Math Meth Oper Res (2002) 56:287–301

Mathematical Methods
of Operations Research

The Shapley value for games on matroids: The dynamic model

J. M. Bilbao[1], T. S. H. Driessen[2], A. Jiménez-Losada[3], and E. Lebrón[4]

[1] Matemática Aplicada II, Escuela Superior de Ingenieros, Camino de los Descubrimientos, 41092 Sevilla, Spain (E-mail: mbilbao@us.es)
[2] Faculty of Mathematical Sciences, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands (E-mail: T.S.H.Driessen@math.utwente.nl)
[3] Matemática Aplicada II (E-mail: hispan@matinc.us.es)
[4] Matemática Aplicada II, Escuela Superior de Ingenieros, Camino de los Descubrimientos, 41092 Sevilla, Spain (E-mail: elebron@platero.eup.us.es)

Manuscript received: August 2001/Final version received: February 2002

Abstract. According to the work of Faigle [3] a static Shapley value for games on matroids has been introduced in Bilbao, Driessen, Jiménez-Losada and Lebrón [1]. In this paper we present a dynamic Shapley value by using a dynamic model which is based on a recursive sequence of static models. In this new model for games on matroids, our main result is that there exists a unique value satisfying analogous axioms to the classical Shapley value. Moreover, we obtain a recursive formula to calculate this dynamic Shapley value. Finally, we prove that its components are probabilistic values.

Mathematics Subject Classification 2000: 91A12

Key words: matroid, cooperative game, Shapley value

1. Introduction

A *matroid* $\mathscr{M}$ on a finite set N is a collection of subsets of N which satisfies the following properties:

(M1) $\varnothing \in \mathscr{M}$.
(M2) If $T \in \mathscr{M}$ and $S \subseteq T$, then $S \in \mathscr{M}$.
(M3) If $S, T \in \mathscr{M}$ and $|T| = |S| + 1$, then there exists $i \in T \backslash S$ such that $S \cup \{i\} \in \mathscr{M}$.

We refer the reader to Welsh [8] and Korte, Lovász and Schrader [4] for a detailed treatment of matroids and their numerous applications in combinatorics and optimization theory.

A cooperative game is a pair (N, v) where N is the finite set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function satisfying $v(\varnothing) = 0$. The subsets of N are called coalitions.

Definition 1.1. *A cooperative game on a matroid* $\mathcal{M}$ *is a pair* $(\mathcal{M}, v)$ *where* $v : \mathcal{M} \to \mathbb{R}$ *satisfies* $v(\emptyset) = 0$.

The coalitions in $\mathcal{M}$ are named *feasible coalitions*. In words, a cooperative game on a matroid represents an evaluation of the potential utility of the feasible coalitions, whereas the non-feasible coalitions are totally ignored because such coalitions are supposed not to be formed anyhow. The real vector space of all games on a matroid $\mathcal{M}$ is denoted by $\Gamma(\mathcal{M})$.

We will use the following concepts of matroid theory in our model. The *rank* function $r : 2^N \to \mathbb{Z}_+$ of a matroid $\mathcal{M}$ on N is defined by

$$r(S) = \max\{|T| : T \subseteq S, T \in \mathcal{M}\}.$$

Given a coalition S, the maximal feasible coalitions contained in S are called *bases* of S. All the bases of one coalition S have the same cardinality $r(S)$. When a feasible coalition B has cardinality $r(N)$, i.e. is a basis of N, then B is called a *basic coalition* of $\mathcal{M}$ and we denote the set of basic coalitions by $\mathcal{B}(\mathcal{M})$. The rank function of a matroid $\mathcal{M}$ is a monotonic and submodular cooperative game.

We also need two operations of matroids: the deletion and the contraction. Let $\mathcal{M}$ be a matroid and $S \in \mathcal{M}$, the *deletion* of S is the new matroid

$$\mathcal{M} \backslash S = \{T \in \mathcal{M} : T \cap S = \emptyset\},$$

i.e. the set of the feasible coalitions which do not contain players of S. The *contraction* of S is the new matroid

$$\mathcal{M}/S = \{T \in \mathcal{M} : T \cap S = \emptyset, T \cup S \in \mathcal{M}\},$$

i.e. the set of the feasible coalitions which can incorporate S in order to form a feasible coalition.

