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Harsanyi power solutions for games on union stable systems

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Abstract This paper analyzes Harsanyi power solutions for cooperative games in which partial cooperation is based on union stable systems. These structures contain as particular cases the widely studied communication graph games and permission structures, among others. In this context, we provide axiomatic characterizations of the Harsanyi power solutions which distribute the Harsanyi dividends proportional to weights determined by a power measure for union stable systems. Moreover, the well-known Myerson value is exactly the Harsanyi power solution for the equal power measure, and on a special subclass of union stable systems the position value coincides with the Harsanyi power solution obtained for the influence power measure.

Keywords Cooperative TU-game · Union stable system · Harsanyi dividend · Power measure · Harsanyi power solution · Myerson value · Position value

1 Introduction

In the classical model of cooperative games with transferable utility, it is generally assumed that there are no restrictions on cooperation. However, in practice, many situations require certain limitations on cooperation. Myerson (1977) used ideas from graph theory and studied how the outcome of a game depends on which players cooperate with each other. He

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assumed that players in a cooperative TU-game are also the nodes in an undirected (communication) graph such that only connected coalitions are feasible and can earn their worth. From a game and communication graph, he defined a restricted game where every coalition earns the sum of the worths of its components (i.e. maximally connected subsets). As solution, he proposed to apply the Shapley value to this restricted game, a solution later called the *Myerson value*. This line of research was then continued by Owen (1986), Borm et al. (1992), and Potters and Reijnierse (1995). Recently, van den Brink et al. (2011a) have generalized the Myerson value by defining the class of *Harsanyi power solutions* for communication graph games that are obtained by applying Harsanyi solutions, introduced and studied as solutions for TU-games in Vasil'ev (1982, 2003) and Derks et al. (2000, 2010), to the restricted game. Whereas the Myerson value allocates the Harsanyi dividends (Harsanyi 1959) of the restricted game equally among the players in the corresponding coalition, in these Harsanyi power solutions the dividends in the restricted game can be distributed among the players according to any positive network power measure.

However, as Myerson himself pointed out partial cooperation cannot always be modeled by a graph. Therefore, Myerson's communication model has been generalized in several directions, for instance towards *conference structures* by Myerson (1980), *hypergraph communication situations* by van den Nouweland et al. (1992) and *union stable systems* by Algaba et al. (2000, 2001). There exists a close relationship between hypergraph communication situations and union stable systems, which has been established in Algaba et al. (2004b).

The union stable system model assumes that if two feasible coalitions have common elements, these will act as intermediaries between the two coalitions in order to establish meaningful cooperation in the union of these coalitions. In other words, players in feasible coalitions can communicate, so players in the intersection of two feasible coalitions make communication in the union coalition possible. This mathematical feature is relevant, and feasible coalition systems coming from communication graphs (Myerson 1977), permission structures (van den Brink 1997; Gilles et al. 1992), systems under precedence constraints (Faigle and Kern 1992), antimatroids (Algaba et al. 2004a) and augmenting systems (Bilbao 2003) verify this condition, being special cases of union stable systems.[1] We remark that, formally, we could translate all results of this paper in terms of hypergraph games but, following the literature, we prefer to present our results in terms of union stable systems, also referring to Sect. 5 of Algaba et al. (2001) for the advantages of this approach.

In the current paper, we argue that in many applications of union stable systems, it is desirable that the power of the players in the network has an effect on the distribution of the dividends in the restricted game. Note that the Myerson value allocates the dividends in the restricted game equally among the players in the corresponding coalitions. To illustrate this point, consider a network of people that are in the Board of Directors of big companies.[2] Typically, such people are members of the board of several companies, see e.g., Mizruchi and Bunting (1981) and Conyon and Muldoon (2006). Besides the influence that a board member has on a company, it is interesting to know what influence board members have on each other when they are member of the board of the same company. Or, even more

[1] Recently, Faigle et al. (2010) have established union stable systems as the more general systems where it is possible to define a meaningful notion of supermodularity that generalizes Shapley's original convexity concept for classical cooperative games.

[2] In modern societies large international firms have an important impact on society, and therefore members of the Boards of Directors of such firms have a repercussion on society and it is interesting to measure, in some sense, this influence.

interesting, what is the influence of a board member of company A on a board member of company B, while they are in no board together, but have a third board member who is sharing a board with each of them. And even there can be influence without such a third member, but more indirect relations. In general, a Board of Directors of a company consists of more than two members, and thus the network where the *basis elements* are the sets of people that belong to the board of one company cannot be modeled by a simple graph. However, if we assume that communication or information can reach a board member of some company B from a board member of another company A, even when these two board members never meet in a common board, but are 'connected' to each other through other board members, then we can consider the boards as the *basis* of a union stable system. The influence a board member has on society depends not only on his/her influence on the firms where he/she is a board member, but also on his/her influence on board members of other firms with whom he/she is 'connected'.

Thus, with the aim of taking into account the role of players in the structure given, when sharing the dividends in the restricted game, we define and axiomatize Harsanyi power solutions for arbitrary union stable systems (generalizing the approach introduced by van den Brink et al. 2011a). So, we allocate the Harsanyi dividends of the restricted game proportional to the power values of the players according to any positive power measure for union stable systems. Such a power measure assigns a non-negative real number to every player in a union stable system, which measures the power or strength of this player in the union stable system considered. Given a power measure, the corresponding sharing system is defined such that for every feasible coalition S the dividends are allocated over the corresponding players proportional to the power measure of the union stable subsystem on S. Special positive power measures for union stable systems are the influence measure and the equal power measure, which generalize the degree and equal power measures, for the particular case of communication graphs. We analyze these Harsanyi power solutions, establishing that for specific power measures they coincide with some well-known solutions in the literature. For example, on the union stable family 2^N, any Harsanyi power solution obtained from a symmetric power measure coincides with the Shapley value, while on the class of all union stable structures the Myerson value (Algaba et al. 2001) coincides with the Harsanyi solution obtained for the equal power measure. In addition, it is interesting to notice that on a special class of union stable systems, denoted by USI^N, and defined as those that are closed under intersection (if the intersection contains at least two elements) and such that every non-unitary feasible coalition (i.e. feasible coalition containing at least two players) can be written in a unique way as a union of non-unitary supports, the Harsanyi power solution obtained for the influence measure is equal to the position value as defined in Algaba et al. (2000). Note that this class contains the sets of connected coalitions in cycle-free communication graphs.

The paper is organized as follows. Section 2 recalls the main concepts on cooperative TU-games and union stable systems which will be used in the following sections. In Sect. 3, we introduce the notion of Harsanyi power solutions for union stable systems. In Sect. 4, we analyze some of their properties. In Sect. 5, we provide axiomatic characterizations of the Harsanyi power solutions on the class USI^N that generalizes the cycle-free communication situations, using the superfluous support property. In Sect. 6, we provide axiomatizations on the class of all games on union stable systems by using either the *superfluous player property* or the *inessential support property* and *connectedness* which gives rise to a new axiomatization of the Myerson value on the class of all union stable structures. Finally, in Sect. 7, we give some special attention to the Myerson value.

2 Preliminaries

2.1 Cooperative TU-games

A *cooperative transferable utility (TU)-game* is a pair (N, v) where $N = \{1, \dots, n\}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$, is a characteristic function.

A distribution of the amount $v(N)$ among the players will be represented by a real-valued vector $x \in \mathbb{R}^n$. Here x_i represents the payoff to player i according to the involved payoff vector x. A *solution* is a real-valued function that assigns a payoff vector to every game (N, v). A solution f satisfies the *efficiency principle* if $\sum_{j\in N} f_j(N, v) = v(N)$.

