N, w_{\mathcal{A}}) = \sum_{\{e' \in N : e \in P^{e'}\}} \left(\sum_{\{S\ e'\text{-block}: e \in S\}} \frac{\lambda_{e'}^S}{|S|} \right) w_{e'},$$

$$Ba_e(N, w_{\mathcal{A}}) = \sum_{\{e' \in N : e \in P^{e'}\}} \left(\sum_{\{S\ e'\text{-block}: e \in S\}} \frac{\lambda_{e'}^S}{2^{|S|-1}} \right) w_{e'}.$$

Proof. The Shapley value is given by

$$Sh_e(N, w_{\mathcal{A}}) = \sum_{\{S \subseteq N. e \in S\}} \frac{\Delta_w^{\mathcal{A}}(S)}{|S|} = \sum_{\{S \text{ block}: e \in S\}} \frac{1}{|S|} \sum_{e' \in Q_S} \lambda_{e'}^S w_{e'}$$
$$= \sum_{\{e' \in N : e \in P^{e'}\}} \left(\sum_{\{S\ e'\text{-block}: e \in S\}} \frac{\lambda_{e'}^S}{|S|} \right) w_{e'},$$

where we have used that if S is an e'-block with $e \in S$ then (like an e'-path such that contains e exists), $e \in P^{e'}$; and if $e' \in N$ with $e \in P^{e'}$ then e belongs to some e'-path which is an e'-block.

The Banzhaf value is obtained in the same form. □

We observe in the above theorem that a player e is able to keep his profit (that is, $Sh_e(N, w_{\mathcal{A}})$ and $Ba_e(N, w_{\mathcal{A}})$ are greater than w_e) if and only if it is an atom. So, for each player $e \in N$ of an interior operator game $(N, w_{\mathcal{A}})$, we interpret the set P^e as the players on whom e depends.

Corollary 11. Given an interior operator game $(N, w_{\mathcal{A}})$ such that $(N, \mathcal{A})$ is a poset antimatroid, the Shapley and the Banzhaf values are, for every $e \in N$,

$$Sh_e(N, w_{\mathcal{A}}) = \sum_{\{e' \in N : e \in P^{e'}\}} \frac{w_{e'}}{|P^{e'}|}, \quad Ba_e(N, w_{\mathcal{A}}) = \sum_{\{e' \in N : e \in P^{e'}\}} \frac{w_{e'}}{2^{|P^{e'}|-1}},$$

where $P^{e'}$ is the unique e'-path.

Example 11. Let $(N, w_{\mathcal{A}})$ be an interior operator game defined by a clan game with clan coalition $C \subsetneq N$. Observe that if $e \in C$ the unique e-path is $\{e\}$ and if $e \notin C$ then $P^e = C \cup \{e\}$. Thus, applying the above corollary, it follows that

$$Sh_e(N, w_{\mathcal{A}}) = \begin{cases} w_e + \dfrac{w(N \setminus C)}{|C| + 1} & \text{if } e \in C, \\ \dfrac{w_e}{|C| + 1} & \text{if } e \notin C. \end{cases}$$

Example 12. Let $(N, w_{\mathcal{A}})$ be an interior operator game defined by an information market game with informed coalition $I \subseteq N$, $|I| \geq 2$. The Shapley value, calculated using the last theorem, is

$$Sh_e(N, w_{\mathcal{A}}) = \begin{cases} w_e + w(N \setminus I) \displaystyle\sum_{k=0}^{|I|-1} \binom{|I| - 1}{k} \frac{1}{k+2} & \text{if } e \in I, \\ w_e \displaystyle\sum_{k=1}^{|I|} \binom{|I|}{k} \frac{1}{k+1} & \text{if } e \notin I. \end{cases}$$