In a classic cooperative game (N, v) it is supposed that the players form the big coalition N (see Driessen [2]). In our model about games on matroids, we suppose that the players cooperate to reach a feasible coalition as big as possible. Thus, when a cooperation structure is defined by a matroid $\mathcal{M}$, we assume that a basic coalition of $\mathcal{M}$ will be formed, unknown beforehand by the players.

Nevertheless, it is possible to bring up different forms of playing. It can occur that the players would only want to study which basic coalition will be formed and how they will share the profits. This is the *static model* which we have already studied in Bilbao et al. [1].

Another possibility named *dynamic model*, is the one that considers the following: once a basic coalition A_1 of $\mathcal{M}$ is formed, the game continues between the players of the set $N \backslash A_1$. These players have in mind to form a new basic coalition A_2 of $\mathcal{M} \backslash A_1$ and once again the game will continue between the players $(N \backslash A_1) \backslash A_2$ and the game will end when all the players have participated (they will determine a partition of N). Note that in this way we suppose that the feasible coalitions which are formed, always belong to the original matroid.

The above procedure describes the dynamic model for games on matroids. The aim is to raise that each player can study "a priori" what coalitions are the most interesting, and also estimate his/her possible benefits.

The paper is organized as follows. In Section 2 we introduce the concepts of a *basic sequence* on a given matroid as well as the *dynamic influence* of a feasible coalition S with respect to a given probability distribution on the set of all basic sequences on the matroid. We present the relationship between the dynamic influence of a feasible coalition S and its static influence with respect to an induced probability distribution on the set of all basic coalitions of the matroid. In Section 3 we introduce, besides three rather standard axioms dealing with linearity, dummy players and symmetric players respectively, a fourth axiom dealing with some kind of efficiency that takes into account the dynamic model in terms of the basic sequences on the matroid. We prove that there exists a unique value, called the *dynamic Shapley value*, satisfying these four axioms.

In addition, we present the relationship between the dynamic Shapley value and the so-called static Shapley value, which has been introduced in Bilbao et al. [1] on the basis of four similar axioms, incorporating a slightly adapted efficiency axiom. The proof of the relationship between the dynamic and static Shapley values (applied to so-called unanimity games) is fully based on the relationship between the dynamic and static influence of any feasible coalition. Finally it is shown that the components of the dynamic Shapley value are probabilistic values in the sense of Weber [7]. A detailed summary of the main results is given in Section 4.

Example 1. Suppose that V is a set of cities where several companies are localized. These companies are interested in building a communication network system connecting some cities of V. Their necessity of communication is represented by a edge-weighted graph $((V,E),w)$ where the weight of each edge is its constructing cost. Let us consider that there are different companies in the same city requiring different services. Under these conditions, in order that the payment of a company only depends on its requirements, we can consider the matroid $\mathscr{M}$ on the players set $N = E$ whose basic coalitions are the spanning trees of the graph (V,E).

The game $(\mathscr{M}, v)$ where $v(S)$ is the sum of edge weights in S, will allow them to study their possibilities and determinate an allocation of total cost. The model can be static or dynamic, depending on whether they want to build only one network or make as many networks as there are necessary to construct each edge of E. The matroid of this example is named graphic matroid (see Korte et al. [4]) and it has already been used by Nagamochi et al. [5] to define the minimum base game on matroids.

Example 2. Let us assume that in the set N of firms of a market (for example, the oil companies which are the owners of all the gasoline stations of a country) there is a group C of big companies whose fusion can end with the competence. In these cases it is usual to restrict the cooperation and consider, for example, that the feasible holdings are the groups of firms which do not contain C. This suggests that we should talk about the matroid

$$\mathscr{M} = \{S \subseteq N : C \nsubseteq S\},$$

and consider the dynamic model of the game because the companies could merge as long as they form feasible coalitions of the matroid.