The most well-known solution is the *Shapley value* (Shapley 1953) given by

$$\Phi_i(N, v) = \sum_{S\subseteq N\setminus\{i\}} \frac{(n - |S| - 1)!(|S|)!}{n!} \big(v\big(S \cup \{i\}\big) - v(S)\big), \qquad \text{for all } i \in N.$$

For each $T \subseteq N$, the *unanimity* game u_T is given by $u_T(S) = 1$, if $T \subseteq S$, and $u_T(S) = 0$, otherwise. It is well-known that the unanimity games u_T, $T \subseteq N$, $T \neq \emptyset$, form a basis of the vectorial space of TU-games on N denoted by $\mathcal{G}^N$, and that each game $v \in \mathcal{G}^N$ can be written as a linear combination of unanimity games in a unique way: $v = \sum_{T\subseteq N,\ T\neq\emptyset} \Delta_v(T)u_T$, where the coefficients $\Delta_v(T)$ are the *Harsanyi dividends* (Harsanyi 1959). By applying the Möbius transformation we obtain that

$$\Delta_v(S) = \sum_{T\subseteq S} (-1)^{|S|-|T|} v(T), \qquad S \subseteq N.$$

The Shapley value belongs to the class of *Harsanyi solutions*, proposed by Vasil'ev (1982, 2003), also known as *sharing values*, see Derks et al. (2000). First, a *sharing system* on N is a system $p = (p^S)_{S\subseteq N, S\neq\emptyset}$, where p^S is an $|S|$-dimensional vector assigning a non-negative share $p_i^S \geq 0$, to every player $i \in S$ with $\sum_{j\in S} p_j^S = 1$, $S \subseteq N$, $S \neq \emptyset$. We denote the collection of all sharing systems on N by P^N. For a game (N, v) and sharing system $p \in P^N$, the corresponding *Harsanyi payoff vector* is the payoff vector $h^p(N, v) \in \mathbb{R}^n$ given by

$$h_i^p(N, v) = \sum_{S\subseteq N,\ i\in S} p_i^S \Delta_v(S), \qquad i \in N,$$

i.e., the payoff $h_i^p(N, v)$ to player $i \in N$ is the sum over all coalitions $S \subseteq N$, containing i, of the share $p_i^S \Delta_v(S)$ of player i in the Harsanyi dividend of coalition S. A *Harsanyi solution* on $\mathcal{G}^N$ assigns for a given sharing system $p \in P^N$ the Harsanyi payoff vector $h^p(N, v)$ to each game (N, v). By definition, the Harsanyi solutions are efficient. The Shapley value is the Harsanyi solution that assigns to any game (N, v) the Harsanyi payoff vector $h^p(N, v)$ with the sharing system p given by $p_i^S = \frac{1}{|S|}$, $S \subseteq N$, $i \in S$.

2.2 Union stable systems and structures

Let $N = \{1, \dots, n\}$ be a finite set of players and $\mathcal{F} \subseteq 2^N$ a set system of feasible coalitions. *Union stable systems* are introduced in Algaba et al. (2000). Formally, the set system $\mathcal{F}$ is called *union stable* if for all $A, B \in \mathcal{F}$ with $A \cap B \neq \emptyset$, it is satisfied that $A \cup B \in \mathcal{F}$.

Many real world situations find its natural framework in these structures. For instance, suppose that player 1 is a homeowner who wants to sell his/her house. Player 1 has signed a contract with a real estate agent, player 2. So, player 1 only can sell his/her house by means of player 2. There are two buyers, players 3 and 4. Notice that in this economic application, the family of feasible coalitions that can generate a surplus are only those which make possible that the seller can sell his/her house. Therefore, the coalitions which can trade are

$$\mathcal{F} = \big\{\{1,2,3\},\{1,2,4\},\{1,2,3,4\}\big\}. \quad (1)$$

An important subclass of union stable systems are communication graphs as considered in Myerson (1977). A *communication graph game* is a triple (N, v, E), where (N, v) is a game and (N, E) is a simple graph. It is easy to see that the set system $\mathcal{F}$, defined by those coalitions which induce connected subgraphs, is a union stable system. However, in practice, a union stable system cannot always be modeled by a communication graph (see van den Brink (2011) for a characterization of the set systems that can be obtained as connected coalitions in a communication graph). For example, the set system $\mathcal{F}$ pointed out above with one seller/two buyers and a real state agent as intermediary, is a union stable system which cannot be the set of connected coalitions in a communication graph.

Let $\mathcal{F}$ be a union stable system and $\mathcal{G} \subseteq \mathcal{F}$. In order to obtain the *basis* of a union stable system (Algaba et al. 2000), the following families are defined inductively

$$\mathcal{G}^{(0)} = \mathcal{G}, \qquad \mathcal{G}^{(m)} = \big\{S \cup T : S, T \in \mathcal{G}^{(m-1)}, S \cap T \neq \emptyset\big\}, \quad (m = 1, 2, \ldots).$$

Notice that $\mathcal{G}^{(0)} \subseteq \mathcal{G}^{(m-1)} \subseteq \mathcal{G}^{(m)} \subseteq \mathcal{F}$, since $\mathcal{G} \subseteq \mathcal{F}$ and $\mathcal{F}$ is union stable. We define $\overline{\mathcal{G}}$ by $\overline{\mathcal{G}} = \mathcal{G}^{(k)}$, where k is the smallest integer such that $\mathcal{G}^{(k+1)} = \mathcal{G}^{(k)}$.

For each union stable family $\mathcal{F}$, it is interesting to find a minimal (with respect to set inclusion relation) subset $\mathcal{B}(\mathcal{F})$ such that $\overline{B(\mathcal{F})} = \mathcal{F}$. Note that the following set is well-defined:

$$\mathcal{E}(\mathcal{F}) = \{G \in \mathcal{F} : G = A \cup B, A \neq G, B \neq G, A, B \in \mathcal{F}, A \cap B \neq \emptyset\}.$$

The set $\mathcal{B}(\mathcal{F}) = \mathcal{F} \setminus \mathcal{E}(\mathcal{F})$ is called the *basis* of $\mathcal{F}$ and the elements of $\mathcal{B}(\mathcal{F})$ are called *supports* of $\mathcal{F}$. We remark that the basis $B(\mathcal{F})$ is the minimal subset of the union stable system $\mathcal{F}$ such that $\overline{B(\mathcal{F})} = \mathcal{F}$, see Algaba et al. (2000).

Let $\mathcal{G} \subseteq 2^N$ be a set system and let $S \subseteq N$. A set $T \subseteq S$ is called a $\mathcal{G}$-*component* of S if $T \in \mathcal{G}$ and there exists no $T' \in \mathcal{G}$ such that $T \subset T' \subseteq S$. Therefore, the $\mathcal{G}$-components of S are the maximal feasible coalitions that belong to $\mathcal{G}$ and are contained in S. We denote by $C_{\mathcal{G}}(S)$ the collection of the $\mathcal{G}$-components of S. Union stable systems can be characterized in terms of the $\mathcal{F}$-components of a coalition in the following way: The set system $\mathcal{F} \subseteq 2^N$ is union stable if and only if for any $S \subseteq N$ with $C_{\mathcal{F}}(S) \neq \emptyset$, the $\mathcal{F}$-components of S are a collection of pairwise disjoint subsets of S, see Algaba et al. (2000). So, if $\mathcal{F}$ is a union stable system, such that for every $i \in N$, there is an $S \in \mathcal{F}$ with $i \in S$, then the $\mathcal{F}$-components form a partition of the player set N.

Let (N, v) be a cooperative game and $\mathcal{F} \subseteq 2^N$ a union stable system. Let $\mathcal{B}(\mathcal{F})$ be the basis of $\mathcal{F}$ and $\mathcal{C}(\mathcal{F}) = \{B \in \mathcal{B}(\mathcal{F}) : |B| \geq 2\}$. If there is no confusion we will just write $\mathcal{B}$ and $\mathcal{C}$. The $\mathcal{F}$-*restricted game*, $v^{\mathcal{F}} : 2^N \to \mathbb{R}$, is defined on the player set N and is given by $v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T)$, where $v^{\mathcal{F}}(S) = 0$, if $C_{\mathcal{F}}(S) = \emptyset$. On the other hand, the *conference game* is defined on the basis of a union stable system[3] (namely, on player set $\mathcal{C}$) and it is the game $(\mathcal{C}, v^{\mathcal{C}})$ where $v^{\mathcal{C}} : 2^{\mathcal{C}} \to \mathbb{R}$ is given by $v^{\mathcal{C}}(\mathcal{A}) = v^{\overline{\mathcal{A}}}(N)$.