Example 13. In Example 10 we observe that the Shapley value assigns to the atoms the following values:

$$\begin{aligned} Sh_1(N, w_{\mathcal{A}}) &= w_1 + \tfrac{3}{10} w_4 + \tfrac{1}{5} w_5, \\ Sh_2(N, w_{\mathcal{A}}) &= w_2 + \tfrac{1}{20} w_4 + \tfrac{7}{60} w_5, \\ Sh_3(N, w_{\mathcal{A}}) &= w_3 + \tfrac{1}{20} w_4 + \tfrac{7}{60} w_5. \end{aligned}$$

The non-atom players obtain the following payoffs

$$\begin{aligned} Sh_4(N, w_{\mathcal{A}}) &= \tfrac{11}{20} w_4 + \tfrac{1}{30} w_5, \\ Sh_5(N, w_{\mathcal{A}}) &= \tfrac{1}{20} w_4 + \tfrac{8}{15} w_5. \end{aligned}$$

5. The Tijs value

For a game $v \in \Gamma(N)$, we define the upper vector $M^{\tau,v} = (M_i^{\tau,v})_{i\in N}$, with components $M_i^{\tau,v} = v\,(N) - v\,(N \setminus \{i\})$, and the lower vector $m^{\tau,v}(v) = (m_i^{\tau,v})_{i\in N}$,

$$m_i^{\tau,v} = \max_{\{S\subseteq N: i\in S\}} [v(S) - M^{\tau,v}\,(S \setminus \{i\})]\,,\, i \in N,$$

where $M^{\tau,v}\,(S \setminus \{i\}) = \sum_{j\in S\setminus\{i\}} M_j^{\tau,v}$. The *Tijs value* or τ-value is defined only on the set $QB\,(N)$ of *quasibalanced games*, i.e., games that satisfy

$$m^{\tau,v} \le M^{\tau,v} \quad \text{and} \quad \sum_{i\in N} m_i^{\tau,v} \le v\,(N) \le \sum_{i\in N} M_i^{\tau,v}.$$

The Tijs value for a game $v \in \Gamma\,(N)$ is the vector

$$\tau(N, v) = m^{\tau,v} + \alpha\,(M^{\tau,v} - m^{\tau,v})$$

with $\alpha \in \mathbb{R}$ such that $\sum_{i\in N} \tau_i(N, v) = v(N)$. First we need to prove that our games are quasi-balanced. Since every balanced game is quasibalanced (see Driessen, 1988, Proposition 3.1 of Chapter III), Theorem 5 allows to prove the claim.

Theorem 12. Let $(N, w_{\mathcal{A}})$ be an interior operator game. Then, for every $e \in N$, the following hold:

1. $M_e^{\tau,w_{\mathcal{A}}} = w(\overline{\{e\}})$,
2. $m_e^{\tau,w_{\mathcal{A}}} = \max\{0, w_e - c_e\}$, where $c_e = \min_{T\in A(e)} \sum_{e'\in T\setminus\{e\}} [w(\overline{\{e'\}}) - w_{e'}]$.

Proof. We will prove both sentences, let $e \in N$:

1. By definition,

$$\begin{aligned} M_e^{\tau,w_{\mathcal{A}}} &= w_{\mathcal{A}}(N) - w_{\mathcal{A}}(N \setminus \{e\}) = w(N) - w\,(int\,(N \setminus \{e\})) \\ &= w(N \setminus int\,(N \setminus \{e\})) = w(\overline{\{e\}}). \end{aligned}$$

2. Recall that

$$m_e^{\tau,w_{\mathcal{A}}} = \max_{\{S\subseteq N: e\in S\}} [w_{\mathcal{A}}(S) - M^{\tau,w_{\mathcal{A}}}\,(S \setminus \{e\})].$$

First, we prove that $m_e^{\tau,w_{\mathcal{A}}} \ge \max\{0, w_e - c_e\}$. Taking $S = \{e\}$, we have

$$w(int(S)) - \sum_{e'\in S\setminus\{e\}} w(\overline{\{e'\}}) = w(\underline{S}) \ge 0.$$

Also, if $T \in A(e)$ is such that $c_e = \sum_{e' \in T \setminus \{e\}} [w(\overline{\{e'\}}) - w_{e'}]$, then

$$\begin{aligned} w(int(T)) - \sum_{e' \in T \setminus \{e\}} w(\overline{\{e'\}}) &= w(T) - \sum_{e' \in T \setminus \{e\}} w(\overline{\{e'\}}) \\ &= w_e - \sum_{e' \in T \setminus \{e\}} [w(\overline{\{e'\}}) - w_{e'}] \\ &= w_e - c_e. \end{aligned}$$

Hence, $m_e^{\tau, w_{\mathcal{A}}} \geq \max\{0, w_e - c_e\}$.