Example 3. An owner has k warehouses which are demanded from a set $N = \{1, \ldots, n\}$ of firms, where $n > k$. Every firm $i \in N$ obtains a profit c_i if it succeeds in storing its products, otherwise it loses everything. Besides that we suppose that each one knows the benefits of the others. Then they have complete information to negotiate theirs profits to get a warehouse. In this situation, we propose the uniform matroid U_n^k (see Korte et al. [4]), whose basic coalitions are all the subsets of N with cardinality k, to allocate the expected profits.

2. The dynamic influence

From now on we assume that the matroid $\mathscr{M}$ is *normal*, i.e., for every $i \in N$ there exists an $S \in \mathscr{M}$ such that $i \in S$.

Definition 2.1. *A basic sequence of* $\mathscr{M}$ *is an ordered set of nonempty feasible coalitions* $(A_1, \ldots, A_k)$ *such that:*

(a) *The first coalition* $A_1 \in \mathscr{B}(\mathscr{M})$.
(b) *If* $k \geq 2$, *then* $A_m \in \mathscr{B}(\mathscr{M} \backslash A_1 \backslash \cdots \backslash A_{m-1})$ *for all* $m = 2, \ldots, k$.
(c) *The matroid* $\mathscr{M} \backslash A_1 \backslash \cdots \backslash A_k = \{\varnothing\}$.

The set of the basic sequences of the matroid is denoted by $\Pi(\mathscr{M})$. A *basic semi-sequence* is an ordered set $(A_1, \ldots, A_s)$ of feasible nonempty coalitions satisfying (a) and (b) in the above definition. So, every basic sequence is a basic semi-sequence. If $\pi = (A_1, \ldots, A_s)$ is a basic semi-sequence of $\mathscr{M}$, then:

(1) $\mathscr{M} \backslash A_1 \backslash \cdots \backslash A_s$ is a matroid.
(2) $A_m \in \mathscr{M}$ for all $m \in \{1, \ldots, s\}$.
(3) $A_q \cap A_m = \varnothing$ for $m, q \in \{1, \ldots, s\}$ with $m \neq q$.

If $\pi = (A_1, \ldots, A_k)$ is a basic sequence then $\bigcup_{m=1}^{k} A_m = N$ and hence π defines a partition of N.

Definition 2.2. *Let* $\mathscr{M}$ *be a matroid. A deletion minor of* $\mathscr{M}$ *is a matroid* $\mathscr{M} \backslash A_1 \backslash \cdots \backslash A_s$ *constructed by a basic semi-sequence* $(A_1, \ldots, A_s)$ *of* $\mathscr{M}$.

We will understand that two deletion minors are different if the semi-sequences which define them are different. Besides that, we will suppose that $\mathscr{M}$ is a deletion minor of itself.

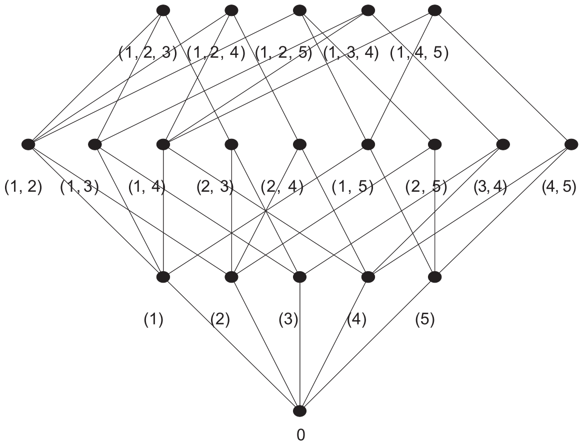
Example 4. Let $\mathscr{M}$ be the matroid with basic coalitions $B_1 = \{1,2,3\}$, $B_2 = \{1,2,4\}$, $B_3 = \{1,2,5\}$, $B_4 = \{1,3,4\}$, $B_5 = \{1,4,5\}$.

The basic sequences are:

$$\pi_1 = (B_1, \{4,5\}) \quad \pi_2 = (B_2, \{3\}, \{5\}) \quad \pi_3 = (B_2, \{5\}, \{3\})$$

$$\pi_4 = (B_3, \{3,4\}) \quad \pi_5 = (B_4, \{2,5\}) \quad \pi_6 = (B_5, \{2,3\})$$

It is possible to see in this example that it can have basic sequences with different number of coalitions.