[3] Although in the beginning of this section, we mentioned that we always take as player set $N = \{1, \ldots, n\}$, the definition of the conference game is the only occasion where we deviate from this. Note that the player set in a conference game is still derived from a structure on N.

Note that the game $(\mathcal{C}, v^{\mathcal{C}})$ is well defined since for each $\mathcal{A} \subseteq \mathcal{C}$, $\overline{\mathcal{A}}$ is a union stable system. The $\mathcal{F}$-restricted game focuses on the role of a player in creating economic possibilities and establishing meaningful communication among the players, whereas the conference game measures the economic value of the grand coalition when specific parts of the cooperation structure are considered.

These games were defined for hypergraphs communication situations in van den Nouweland et al. (1992), whereas a study about the relationship with those arising from union stable systems can be found in Algaba et al. (2004b). Note that the two above definitions extend the *point game* (introduced by Myerson 1977) and the *arc game* (introduced by Borm et al. 1992) for communication graph games, where for a communication graph game (N, v, E), we have that $\mathcal{C} = \{\{i, j\} : \{i, j\} \in E\}$.

A *union stable cooperation structure* is a triple $(N, v, \mathcal{F})$ where $N = \{1, \dots, n\}$ is the set of players, (N, v) is a TU-game and $\mathcal{F}$ is a union stable system. For convenience, we assume from now on that the underlying game (N, v) is zero-normalized, i.e., $v(\{i\}) = 0$, for all $i \in N$.

The set of all union stable cooperation structures on N will be denoted by $US^N = \{(N, v, \mathcal{F}) : \mathcal{F} \text{ is union stable}\}$.

A feasible coalition is called *non-unitary* when it contains at least two players. We denote by USI^N the special subclass of US^N where the following two conditions are satisfied:

(1) For all $S, T \in \mathcal{F}$ with $|S \cap T| \geq 2$ we have $S \cap T \in \mathcal{F}$.
(2) All non-unitary feasible coalitions can be written in a unique way as a union of non-unitary supports.

Notice that this subclass of union stable cooperation structures generalizes those communication graph games for which the graphs do not contain cycles.

2.3 Allocation rules for union stable cooperation structures

An *allocation rule* on US^N is a map γ that assigns to each union stable cooperation structure $(N, v, \mathcal{F})$ a payoff vector $\gamma(N, v, \mathcal{F}) \in \mathbb{R}^n$.

Both the Myerson value and the position value are defined from the Shapley value (Shapley 1953), but using the $\mathcal{F}$-restricted game and the conference game, respectively, which were defined in the previous subsection. The Myerson value was introduced in Myerson (1977) and later extended in Myerson (1980). Myerson pointed out the need to generalize this model towards restricted cooperation situations which cannot be modeled by a graph. This idea has been studied by van den Nouweland et al. (1992) and Algaba et al. (2001). Given a union stable cooperation structure $(N, v, \mathcal{F})$, the *Myerson value*, denoted by $\mu(N, v, \mathcal{F}) \in \mathbb{R}^n$, is defined by

$$\mu(N, v, \mathcal{F}) = \Phi\big(N, v^{\mathcal{F}}\big).$$

The position value for communication graph situations was first introduced in Meessen (1988) and studied in Borm et al. (1992). This value was extended to hypergraph communication situations in van den Nouweland et al. (1992) and it is defined on union stable cooperation structures in Algaba et al. (2000). Let $(N, v, \mathcal{F})$ be a union stable cooperation structure. The *position value*, denoted by $\pi(N, v, \mathcal{F}) \in \mathbb{R}^n$, is given by

$$\pi_i(N, v, \mathcal{F}) = \sum_{C \in \mathcal{C}_i(\mathcal{F})} \frac{1}{|C|} \Phi_C\big(\mathcal{C}, v^{\mathcal{C}}\big), \quad \text{for } i \in N,$$

where $\mathcal{C}_i(\mathcal{F}) = \{C \in \mathcal{C} : i \in C\}$. When there is no confusion we will often write $\mathcal{C}_i$ instead of $\mathcal{C}_i(\mathcal{F})$.

3 Harsanyi power solutions for union stable cooperation structures

In van den Brink et al. (2011a) the class of Harsanyi power solutions for communication graph games is introduced. A main purpose of the current paper is to extend these solutions to the union stable cooperation structures since in the introduction we illustrated that many networks in society cannot be modeled by a simple graph.

We denote the class of all union stable systems on N by $\mathcal{U}^N$. A *power measure* for union stable systems is a function $\sigma:\mathcal{U}^N \to \mathbb{R}^n_+$ that assigns to any union stable system $\mathcal{F}$, a non-negative vector $\sigma(N,\mathcal{F}) \in \mathbb{R}^n_+$, yielding the non-negative power $\sigma_i(N,\mathcal{F})$ of player $i \in S$ in the union stable system $\mathcal{F}$. Two players $i, j \in N$ are symmetric in the union stable system $\mathcal{F}$ if for all $S \in \mathcal{F}$ it holds that [$i \in S$ if and only if $j \in S$]. A power measure is *symmetric* if for any union stable system $\mathcal{F}$ and players $i, j \in N$ that are symmetric in $\mathcal{F}$, we have $\sigma_i(N,\mathcal{F}) = \sigma_j(N,\mathcal{F})$. It is *positive* if for any $\mathcal{F} \in \mathcal{U}^N$, the power of player i is positive if and only if $i \in M$, for some $M \in C_{\mathcal{F}}(N)$, with $|M| \geq 2$ (or, equivalently $i \in C$, for some $C \in \mathcal{C}(\mathcal{F})$). Thus, player i has zero power if $i \notin \bigcup_{M \in C_{\mathcal{F}}(N),\ |M| \geq 2} M$. Throughout this paper, we only consider *positive* power measures. For a union stable system $\mathcal{F}$ and a coalition $S \subseteq N$, we denote by $\mathcal{F}_S = \{T \in \mathcal{F} : T \subseteq S\}$, the union stable subsystem of those feasible coalitions for which all players belong to S. Note that $\mathcal{F}_S$ is union stable. We denote by $\mathcal{C}_S$ the supports in $\mathcal{F}$ with at least two players that only contains players of S.

One of the best known power measures for simple graphs is the degree measure which assigns to every player in a communication graph its number of neighbors. This degree measure is generalized for union stable systems by Algaba et al. (2000) who define the *influence* of a player i by

$$I_i(N,\mathcal{F}) = \sum_{C \in \mathcal{C}_i} 1/|C|. \tag{2}$$

Note that, according to this influence measure, the power of a player is less when it is in a bigger support. In case the influence would not depend on the number of players in a support, another generalization of the degree measure, would just assign to every player the number of supports which it belongs to, yielding $\overline{I}_i(N,\mathcal{F}) = |\mathcal{C}_i|$, $i \in N$.

Given a positive power measure σ, we define the corresponding *Harsanyi power solution* $\varphi^\sigma : US^N \to \mathbb{R}^n$ on the class of games on union stable systems by

$$\varphi_i^\sigma(N,v,\mathcal{F}) = \sum_{\substack{T \subseteq N,\ i \in T \\ \sum_{j \in T} \sigma_j(N,\mathcal{F}_T) > 0}} \frac{\sigma_i(T,\mathcal{F}_T)}{\sum_{j \in T} \sigma_j(T,\mathcal{F}_T)} \Delta_{v^{\mathcal{F}}}(T).$$

Note that $\sum_{j \in T} \sigma_j(T,\mathcal{F}_T) = 0$ if and only if $\mathcal{C}(\mathcal{F}_T) = \emptyset$, in which case, $\Delta_{v^{\mathcal{F}}}(T) = 0$.