On the other hand, if $S \subseteq N$ we must have

$$w(int(S)) - \sum_{e' \in S \setminus \{e\}} w(\overline{\{e'\}}) \leq w(int(S)) - \sum_{e' \in int(S) \setminus \{e\}} w(\overline{\{e'\}}).$$

If $e \notin int(S)$,

$$w(int(S)) - \sum_{e' \in int(S) \setminus \{e\}} w(\overline{\{e'\}}) = w(int(S)) - \sum_{e' \in int(S)} w(\overline{\{e'\}}) \leq 0,$$

but if $e \in int(S)$, so there exists $T \in A(e)$ verifying $T \subseteq int(S)$. Thus,

$$\begin{aligned} & w(int(S)) - \sum_{e' \in int(S) \setminus \{e\}} w(\overline{\{e'\}}) = w_e + w(int(S) \setminus \{e\}) - \sum_{e' \in int(S) \setminus \{e\}} w(\overline{\{e'\}}) \\ & = w_e - \sum_{e' \in int(S) \setminus \{e\}} [w(\overline{\{e'\}}) - w_{e'}] \\ & \leq w_e - \sum_{e' \in T \setminus \{e\}} [w(\overline{\{e'\}}) - w_{e'}] \leq w_e - c_e. \end{aligned}$$

Therefore, $m_e^{\tau, w_{\mathcal{A}}} \leq \max\{0, w_e - c_e\}$. □

We note that the maximum expected payoff for a player is the sum of the profits he controls. The lower vector is determined only checking the paths of the player. In some cases, this calculation is easier.

Corollary 13. An interior operator game $(N, w_{\mathcal{A}})$, where $(N, \mathcal{A})$ is a coatomic antimatroid, verifies $C(N, w_{\mathcal{A}}) = \{\tau(N, w_{\mathcal{A}})\} = \{w\}$.

Proof. We will use the Proposition 3.4 of the Chapter III in Driessen (1988). It is sufficient that $M^{\tau, w_{\mathcal{A}}}(S) \geq w_{\mathcal{A}}(S)$ for all $S \subseteq N$ and $M^{\tau, w_{\mathcal{A}}}(N) = w_{\mathcal{A}}(N)$ to conclude that $C(w_{\mathcal{A}}) = \{\tau(N, w_{\mathcal{A}})\}$. As $\mathcal{A}$ is coatomic, then $\overline{\{e\}} = \{e\}$ for every $e \in N$. Hence, if $S \subseteq N$

$$M^{\tau, w_{\mathcal{A}}}(S) = \sum_{e \in S} w(\overline{\{e\}}) = w(S) \geq w(int(S)) = w_{\mathcal{A}}(S)$$

because $w \in \mathbb{R}_{+}^{N}$, and

$$M^{\tau, w_{\mathcal{A}}}(N) = \sum_{e \in N} w(\overline{\{e\}}) = w(N) = w_{\mathcal{A}}(N).$$

Then $C(N, w_{\mathcal{A}}) = \{\tau(N, w_{\mathcal{A}})\}$ and since $w \in C(N, w_{\mathcal{A}})$ we get the claim. □

Example 14. Let $(N, w_{\mathcal{A}})$ be an interior operator game defined by an information market game with informed coalition $I \subseteq N$, $|I| \geq 2$. Then $\tau(N, w_{\mathcal{A}}) = w$ using Corollary 13.