We will denote by $\mathscr{D}(\mathscr{M})$ the set of probability distributions on the set $\Pi(\mathscr{M})$ of basic sequences on the matroid $\mathscr{M}$.

Definition 2.3. *Let $\mathscr{M}$ be a matroid and let $D \in \mathscr{D}(\mathscr{M})$ be a probability distribution on $\Pi(\mathscr{M})$. The dynamic influence of $S \in \mathscr{M}$ with respect to D is the sum of probabilities of basic sequences containing a coalition that contains S, i.e.*

$$w^D(S) = \sum_{\pi \in \Pi_S(\mathscr{M})} D(\pi),$$

where $\Pi_S(\mathscr{M})$ denotes the set of basic sequences $\pi = (A_1, \dots, A_k)$ such that $S \subseteq A_m$ for some $m \in \{1, \dots, k\}$.

We recall the concept of influence of a feasible coalition for the static model introduced in Bilbao et al. [1]. With respect to a given probability distribution $P_{\mathscr{M}}$ on $\mathscr{B}(\mathscr{M})$, the *static influence* of $S \in \mathscr{M}$ is the sum of the probabilities of basic coalitions containing coalition S, i.e.

$$w^{P_{\mathscr{M}}}(S) = \sum_{B \in \mathscr{B}_S(\mathscr{M})} P_{\mathscr{M}}(B),$$

where $\mathscr{B}_S(\mathscr{M}) = \{B \in \mathscr{B}(\mathscr{M}) : S \subseteq B\}$.

Note that the dynamic influence of every one-player coalition is equal to one because every basic sequence yields a partition of N. The main result of this section concerns the relationship between the dynamic and static influence. For that purpose we prove a preliminary lemma about the decomposition of probability distributions on $\Pi(\mathscr{M})$. We denote by $\mathscr{P}(\mathscr{M})$ the set of probability distributions on $\mathscr{B}(\mathscr{M})$.

Lemma 2.1. *A map $D : \Pi(\mathscr{M}) \to \mathbb{R}$ is a probability distribution on $\Pi(\mathscr{M})$ if and only if there exists a unique probability distribution $P_{\mathscr{M}'} \in \mathscr{P}(\mathscr{M}')$ for each nonempty deletion minor $\mathscr{M}'$ such that*

$$D(\pi) = P_{\mathscr{M}}(A_1) \prod_{m=2}^{k} P_{\mathscr{M} \backslash A_1 \backslash \cdots \backslash A_{m-1}}(A_m), \tag{1}$$

for every basic sequence $\pi = (A_1, A_2, \ldots, A_k)$ *of* $\mathscr{M}$.

Proof. ($\Rightarrow$) Consider the probability distribution D. We will define a probability on the basic coalitions of any deletion minor of $\mathscr{M}$. First we consider the real function $P_{\mathscr{M}}$ on $\mathscr{B}(\mathscr{M})$, defined by

$$P_{\mathscr{M}}(B) = \sum_{\{\pi \in \Pi(\mathscr{M}) : \pi = (B, \ldots)\}} D(\pi) \geq 0.$$

Because

$$\sum_{B \in \mathscr{B}(\mathscr{M})} P_{\mathscr{M}}(B) = \sum_{B \in \mathscr{B}(\mathscr{M})} \sum_{\{\pi \in \Pi(\mathscr{M}) : \pi = (B, \ldots)\}} D(\pi) = \sum_{\pi \in \Pi(\mathscr{M})} D(\pi) = 1,$$

$P_{\mathscr{M}}$ is a probability distribution on $\mathscr{B}(\mathscr{M})$.