In the following, for union stable system $\mathcal{F}$ and coalition T such that $\sum_{j \in T} \sigma_j(N,\mathcal{F}_T) \neq 0$, we denote

$$p_i^{\mathcal{F},T}(\sigma) = \frac{\sigma_i(T,\mathcal{F}_T)}{\sum_{j \in T} \sigma_j(T,\mathcal{F}_T)}.$$

So, the Harsanyi power solution φ^σ assigns to each union stable structure $(N,v,\mathcal{F})$ the Harsanyi solution (or sharing value) of the corresponding $\mathcal{F}$-restricted game $(N, v^{\mathcal{F}})$, where any dividend $\Delta_{v^{\mathcal{F}}}(T)$ of coalition T in the restricted game is distributed to the players in T proportional to their powers in $\mathcal{F}_T$. Observe that the shares do not matter when all powers are zero, because that can only happen when the only feasible coalitions of T are singletons, and thus the dividend of T in $(N, v^{\mathcal{F}})$ is equal to zero. Therefore, for notational convenience, in the following we take $p_i^{\mathcal{F},T}(\sigma) = \frac{1}{|T|}$, for all $i \in T$ whenever $\sum_{j \in T} \sigma_j(N,\mathcal{F}_T) = 0$.

In this paper, we will provide axiomatizations of the Harsanyi power solutions on the class US^N as well as on the subclass USI^N. Before doing that, we state the following lemma, generalizing a result of van den Brink et al. (2011a), establishing that dividends of non-feasible coalitions are zero.

Lemma 1 *Let* $(N, v, \mathcal{F}) \in US^N$ *be a union stable structure. Then* $\Delta_{v^{\mathcal{F}}}(S) = 0$, *for all* $S \notin \mathcal{F}$.

Proof Let $S \notin \mathcal{F}$. We prove the lemma by induction on the cardinality of S. If $|S| = 1$, then $\Delta_{v^{\mathcal{F}}}(S) = v^{\mathcal{F}}(S) = 0$, by zero-normality. Proceeding by induction suppose that $\Delta_{v^{\mathcal{F}}}(T) = 0$ whenever $T \notin \mathcal{F}$ with $|T| < |S|$. Then,

$$\begin{aligned}\Delta_{v^{\mathcal{F}}}(S) &= v^{\mathcal{F}}(S) - \sum_{T\subset S} \Delta_{v^{\mathcal{F}}}(T)\\ &= \sum_{T\in C_{\mathcal{F}}(S)} v^{\mathcal{F}}(T) - \sum_{\substack{T\subset S\\ T\in\mathcal{F}}} \Delta_{v^{\mathcal{F}}}(T)\\ &= \sum_{T\in C_{\mathcal{F}}(S)} \sum_{\substack{H\subseteq T\\ H\in\mathcal{F}}} \Delta_{v^{\mathcal{F}}}(H) - \sum_{\substack{T\subset S\\ T\in\mathcal{F}}} \Delta_{v^{\mathcal{F}}}(T) = 0.\end{aligned}$$

□

4 Properties

First, we recall some standard axioms for allocation rules. An allocation rule $\gamma : US^N \to \mathbb{R}^n$ is called *component-efficient* if for all $(N, v, \mathcal{F}) \in US^N$ and $M \in C_{\mathcal{F}}(N)$, we have $\sum_{i\in M} \gamma_i(N, v, \mathcal{F}) = v(M)$.

An allocation rule $\gamma : US^N \to \mathbb{R}^n$ satisfies *component dummy* if for all $i \notin \bigcup_{M\in C_{\mathcal{F}}(N)} M$, we have $\gamma_i(N, v, \mathcal{F}) = 0$.

An allocation rule $\gamma : US^N \to \mathbb{R}^n$ is *additive* if

$$\gamma(N, v + w, \mathcal{F}) = \gamma(N, v, \mathcal{F}) + \gamma(N, w, \mathcal{F}),$$

for all $(N, v, \mathcal{F}), (N, w, \mathcal{F}) \in US^N$.

Theorem 1 *Let* σ *be a positive power measure. The Harsanyi power solution* $\varphi^\sigma : US^N \to \mathbb{R}^n$ *is an allocation rule that satisfies component efficiency, component-dummy and additivity.*

Proof Since $\Delta_{v^{\mathcal{F}}}(S) = 0$, if there is no $T \in \mathcal{F}$ such that $S \subseteq T$, for every component $M \in C_{\mathcal{F}}(N)$ we have

$$\begin{aligned}\sum_{i\in M} \varphi_i^\sigma(N, v, \mathcal{F}) &= \sum_{i\in M} \sum_{\substack{T\subseteq N\\ i\in T}} p_i^{\mathcal{F},T}(\sigma)\Delta_{v^{\mathcal{F}}}(T) = \sum_{T\subseteq M} \sum_{i\in T} p_i^{\mathcal{F},T}(\sigma)\Delta_{v^{\mathcal{F}}}(T)\\ &= \sum_{T\subseteq M} \Delta_{v^{\mathcal{F}}}(T) = v^{\mathcal{F}}(M) = v(M),\end{aligned}$$

where the second equality follows from Lemma 1 and the fact that coalitions that contain players from different $\mathcal{F}$-components are not feasible, showing that φ^σ satisfies component efficiency.

It is obvious that φ^σ satisfies component dummy since $\Delta_{v^{\mathcal{F}}}(S) = 0$, for all $S \subseteq N$ with $i \in S$ and $i \notin \bigcup_{M\in C_{\mathcal{F}}(N)} M$.

For all $v, w \in \mathcal{G}^N$ and all $\mathcal{F} \in \mathcal{U}^N$, $(v+w)^{\mathcal{F}}(S) = v^{\mathcal{F}}(S) + w^{\mathcal{F}}(S)$, and thus $\Delta_{(v+w)^{\mathcal{F}}}(S) = \Delta_{v^{\mathcal{F}}}(S) + \Delta_{w^{\mathcal{F}}}(S)$, for all $S \subseteq N$. Then

$$\begin{aligned}\varphi_i^\sigma(N, v+w, \mathcal{F}) &= \sum_{S\subseteq N, i\in S} p_i^{\mathcal{F},S}(\sigma)\Delta_{(v+w)^{\mathcal{F}}}(S)\\ &= \sum_{S\subseteq N, i\in S} p_i^{\mathcal{F},S}(\sigma)\big(\Delta_{v^{\mathcal{F}}}(S) + \Delta_{w^{\mathcal{F}}}(S)\big)\\ &= \varphi_i^\sigma(N, v, \mathcal{F}) + \varphi_i^\sigma(N, w, \mathcal{F}),\end{aligned}$$

showing that φ^σ satisfies additivity. □

Next, we generalize point unanimity as used in Algaba et al. (2012) to axiomatize the Myerson value, in a similar way as the communication ability property for communication graph games of Borm et al. (1992) is generalized in van den Brink et al. (2011a). Let $D(\mathcal{F}) = \{i \in N : \mathcal{C}_i \neq \emptyset\}$ be those players who belong to at least one non-unitary support. A union stable structure $(N, v, \mathcal{F})$ is called point unanimous if there is an $\alpha \in \mathbb{R}$ such that $v^{\mathcal{F}}(S) = 0$, whenever $D(\mathcal{F}) \not\subseteq S$ and $v^{\mathcal{F}}(S) = \alpha$, otherwise, i.e., the $\mathcal{F}$-restricted game is a multiple of the unanimity game on $D(\mathcal{F})$. When there is no confusion, we will often write D instead of $D(\mathcal{F})$.

An allocation rule γ satisfies *point unanimity* if all players that belong to at least one non-unitary support earn the same in any point unanimous union stable structure, while the other players earn zero. As mentioned above, this axiom is used in Algaba et al. (2012) to axiomatize the Myerson value. Using power measures for union stable systems, we generalize this axiom in a similar way as the communication ability property for communication graph games of Borm et al. (1992) is modified and generalized in van den Brink et al. (2011a). So, taking into account the power or centrality of players in a union stable system, for a given power measure σ, we modify this axiom by requiring that the payoffs are allocated proportional to the players' power in the given union stable system.

σ-point unanimity. If $(N, v, \mathcal{F})$ is point unanimous, then there is $\alpha \in \mathbb{R}$ such that $f(N, v, \mathcal{F}) = \alpha\sigma(N, \mathcal{F})$.