Corollary 14. Let $(N, w_{\mathcal{A}})$ be an interior operator game, and $e \in N$. Then,

1. If $e \in a(\mathcal{A})$, $m_e^{\tau, w_{\mathcal{A}}} = w_e$,
2. If $|P_e| \geq 2$, $m_e^{\tau, w_{\mathcal{A}}} = 0$.

Proof. These are consequence of the above theorem.

1. When $e \in a(\mathcal{A})$, $c_e = 0$. Hence, $m_e^{\tau, w_{\mathcal{A}}} = w_e$.
2. If $|P_e| \geq 2$ then $P_e \neq \{e\}$ and we have $e \notin a(\mathcal{A})$. Thus for all $T \in A(e)$, there exists $e^* \in T$, $e^* \neq e$.

If $T \in A(e)$ is such that $c_e = \sum_{e' \in T \setminus \{e\}} [w(\overline{\{e'\}}) - w_e]$, it is obtained $w_e - c_e \leq 0$ because

$$w_e - c_e = -\sum_{e' \in T \setminus \{e, e^*\}} [w(\overline{\{e'\}}) - w_{e'}] - [w(\overline{\{e^*\}}) - w_{e^*} - w_e],$$

where we have used that $e^* \in P_e$ and the last proposition to obtain $e \in \overline{\{e^*\}}$. We conclude that $m_e^{\tau, w_{\mathcal{A}}} = 0$. □

Corollary 15. An interior operator game $(N, w_{\mathcal{A}})$, where $(N, \mathcal{A})$ is an antimatroid satisfying $|P_e| \geq 2$ for all $e \notin a(\mathcal{A})$, verifies

$$\tau_e(N, w_{\mathcal{A}}) = p_{\mathcal{A}} w\left(\overline{\{e\}}\right) + (1 - p_{\mathcal{A}}) w(int(e)),$$

with

$$p_{\mathcal{A}} = \frac{w(N) - w(a(\mathcal{A}))}{w(\mathcal{A}) - w(a(\mathcal{A}))}, \quad w(\mathcal{A}) = \sum_{e \in N} w(\overline{\{e\}}).$$

Proof. By the above corolary the lower vector is

$$m_e^{\tau, w_{\mathcal{A}}} = \begin{cases} w_e & \text{if } e \in a(\mathcal{A}), \\ 0 & \text{if } e \notin a(\mathcal{A}). \end{cases}$$

Then the Tijs value is a vector in this convex form

$$\tau_e(N, w_{\mathcal{A}}) = \begin{cases} (1-\alpha)w_e + \alpha w(\overline{\{e\}}) & \text{if } e \in a(\mathcal{A}), \\ \alpha w(\overline{\{e\}}) & \text{if } e \notin a(\mathcal{A}). \end{cases}$$
$$= \alpha w(\overline{\{e\}}) + (1-\alpha)w(int(e)).$$

The value $\tau(N, w_{\mathcal{A}})$ is an efficient vector, therefore

$$\sum_{e \in N} \tau_e(N, w_{\mathcal{A}}) = (1-\alpha)w(a(\mathcal{A})) + \alpha w(\mathcal{A}) = w(N)$$

and the coefficient $\alpha = p_{\mathcal{A}}$. □

In the last antimatroids the Tijs value is a convex combination of the worths of the closure and the interior for a player. The coefficient $p_{\mathcal{A}}$ is the quotient obtained by dividing the maximum expected gap into the real gap of the atoms.

Example 15. Let $(N, w_{\mathcal{A}})$ an interior operator game defined by a clan game with clan coalition $C \subsetneq N$. In this case, $A(e) = C \cup \{e\}$ if $e \notin a(\mathcal{A})$. Then, by Corollary 15 the convex coefficient is

$$p_{\mathcal{A}} = \frac{1}{|C|+1},$$

and the Tijs value coincides with the Shapley value

$$\tau_e(N, w_{\mathcal{A}}) = \begin{cases} w_e + \dfrac{w(N \setminus C)}{1+|C|} & \text{if } e \in C, \\ \dfrac{w_e}{1+|C|} & \text{if } e \notin C. \end{cases}$$

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