Now, for each deletion minor $\mathscr{M} \backslash A_1$ of $\mathscr{M}$ such that $P_{\mathscr{M}}(A_1) \neq 0$, we define

$$P_{\mathscr{M} \backslash A_1}(B) = \frac{1}{P_{\mathscr{M}}(A_1)} \sum_{\{\pi \in \Pi(\mathscr{M}) : \pi = (A_1, B, \ldots)\}} D(\pi)$$

for all $B \in \mathscr{B}(\mathscr{M} \backslash A_1)$. Obviously $P_{\mathscr{M} \backslash A_1}$ is a probability on $\mathscr{B}(\mathscr{M} \backslash A_1)$. If $P_{\mathscr{M}}(A_1) = 0$, we take for $P_{\mathscr{M} \backslash A_1}$ any probability distribution on $\mathscr{B}(\mathscr{M} \backslash A_1)$.

Finally, if $\mathscr{M}_i = \mathscr{M} \backslash A_1 \backslash \cdots \backslash A_i$ and $\mathscr{M}_{i+1} = \mathscr{M} \backslash A_1 \backslash \cdots \backslash A_i \backslash A_{i+1}$ are nonempty deletion minors of $\mathscr{M}$, we will construct the probability $P_{\mathscr{M}_{i+1}}$ using the probability $P_{\mathscr{M}_i}$, by the following recursive procedure. If $K(\mathscr{M}_{i+1}) = P_{\mathscr{M}}(A_1) P_{\mathscr{M}_1}(A_2) \cdots P_{\mathscr{M}_i}(A_{i+1}) \neq 0$, then

$$P_{\mathscr{M}_{i+1}}(B) = \frac{1}{K(\mathscr{M}_{i+1})} \sum_{\{\pi \in \Pi(\mathscr{M}) : \pi = (A_1, \ldots, A_i, A_{i+1}, B \ldots)\}} D(\pi)$$

for all $B \in \mathscr{B}(\mathscr{M}_{i+1})$. In other case, $P_{\mathscr{M}_{i+1}}$ is any probability distribution on $\mathscr{B}(\mathscr{M}_{i+1})$.

Now we can assure that these probabilities allow us to express D like in (1). Note that if $\pi = (A_1, \ldots, A_k)$ is a basic sequence of $\mathscr{M}$, the expression of

$$P_{\mathscr{M}}(A_1) P_{\mathscr{M} \backslash A_1}(A_2) P_{\mathscr{M} \backslash A_1 \backslash A_2}(A_3) \cdots P_{\mathscr{M} \backslash A_1 \backslash \cdots \backslash A_{k-1}}(A_k)$$

is such that or each factor is the denominator of the following one and the last is $D(\pi)$ or the result is zero and also $D(\pi) = 0$.

($\Leftarrow$) We use induction on the number of players. When there is only one player the result is trivial for the unique possible matroid. We suppose the result is true for every matroid with less than n players and will prove it for n. We consider the function

$$D(\pi) = P_{\mathscr{M}}(A_1) \prod_{m=2}^{k} P_{\mathscr{M}\backslash A_1 \backslash \cdots \backslash A_{m-1}}(A_m),$$

for $\pi = (A_1, A_2, \ldots, A_k) \in \Pi(\mathscr{M})$. For each basic sequence $\pi = (A_1, A_2, \ldots, A_k)$ of $\mathscr{M}$ we have that $\pi' = (A_2, \ldots, A_k)$ is a basic sequence of $\mathscr{M}\backslash A_1$, and conversely. By induction the map $D_{\mathscr{M}\backslash A_1} : \Pi(\mathscr{M}\backslash A_1) \to \mathbb{R}$ such that

$$D_{\mathscr{M}\backslash A_1}(\pi') = \prod_{m=2}^{k} P_{\mathscr{M}\backslash A_1 \backslash \cdots \backslash A_{m-1}}(A_m) \tag{2}$$

for all basic sequence $\pi' = (A_2, \ldots, A_k)$ of $\mathscr{M}\backslash A_1$, is a probability distribution on $\Pi(\mathscr{M}\backslash A_1)$. Thus, we obtain