Observe that σ-point unanimity reduces to point unanimity when we take the equal power measure $\sigma = E$ given by

$$E_i(N, \mathcal{F}) = \begin{cases} 1, & \text{if } \mathcal{C}_i \neq \emptyset,\\ 0, & \text{if } \mathcal{C}_i = \emptyset.\end{cases}$$

Alternatively, any power measure can be used, such as the influence measure I or its alternative $\overline{I}$. Every Harsanyi power solution satisfies its corresponding σ-point unanimity.

Proposition 1 *Let σ be a positive power measure. The Harsanyi power solution φ^σ satisfies σ-point unanimity on US^N.*

Proof When $(N, v, \mathcal{F})$ is point unanimous, then $v^{\mathcal{F}} = v^{\mathcal{F}}(N)u_D$ and φ^σ is obtained by distributing the unique non-zero dividend $\Delta_{v^{\mathcal{F}}}(N) = v^{\mathcal{F}}(N)$ among the players in D according to the σ-measure, showing that φ^σ satisfies σ-point unanimity. □

5 Axiomatic characterizations on *USI*^{N}

Algaba et al. (2012) axiomatized the Myerson value on the class USI^N by adding the superfluous support property to the properties of component efficiency, component-dummy, additivity and point unanimity, where the axioms are defined on the subclass USI^N similar as they are defined before on US^N.

The support $H \in \mathcal{C}$ is called *superfluous* for $(N, v, \mathcal{F}) \in US^N$ if $v^{\mathcal{C}}(\mathcal{A}) = v^{\mathcal{C}}(\mathcal{A} \setminus \{H\})$, for all $\mathcal{A} \subseteq \mathcal{C}$, i.e., if support H is a null player in the conference game. An allocation rule $\gamma : US^N \to \mathbb{R}^N$ has the *superfluous support property* if $\gamma(N, v, \mathcal{F}) = \gamma(N, v, \overline{\mathcal{B} \setminus \{H\}})$, for all $(N, v, \mathcal{F}) \in US^N$ and for every superfluous support $H \in \mathcal{C}$ for $(N, v, \mathcal{F})$.

On the class USI^N, all Harsanyi power solutions satisfy the superfluous support property.

Theorem 2 *Let σ be a positive power measure. The Harsanyi power solution φ^σ satisfies the superfluous support property on USI^N.*

Proof Since it is assumed that any game is zero-normalized, it follows from the definition of the $\mathcal{F}$-restricted game that, if H is a superfluous support, then $v^{\mathcal{F}} = v^{\overline{\mathcal{B} \setminus \{H\}}}$, and thus

$$\Delta_{v^{\mathcal{F}}}(S) = \Delta_{v^{\overline{\mathcal{B} \setminus \{H\}}}}(S), \quad S \subseteq N.$$

When $H \nsubseteq S$, then $\mathcal{F}_S = (\overline{\mathcal{B} \setminus \{H\}})_S$ and so $\sigma_i(S, \mathcal{F}_S) = \sigma_i(S, (\overline{\mathcal{B} \setminus \{H\}})_S)$ for all $i \in S$, implying that the share of $i \in S$ in $\Delta_{v^{\mathcal{F}}}(S)$ is equal to the share of i in $\Delta_{v^{\overline{\mathcal{B} \setminus \{H\}}}}(S)$.

When $H \subseteq S$, then $S \notin \overline{\mathcal{B} \setminus \{H\}}$ (since otherwise, if S can be written as a union of supports in $\mathcal{B} \setminus \{H\}$, then in $\mathcal{F}$ it can be written as the union of these supports and H), and thus $\Delta_{v^{\overline{\mathcal{B} \setminus \{H\}}}}(S) = 0$ by Lemma 1. So, $\Delta_{v^{\overline{\mathcal{B} \setminus \{H\}}}}(S) = \Delta_{v^{\mathcal{F}}}(S) = 0$ and the shares don't matter. Hence

$$\varphi^\sigma(N, v, \mathcal{F}) = \varphi^\sigma\big(N, v, \overline{\mathcal{B} \setminus \{H\}}\big),$$

showing that φ^σ satisfies the superfluous support property on USI^N. □

Note that in the proof above, it is crucial that for any $(N, v, \mathcal{F}) \in USI^N$ there cannot be a support H such that also a proper subset of H is a support in $\mathcal{F}$.[4]

Similar as shown for the Myerson value in Algaba et al. (2012) (i.e., for $\sigma = E$), it can be shown that on USI^N, the above five axioms characterize the Harsanyi power solution that corresponds to the power measure that is applied to σ-point unanimity.[5]

Theorem 3 *Let σ be a positive power measure. The Harsanyi power solution $\varphi^\sigma : USI^N \to \mathbb{R}^n$ is the unique allocation rule that satisfies component efficiency, component dummy, additivity, the superfluous support property and σ-point unanimity.*

Besides weakening the communication ability property by using point unanimous games instead of point anonymous games, in van den Brink et al. (2011a) also the degree property, used in Borm et al. (1992) to axiomatize the position value for communication graph games,

[4]Consider, for example the union stable system $\mathcal{F} = \{\{1,2\}, \{1,2,3\}\}$. Note that $(N, v, \mathcal{F}) \notin USI^N$ since $\{1,2,3\}$ is a support and it can also be written as the union of the supports $\{1,2\}$ and $\{1,2,3\}$. Further, consider the unanimity game $v = u_{\{1,2,3\}}$. Then the influence measure on $N = \{1,2,3\}$ yields $I(N, \mathcal{F}) = (5/6, 5/6, 1/3)$, so $\varphi^I(N, v, \mathcal{F}) = (5/6, 5/6, 1/3)$. Deleting from $\mathcal{F}$ the superfluous support $\{1,2\}$, we obtain the union stable system $\mathcal{F}' = \{\{1,2,3\}\}$. Now the influence measure is given by $I(N, \mathcal{F}') = (1/3, 1/3, 1/3)$, so $\varphi^I(N, v, \mathcal{F}') = (1/3, 1/3, 1/3)$, showing that the outcome changed although we only deleted a superfluous support. Also, on $(N, v, \mathcal{F})$ the position value is not equal to $\varphi^I(N, v, \mathcal{F})$ since $\pi(N, v, \mathcal{F}) = (1/3, 1/3, 1/3)$.

[5]The proof can be obtained from the corresponding proof for the Myerson value in Algaba et al. (2012) by applying σ-point unanimity instead of point unanimity at every occasion where this is used.

is weakened in a similar way. The same we can do for the influence property, defined using the influence measure discussed before (see (2)) and used in Algaba et al. (2000) to characterize the position value for games on union stable systems. The triple $(N, v, \mathcal{F}) \in US^N$ is called *support unanimous* if there is an $\alpha \in \mathbb{R}$ such that $v^{\mathcal{C}}(\mathcal{A}) = 0$, if $\mathcal{C} \not\subseteq \mathcal{A}$, and $v^{\mathcal{C}}(\mathcal{A}) = \alpha$, otherwise, i.e., if the conference game $(\mathcal{C}, v^{\mathcal{C}})$ is a multiple of the unanimity game on $\mathcal{C}$.[6]

σ-influence property. If $(N, v, \mathcal{F})$ is support unanimous, then there is $\alpha \in \mathbb{R}$ such that $f(N, v, \mathcal{F}) = \alpha\sigma(N, \mathcal{F})$.

The next lemma states that (i) on the class of all union stable structures support unanimity implies point unanimity, and (ii) on the subclass USI^N these properties are equivalent, which extends a result in van den Brink et al. (2011a).

Lemma 2

(i) *Let $(N, v, \mathcal{F}) \in US^N$. If $(N, v, \mathcal{F})$ is support unanimous, then $(N, v, \mathcal{F})$ is also point unanimous.*

(ii) *Let $(N, v, \mathcal{F}) \in USI^N$. Then $(N, v, \mathcal{F})$ is support unanimous if and only if $(N, v, \mathcal{F})$ is point unanimous.*

Proof (i) Let $(N, v, \mathcal{F}) \in US^N$ be support unanimous. If $D = \emptyset$, then $\mathcal{C} = \emptyset$ implying that $v^{\mathcal{F}}(S) = 0$ for all $S \subseteq N$ (by zero-normality of v), and thus $(N, v, \mathcal{F}) \in US^N$ is point unanimous. Next, suppose that $D \neq \emptyset$. By definition of support unanimity, $v^{\mathcal{C}} = v^{\mathcal{F}}(N)u_{\mathcal{C}}$, and thus $v^{\overline{\mathcal{B}\setminus\{H\}}}(N) = v^{\mathcal{C}}(\overline{\mathcal{B}\setminus\{H\}}) = 0$ for all $H \in \mathcal{C}$. For $T \subseteq N$, we distinguish two cases:

Case 1. Let T be such that $D \not\subseteq T$. Then,

$$v^{\mathcal{F}_T}(N) = \sum_{S\in C_{\mathcal{F}_T}(N)} v(S) = \sum_{S\in C_{\mathcal{F}_T}(T)} v(S) = v^{\mathcal{F}_T}(T) = v^{\mathcal{F}}(T),$$

where the second equality follows from zero-monotonicity of (N, v) and the fact that all players that do not belong to T are dummies in $(N, \mathcal{F}_T)$. From $v^{\mathcal{C}} = v^{\mathcal{F}}(N)u_{\mathcal{C}}$, it further follows that $v^{\mathcal{F}_T}(N) = 0$ since $\mathcal{F}_T$ is a proper subset of $\mathcal{F}$. Hence

$$v^{\mathcal{F}}(T) = v^{\mathcal{F}_T}(N) = 0.$$

Case 2. Let T be such that $D \subseteq T$. By definition of $v^{\mathcal{F}}$ and $C_{\mathcal{F}}(T) = C_{\mathcal{F}}(N)$,

$$v^{\mathcal{F}}(T) = \sum_{S\in C_{\mathcal{F}}(T)} v(S) = \sum_{S\in C_{\mathcal{F}}(N)} v(S) = v^{\mathcal{F}}(N).$$

From these two cases we conclude that $v^{\mathcal{F}} = v^{\mathcal{F}}(N)u_D$, which proves (i).

(ii) The 'only if' part follows from (i). To prove the 'if' part, let $(N, v, \mathcal{F}) \in USI^N$ be point unanimous. Note that if we delete a support H of $\mathcal{F}$ with $(N, v, \mathcal{F}) \in USI^N$, then the set $D \notin \overline{\mathcal{B}\setminus\{H\}}$. Thus, if $v^{\mathcal{F}} = v^{\mathcal{F}}(N)u_D$, then $v^{\mathcal{C}}(\mathcal{C}) = v^{\mathcal{F}}(N)$ and $v^{\mathcal{C}}(\mathcal{A}) = v^{\overline{\mathcal{A}}}(N) = 0$, for all $\mathcal{A} \subset \mathcal{C}$, implying that $v^{\mathcal{C}} = v^{\mathcal{F}}(N)u_{\mathcal{C}}$. □

From this lemma, it immediately follows that σ-point unanimity implies the σ-influence property whereas on the subclass USI^N, σ-point unanimity and the σ-influence property are equivalent. As a corollary from Theorem 3 and Lemma 2 (ii), we immediately obtain the following result.

[6]The influence property used in Algaba et al. (2000) to axiomatize the position value, is obtained by taking $\sigma = I$, and moreover applying this to all *support anonymous* union stable structures which include the support unanimous union stable structures, i.e. it states that for each $(N, v, \mathcal{F}) \in US^N$ that is *support anonymous*, the payoffs to the players are proportional to their influence. This axiomatization still holds when the influence property is weakened by requiring it only for *support unanimous* union stable structures.

Corollary 1 *Let σ be a positive power measure. The Harsanyi power solution $\varphi^\sigma : USI^N \rightarrow \mathbb{R}^n$ is the unique allocation rule that satisfies component efficiency, component-dummy, additivity, the superfluous support property and the σ-influence property.*

For $\sigma = I$, Theorem 3 generalizes the axiomatization of the position value for communication graph games in Borm et al. (1992) (using support unanimous instead of support anonymous graph games), while for $\sigma = E$, this generalizes the axiomatization of the Myerson value being a corollary from van den Brink et al. (2011a).

Taking the influence measure, $\sigma = I$, it follows from Algaba et al. (2000) that on USI^N, the position value is a Harsanyi power solution.

Corollary 2 *If $(N, v, \mathcal{F}) \in USI^N$ then $\varphi^I(N, v, \mathcal{F}) = \pi(N, v, \mathcal{F})$.*

Corollary 2 shows that to define and compute the position value on USI^N, we do not need the conference game, since it is a Harsanyi solution applied to the $\mathcal{F}$-restricted game $(N, v^{\mathcal{F}})$. However, for arbitrary union stable systems the position value is not equal to φ^I as is already shown in van den Brink et al. (2011a) for communication graph games. Clearly, since the influence measure is symmetric, it follows that φ^I yields the Shapley value when there are no restrictions to the cooperation, i.e., if $\mathcal{F} = 2^N$. On the other hand, it is well-known that the position value does not generalize the Shapley value in general, i.e., $\pi(v, 2^N)$ need not be equal to $Sh(N, v)$.

As further corollaries we immediately obtain axiomatizations of the Myerson value and position value, of which some are already known.

Corollary 3 *On the class USI^N, it holds that:*

(i) *The position value $\pi : USI^N \rightarrow \mathbb{R}^n$ is the unique allocation rule that satisfies component efficiency, component-dummy, additivity, the superfluous support property and I-point unanimity.*
(ii) *The Myerson value $\mu : USI^N \rightarrow \mathbb{R}^n$ is the unique allocation rule that satisfies component efficiency, component-dummy, additivity, the superfluous support property and E-point unanimity.*
(iii) *The position value $\pi : USI^N \rightarrow \mathbb{R}^n$ is the unique allocation rule that satisfies component efficiency, component-dummy, additivity, the superfluous support property and the I-influence property.*
(iv) *The Myerson value $\mu : USI^N \rightarrow \mathbb{R}^n$ is the unique allocation rule that satisfies component efficiency, component-dummy, additivity, the superfluous support property and the E-influence property.*

Part (iii) is already known from Algaba et al. (2000), while part (ii) is shown in Algaba et al. (2012). Similarly as stated in van den Brink et al. (2011a) for cycle-free communication graph games, we may conclude that both the position value and the Myerson value on USI^N can be characterized by some influence property and some point unanimity property. Therefore, the difference between them is in the power measure that is used. For the position value this is the I-influence measure, whereas for the Myerson value it is the E-influence measure.

Let $USU^N \subset US^N$ be the class of support unanimous union stable structures. Next, we show that on this class the position value is equal to the Harsanyi power solution for the I-influence measure.

Proposition 2 *Let σ be a positive power measure. The Harsanyi power solution $\varphi^\sigma : USU^N \to \mathbb{R}^n$ is the unique allocation rule that satisfies component efficiency, component dummy and the σ-influence property.*

Proof Let $(N, v, \mathcal{F}) \in USU^N$ be a support unanimous union stable structure. If $D = \emptyset$, the payoffs are determined by component-dummy. Suppose $D \neq \emptyset$. Then $\varphi^\sigma(N, v, \mathcal{F}) = \alpha\sigma(N, \mathcal{F})$ by the σ-influence property. Let $\gamma(N, v, \mathcal{F})$ be another allocation rule satisfying component efficiency, component dummy and the σ-influence property. As $(N, v, \mathcal{F})$ is a support unanimous union stable structure, $\gamma(N, v, \mathcal{F}) = \beta\sigma(N, \mathcal{F})$. In order to prove that $\beta = \alpha$, consider $M \in C_{\mathcal{F}}(N)$.