$$\begin{aligned}
\sum_{\pi \in \Pi(\mathscr{M})} D(\pi) &= \sum_{\{\pi \in \Pi(\mathscr{M}) : \pi = (A_1, \ldots, A_k)\}} P_{\mathscr{M}}(A_1) \prod_{m=2}^{k} P_{\mathscr{M}\backslash A_1 \backslash \cdots \backslash A_{m-1}}(A_m) \\
&= \sum_{\{\pi \in \Pi(\mathscr{M}) : \pi = (A_1, \pi'),\, \pi' \in \Pi(\mathscr{M}\backslash A_1)\}} P_{\mathscr{M}}(A_1) D_{\mathscr{M}\backslash A_1}(\pi') \\
&= \sum_{A_1 \in \mathscr{B}(\mathscr{M})} P_{\mathscr{M}}(A_1) \sum_{\pi' \in \Pi(\mathscr{M}\backslash A_1)} D_{\mathscr{M}\backslash A_1}(\pi') \\
&= \sum_{A_1 \in \mathscr{B}(\mathscr{M})} P_{\mathscr{M}}(A_1) = 1.
\end{aligned} \tag{3}$$

and, then, D is a probability distribution on $\Pi(\mathscr{M})$. $\square$

We deduce from the above proof, that every probability $D \in \mathscr{D}(\mathscr{M})$ determines, for each basic coalition $A_1 \in \mathscr{B}(\mathscr{M})$, the probability distribution $D_{\mathscr{M}\backslash A_1} \in \mathscr{D}(\mathscr{M}\backslash A_1)$ by (2). Besides that, if $\pi = (A_1, A_2, \ldots, A_k)$ is in $\Pi(\mathscr{M})$ then

$$D(\pi) = P_{\mathscr{M}}(A_1) D_{\mathscr{M}\backslash A_1}(\pi'), \tag{4}$$

where $\pi' = (A_2, \ldots, A_k) \in \Pi(\mathscr{M}\backslash A_1)$.

Finally, from (3) we obtain for $A_1 \in \mathscr{B}(\mathscr{M})$,

$$\sum_{\{\pi \in \Pi(\mathscr{M}) : \pi = (A_1, \ldots)\}} D(\pi) = P_{\mathscr{M}}(A_1). \tag{5}$$

In Lemma 2.1 we use one probability for every deletion minor to obtain a probability distribution on the basic sequences of a matroid $\mathscr{M}$. For instance, we can take the equitable distribution for all deletion minors and obtain, for all $\pi = (A_1, A_2, \ldots, A_k)$ of $\Pi(\mathscr{M})$,

$$D(\pi) = \frac{1}{b(\mathscr{M})} \prod_{m=1}^{k-1} \frac{1}{b(\mathscr{M}\backslash A_1 \backslash \cdots \backslash A_m)},$$

where $b(\mathscr{M}\backslash A_1\backslash\cdots\backslash A_m)$ is the number of basic coalitions in the deletion minor $\mathscr{M}\backslash A_1\backslash\cdots\backslash A_m$.

We can also propose that the probabilities on the deletion minors are given by an initial probability on the basic coalitions of the original matroid. Thus, we take $P_{\mathscr{M}} \in \mathscr{P}(\mathscr{M})$ and define the induced probability distribution on $\mathscr{M}\backslash A_1$ as

$$P_{\mathscr{M}\backslash A_1}(B) = \frac{w^{P_{\mathscr{M}}}(B)}{\sum_{B\in\mathscr{B}(\mathscr{M}\backslash A_1)} w^{P_{\mathscr{M}}}(B)},$$

for all $B \in \mathscr{B}(\mathscr{M}\backslash A_1)$. In the same form the rest of the probabilities are defined by induction and using (1) we give D built by $P_{\mathscr{M}}$.

Definition 2.4. *Let $\mathscr{M}$ be a matroid on N. A feasible coalition $S \in \mathscr{M}$ is called an isthmus coalition of $\mathscr{M}$ if $S \cap B \neq \emptyset$ for all $B \in \mathscr{B}(\mathscr{M})$. A player $i \in N$ is an isthmus player of $\mathscr{M}$ if $i \in B$ for all $B \in \mathscr{B}(\mathscr{M})$.*

The following theorem allows to calculate the dynamic influence of coalitions for a distribution D by recurrence using the static influences of the deletion minors, with the probabilities which define D by the formula (1).

Theorem 2.2. *Let $\mathscr{M}$ be a matroid and $D \in \mathscr{D}(\mathscr{M})$ a probability distribution such that for every basic sequence $\pi = (A_1, A_2, \ldots, A_k) \in \Pi(\mathscr{M})$ we have*

$$D(\pi) = P_{\mathscr{M}}(A_1)\prod_{m=2}^{k} P_{\mathscr{M}\backslash A_1\backslash\cdots\backslash A_{m-1}}(A_m).$$

Then, for every $S \in \mathscr{M}$, its dynamic influence $w^D(S)$ with respect to D is given by

$$w^D(S) = \begin{cases} w^{P_{\mathscr{M}}}(S), & \text{if } S \text{ is isthmus,} \\ w^{P_{\mathscr{M}}}(S) + \sum_{B\in\mathscr{B}(\mathscr{M}\backslash S)} P_{\mathscr{M}}(B) w^{D_{\mathscr{M}\backslash B}}(S), & \text{otherwise,} \end{cases} \tag{6}$$

where $D_{\mathscr{M}\backslash B}$ is the probability distribution on $\Pi(\mathscr{M}\backslash B)$ that originates from D according to formula (2).

Proof. Let $S \in \mathscr{M}$. In the case that S is an isthmus coalition in the matroid $\mathscr{M}$, we have that $\Pi_S(\mathscr{M}) = \bigcup_{B\in\mathscr{B}_S(\mathscr{M})}\{\pi \in \Pi(\mathscr{M}) : \pi = (B, \ldots)\}$. Then

$$\begin{aligned} w^D(S) &= \sum_{\pi\in\Pi_S(\mathscr{M})} D(\pi) \\ &= \sum_{B\in\mathscr{B}_S(\mathscr{M})} \sum_{\{\pi\in\Pi(\mathscr{M}):\pi=(B,\ldots)\}} D(\pi) \\ &= \sum_{B\in\mathscr{B}_S(\mathscr{M})} P_{\mathscr{M}}(B) = w^{P_{\mathscr{M}}}(S), \end{aligned}$$

using (5).

If S is not an isthmus in $\mathscr{M}$ then,

$$\begin{aligned}
w^{P_{\mathscr{M}}}(S) &+ \sum_{B \in \mathscr{B}(\mathscr{M}\backslash S)} P_{\mathscr{M}}(B) w^{D_{\mathscr{M}\backslash B}}(S) \\
&= \sum_{B \in \mathscr{B}_S(\mathscr{M})} P_{\mathscr{M}}(B) + \sum_{B \in \mathscr{B}(\mathscr{M}\backslash S)} P_{\mathscr{M}}(B) \sum_{\pi' \in \Pi_S(\mathscr{M}\backslash B)} D_{\mathscr{M}\backslash B}(\pi') \\
&= \sum_{\{\pi \in \Pi(\mathscr{M}) : \pi = (B,\ldots),\, B \in \mathscr{B}_S(\mathscr{M})\}} D(\pi) \\
&\quad + \sum_{B \in \mathscr{B}(\mathscr{M}\backslash S)} \sum_{\{\pi = (B,\pi') : \pi' \in \Pi_S(\mathscr{M}\backslash B)\}} P_{\mathscr{M}}(B) D_{\mathscr{M}\backslash B}(\pi') \\
&= \sum_{\{\pi \in \Pi(\mathscr{M}) : \pi = (B,\ldots),\, B \in \mathscr{B}_S(\mathscr{M})\}} D(\pi) \\
&\quad + \sum_{\{\pi \in \Pi_S(\mathscr{M}) : \pi = (B,\ldots),\, B \in \mathscr{B}(\mathscr{M}\backslash S)\}} D(\pi) \\
&= \sum_{\pi \in \Pi_S(\mathscr{M})} D(\pi) = w^D(S),
\end{aligned}$$

where, in the first term of the second equality we use (5). Besides that, the second term of the third equality is obtained by (4). □