Applying component-efficiency, it holds that $\sum_{i\in M} \gamma_i(N, v, \mathcal{F}) = v(M) = \sum_{i\in M} \varphi_i^\sigma(N, v, \mathcal{F})$. Therefore, $\sum_{i\in M} \beta\sigma_i(N, \mathcal{F}) = \sum_{i\in M} \alpha\sigma_i(N, \mathcal{F})$. Hence, it is satisfied $(\beta - \alpha)\sum_{i\in M} \sigma_i(N, \mathcal{F}) = 0$. Since $\sum_{i\in M} \sigma_i(N, \mathcal{F}) \neq 0$ if $M \in C_{\mathcal{F}}(N)$, we have that $\beta = \alpha$. Thus, $\gamma(N, v, \mathcal{F}) = \varphi^\sigma(N, v, \mathcal{F})$. □

Since the position value satisfies the I-influence property (see Algaba et al. 2000), we obtain the following corollary.

Corollary 4 *For every support unanimous union stable structure $(N, v, \mathcal{F})$, we have $\pi(N, v, \mathcal{F}) = \varphi^I(N, v, \mathcal{F})$.*

6 Axiomatic characterizations on US^N

Since Harsanyi power solutions do not satisfy the superfluous support property on the class of all union stable structures, in order to provide an axiomatic characterization on the class of all union stable structures, we will introduce other axioms and provide two axiomatizations. The first one generalizes the corresponding axiomatization of Harsanyi power solutions for communication graph games given in van den Brink et al. (2011a), the second one extends an axiomatization of the Myerson value for games on union stable systems studied in Algaba et al. (2001).

An allocation rule $\gamma : US^N \to \mathbb{R}^N$ satisfies the *inessential support property* if, for every union stable system $\mathcal{F}$, $c \in \mathbb{R}$, $T \in \mathcal{F}$ with $T \neq \varnothing$, and $H \in \mathcal{C}$ such that $H \not\subseteq T$, we have

$$\gamma(N, cu_T, \mathcal{F}) = \gamma\big(N, cu_T, \overline{\mathcal{B} \setminus \{H\}}\big),$$

where $cu_T(S) = c$, if $T \subseteq S$ and $cu_T(S) = 0$, otherwise.

The inessential support property yields that given a unanimity game on a nonempty feasible coalition T, the solution does not depend on those supports which contain a player outside of coalition T. In other words, the essential supports for the unanimity games on a nonempty feasible coalition T are those whose agents are all in T.

An allocation rule $\gamma : US^N \to \mathbb{R}^N$ satisfies *connectedness* if it satisfies $\gamma(N, v, \mathcal{F}) = \gamma(N, w, \mathcal{F})$ whenever $v^{\mathcal{F}} = w^{\mathcal{F}}$. Connectedness means that the solution only depends on the feasible coalitions.

Proposition 3 *Let σ be a positive power measure. The Harsanyi power solution $\varphi^\sigma : US^N \to \mathbb{R}^n$ satisfies the inessential support property and connectedness.*

Proof In order to prove the inessential support property, let $v = cu_T$, $c \in \mathbb{R}$, with $T \in \mathcal{F}$, $T \neq \varnothing$ and $H \in \mathcal{C}$ such that $H \not\subseteq T$. Since $T \in \overline{\mathcal{B} \setminus \{H\}}$,

$$\Delta_{v^{\mathcal{F}}}(T) = \Delta_{v^{\overline{\mathcal{B}\setminus\{H\}}}}(T) = c \quad \text{and} \quad \Delta_{v^{\mathcal{F}}}(S) = \Delta_{v^{\overline{\mathcal{B}\setminus\{H\}}}}(S) = 0, \text{ for all } S \neq T.$$

In addition to this, $(T, \mathcal{F}_T) = (T, (\overline{\mathcal{B} \setminus \{H\}})_T)$, and thus

$$\sigma(T, \mathcal{F}_T) = \sigma\big(T, \big(\overline{\mathcal{B} \setminus \{H\}}\big)_T\big).$$

Therefore,

$$\varphi^\sigma(N, v, \mathcal{F}) = \varphi^\sigma\big(N, v, \overline{\mathcal{B} \setminus \{H\}}\big),$$

showing that φ^σ satisfies the inessential support property.

Connectedness holds straightforward, since $\Delta_{v^\mathcal{F}}(S) = \Delta_{w^\mathcal{F}}(S)$, for all coalition $S \subseteq N$, whenever $v^\mathcal{F} = w^\mathcal{F}$. □

Using these two properties, we obtain the following axiomatic characterization of the Harsanyi power solutions on US^N, generalizing the corresponding result of van den Brink et al. (2011a).

Theorem 4 *Let σ be a positive power measure. The Harsanyi power solution $\varphi^\sigma : US^N \to \mathbb{R}^n$ is the unique allocation rule that satisfies component efficiency, component-dummy, additivity, σ-point unanimity, the inessential support property and connectedness.*

Proof It follows from Theorems 1 and Propositions 1 and 3 that φ^σ is an allocation rule that satisfies the six properties on US^N. To show uniqueness, let $(N, v, \mathcal{F}) \in US^N$ and let $\gamma : US^N \to \mathbb{R}^n$ be an allocation rule that satisfies the above properties.

To prove uniqueness, as γ is additive and the game (N, v) is zero-normalized, it is sufficient to show that for all $T \subseteq N$ with $|T| \geq 2$, and $\beta \in \mathbb{R}$, $\gamma(N, \beta u_T, \mathcal{F})$ is uniquely determined. To prove this, fix $T \subseteq N$ with $|T| \geq 2$. We distinguish two cases:

Case 1. Suppose that $T \in \mathcal{F}$. Then $(\beta u_T)^\mathcal{F} = \beta u_T$. From the inessential support property, $\gamma(N, \beta u_T, \mathcal{F}) = \gamma(N, \beta u_T, \mathcal{F}_T)$. Moreover, $(N, \beta u_T, \mathcal{F}_T)$ is a point unanimous union stable structure. So, σ-point unanimity yields that there exists $\alpha \in \mathbb{R}$ such that

$$\gamma(N, \beta u_T, \mathcal{F}) = \gamma(N, \beta u_T, \mathcal{F}_T) = \alpha\sigma(N, \mathcal{F}_T).$$

Applying efficiency, it holds

$$\sum_{i \in T} \gamma_i(N, \beta u_T, \mathcal{F}) = \sum_{i \in T} \alpha\sigma_i(N, \mathcal{F}_T) = \alpha \sum_{i \in T} \sigma_i(N, \mathcal{F}_T) = \beta,$$

and thus $\alpha = \frac{\beta}{\sum_{i \in T} \sigma_i(N, \mathcal{F}_T)}$. We conclude that

$$\gamma_i(N, \beta u_T, \mathcal{F}) = \begin{cases} \frac{\beta\sigma_i(N, \mathcal{F}_T)}{\sum_{i \in T} \sigma_i(N, \mathcal{F}_T)}, & \text{if } i \in T, \\ 0, & \text{otherwise,} \end{cases}$$

and thus, $\gamma(N, \beta u_T, \mathcal{F}) = \varphi^\sigma(N, \beta u_T, \mathcal{F})$.

Case 2. Suppose that $T \notin \mathcal{F}$. Let $\mathcal{S} = \{S \in \mathcal{F} : T \subset S\}$ be the collection of feasible subsets of N that contain T, and let $c_R \in \mathbb{R}$, $R \in \mathcal{S}$, be such that,

$$(\beta u_T)^\mathcal{F} = \sum_{R \in \mathcal{S}} c_R u_R,$$

(see van den Brink et al. (2011b) to determine $c_R \in \mathbb{R}$ and $R \in \mathcal{S}$).[7] Applying Case 1 to each feasible coalition $R \in \mathcal{S}$, it is satisfied that

$$\gamma(N, c_R u_R, \mathcal{F}) = \varphi^\sigma(N, c_R u_R, \mathcal{F}).$$

[7] Although van den Brink et al. (2011b) is about games on union closed systems, the proof is the same.

By additivity of γ and φ^σ it follows that $\gamma(N, \beta u_T, \mathcal{F}) = \varphi^\sigma(N, \beta u_T, \mathcal{F})$. Again by additivity, $\gamma(N, v, \mathcal{F}) = \varphi^\sigma(N, v, \mathcal{F})$. □

As the Myerson value is the Harsanyi power solution for the equal power measure on US^N, as a result we get a new characterization for the Myerson value on US^N.

Corollary 5 *The Myerson value $\mu : US^N \to \mathbb{R}^n$ is the unique allocation rule that satisfies component efficiency, component-dummy, additivity, E-point unanimity, the inessential support property and connectedness on US^N.*

Algaba et al. (2001) provide an axiomatization of the Myerson value using the superfluous player property, generalizing a result for communication graph games in van den Nouweland (1993). A player $i \in N$ is called superfluous for $(N, v, \mathcal{F}) \in US^N$ if $v^{\mathcal{F}}(S) = v^{\mathcal{F}}(S \setminus \{i\})$, for all $S \subseteq N$, i.e., it is a null player in the $\mathcal{F}$-restricted game $(N, v^{\mathcal{F}})$. An allocation rule γ satisfies the *superfluous player property* if for all $(N, v, \mathcal{F})$, and every player $i \in N$ that is superfluous for $(N, v, \mathcal{F})$, it holds $\gamma(N, v, \mathcal{F}) = \gamma(N, v, \mathcal{F}_{N\setminus\{i\}})$, where $\mathcal{F}_{N\setminus\{i\}} = \{F \in \mathcal{F} : F \subseteq N \setminus \{i\}\}$.

It can be shown, in a similar way, that every Harsanyi power solution satisfies this property.

Theorem 5 *Let σ be a positive power measure. The Harsanyi power solution $\varphi^\sigma : US^N \to \mathbb{R}^n$ satisfies the superfluous player property.*

Proof Let $i \in N$ be a superfluous player in $(N, v, \mathcal{F})$. We have to prove $\varphi^\sigma(N, v, \mathcal{F}) = \varphi^\sigma(N, v, \mathcal{F}_{N\setminus\{i\}})$. Since i is a null player in the game $(N, v^{\mathcal{F}})$, $\varphi_i^\sigma(N, v, \mathcal{F}) = 0$. Since $i \notin \bigcup_{M \in C_{\mathcal{F}_{N\setminus\{i\}}}(N)} M$, by φ^σ satisfying component dummy we have $\varphi_i^\sigma(N, v, \mathcal{F}_{N\setminus\{i\}}) = 0$.

For the other players it is sufficient to show that $v^{\mathcal{F}}(S) = v^{\mathcal{F}_{N\setminus\{i\}}}(S)$, or equivalently that $v^{\mathcal{F}}(S \setminus \{i\}) = v^{\mathcal{F}_{N\setminus\{i\}}}(S)$ for all $S \subseteq N$, whenever i is a superfluous player in $(N, v, \mathcal{F})$. Since $C_{\mathcal{F}}(S \setminus \{i\}) = C_{\mathcal{F}_{N\setminus\{i\}}}(S)$, we have, for all $S \subseteq N$

$$v^{\mathcal{F}}\big(S \setminus \{i\}\big) = \sum_{T \in C_{\mathcal{F}}(S\setminus\{i\})} v(T) = \sum_{T \in C_{\mathcal{F}_{N\setminus\{i\}}}(S)} v(T) = v^{\mathcal{F}_{N\setminus\{i\}}}(S).$$

□

In Algaba et al. (2001, Theorem 3.7) the Myerson value is axiomatized by component efficiency, component dummy, additivity, the superfluous player property and *point anonymity*. The following result can be obtained from the corresponding proof in Algaba et al. (2001) by applying σ-point unanimity instead of point anonymity each time that it is used.

Theorem 6 *Let σ be a positive power measure. The Harsanyi power solution $\varphi^\sigma : US^N \to \mathbb{R}^n$ is the unique allocation rule that satisfies component efficiency, component-dummy, additivity, the superfluous player property and σ-point unanimity.*

Lemma 3.6 of Algaba et al. (2001) states that for every allocation rule that satisfies component efficiency, component-dummy, additivity and the superfluous player property, the payoff allocation to any game on a union stable system equals the payoff allocation assigned to the $\mathcal{F}$-restricted game on the same union stable system. But then it satisfies connectedness. So, replacing the inessential support property by the superfluous player property in Theorem 4, also we can do without connectedness.

7 The Myerson value

We used σ-point unanimity and the σ-influence property to characterize the corresponding Harsanyi power solution φ^σ. These properties are based on point unanimous, respectively support unanimous, union stable structures. In Algaba et al. (2001) is shown that the Myerson value satisfies point anonymity, which is stronger than E-point unanimity, using point anonymous games. A union stable structure $(N, v, \mathcal{F})$ is called *point anonymous* if there exists $f : \{0, 1, \ldots, |D(\mathcal{F})|\} \to \mathbb{R}$, such that $v^{\mathcal{F}}(S) = f(|S \cap D(\mathcal{F})|)$, for all $S \subseteq N$. Obviously, a point unanimous union stable structure is point anonymous (with function $f : \{0, 1, \ldots, |D|\} \to \mathbb{R}$ such that $f(k) = 0$, for all $k \in \{0, 1, \ldots, |D| - 1\}$). An allocation rule γ satisfies *point anonymity* if all players that belong to at least one non-unitary support earn the same in any point anonymous union stable structure, while the other players earn zero. Although the Myerson value satisfies E-point anonymity, in general, Harsanyi power solutions φ^σ do not satisfy the corresponding σ-point anonymity property, stating that in a point anonymous union stable structure the payoffs are allocated proportional to power measure σ.[8]

In Algaba et al. (2012) an axiomatization of the Myerson value by component efficiency, component-dummy, additivity, point unanimity and the strong superfluous support property is given. This last property states that the payoff allocation does not change after deleting a support whose absence does not influence the $\mathcal{F}$-restricted game. This property implies the superfluous support property, and, therefore, it is not satisfied by all Harsanyi power solutions, since they already do not satisfy the superfluous support property on US^N, in general.

We end this section with an example illustrating some solutions mentioned in this paper.

Example 1 Consider the player set $N = \{1, 2, 3, 4\}$ and the union stable system given by $\mathcal{F} = \{\{1, 2\}, \{1, 2, 3\}, \{2, 3, 4\}, N\}$. Let $v : 2^N \to \mathbb{R}$ be defined by $v = u_{\{1,2,3\}}$, for all $S \subseteq N$. Then,

$$\mathcal{B} = \mathcal{C} = \big\{\{1, 2\}, \{1, 2, 3\}, \{2, 3, 4\}\big\}.$$

Notice that $v^{\mathcal{F}} = u_{\{1,2,3\}}$ and $(N, v, \mathcal{F}) \notin USI^N$ since $N \in \mathcal{F}$ and it can be written as the union of the supports $\{1, 2\} \cup \{2, 3, 4\}$ as well as the union of the supports $\{1, 2, 3\} \cup \{2, 3, 4\}$, i.e., the non-unitary feasible coalition N does not admit a unique expression as a union of non-unitary supports. Note that the second condition which defines the subclass of union stable structures USI^N is not satisfied either since $\{1, 2, 3\} \cap \{2, 3, 4\} = \{2, 3\} \notin \mathcal{F}$. Computing the Myerson value, the position value, the Harsanyi power solution obtained for the E-influence measure and the Harsanyi power solution obtained for the I-influence measure, we obtain

$$\mu(N, v, \mathcal{F}) = \varphi^E(N, v, \mathcal{F}) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right).$$

$$\pi(N, v, \mathcal{F}) = \left(\frac{11}{36}, \frac{13}{36}, \frac{10}{36}, \frac{2}{36}\right).$$

$$\varphi^I(N, v, \mathcal{F}) = \left(\frac{5}{12}, \frac{5}{12}, \frac{2}{12}, 0\right).$$

[8] A similar remark can be made about strengthening the σ-influence property using *support anonymous* union stable structures, i.e. triples $(N, v, \mathcal{F}) \in US^N$ such that there exists a function $f : \{0, 1, \ldots, |\mathcal{C}|\} \to \mathbb{R}$ with $v^{\mathcal{C}}(\mathcal{A}) = f(|\mathcal{A}|)$, for all $\mathcal{A} \subseteq \mathcal{C}$. In Algaba et al. (2000) is shown that the position value satisfies support anonymity, which is used to characterize this solution on USI^N.

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