3. The dynamic Shapley value

In the setting of the dynamic approach to a given matroid, as developed in Section 2, we aim to introduce axiomatically a unique value fully determined by four axioms. In comparison to the axioms of the classical Shapley value (cf. Shapley [6]) and the static Shapley value for games on matroids (cf. Bilbao et al. [1, Theorem 4.2]), only the efficiency axiom needs to be adapted slightly in accordance with the cornerstone of the dynamic model, being the set of basic sequences on the matroid. Notice that for any basic sequence of the form $\pi = (A_1, \ldots, A_k)$, the associated overall benefits in a game $v \in \Gamma(\mathscr{M})$ are given by $v(A_1) + \cdots + v(A_k)$, whereas its probability is given by $D(\pi)$ according to any probability distribution D on $\Pi(\mathscr{M})$.

Let $\Psi : \Gamma(\mathscr{M}) \to \mathbb{R}^N$, $\Psi = (\Psi_i)_{i \in N}$, be a value on the vector space $\Gamma(\mathscr{M})$ consisting of games on the matroid $\mathscr{M}$.

Axiom 1. (Linearity in the game space)
$\Psi(\alpha v + \beta w) = \alpha \Psi(v) + \beta \Psi(w)$ for all $v, w \in \Gamma(\mathscr{M})$ and $\alpha, \beta \in \mathbb{R}$.

A player $i \in N$ is called *dummy* in the game $v \in \Gamma(\mathscr{M})$ whenever it holds $v(S \cup \{i\}) - v(S) = v(\{i\})$ for all $S \in \mathscr{M}/i$.

Axiom 2. (Dummy player property)
For every dummy player i in a game $v \in \Gamma(\mathscr{M})$, it holds $\Psi_i(v) = v(\{i\})$.

For every feasible coalition $S \in \mathscr{M}$ we define the *unanimity game* u_S

$$u_S(T) = \begin{cases} 1, & T \in \mathscr{M},\ S \subseteq T, \\ 0, & \text{otherwise.} \end{cases}$$

Axiom 3. (Symmetry property applied to unanimity games)
For each $S \in \mathscr{M}$ and each pair $i, j \in S$ we have $\Psi_i(u_S) = \Psi_j(u_S)$.

Axiom 4. (Dynamic efficiency)
For each $D \in \mathscr{D}(\mathscr{M})$ and each $v \in \Gamma(\mathscr{M})$, it holds

$$\sum_{i \in N} \Psi_i(v) = \sum_{\pi \in \Pi(\mathscr{M})} D(\pi)(v(A_1) + \cdots + v(A_k)),$$

where $\pi = (A_1, \ldots, A_k) \in \Pi(\mathscr{M})$.

The next theorem states that there exists a unique value satisfying the above four axioms: the *dynamic Shapley value*. Moreover, we provide a recursive formula to calculate the dynamic Shapley value using a sequence of static Shapley values. We now recall some formulas for the static Shapley value obtained in Bilbao et al. [1].

Given a matroid $\mathscr{M}$ and a probability distribution over the set of basic coalitions $P_{\mathscr{M}} \in \mathscr{P}(\mathscr{M})$, the static Shapley value $Sh^{P_{\mathscr{M}}} = (Sh_i^{P_{\mathscr{M}}})_{i \in N}$ defined on $\Gamma(\mathscr{M})$ is given by

$$Sh_i^{P_{\mathscr{M}}}(v) = \sum_{B \in \mathscr{B}_i(\mathscr{M})} P_{\mathscr{M}}(B) Sh_i^B(v_B),$$

where $Sh_i^B(v_B)$ is the classical Shapley value for the cooperative game $v_B : 2^B \to \mathbb{R}$. Moreover, for the unanimity games u_S with $S \in \mathscr{M}$, this value satisfies

$$Sh_i^{P_{\mathscr{M}}}(u_S) = \begin{cases} \dfrac{w^{P_{\mathscr{M}}}(S)}{|S|}, & \text{if } i \in S, \\ 0, & \text{if } i \notin S. \end{cases} \tag{7}$$

Theorem 3.1. *Let $\mathscr{M}$ be a matroid and $D \in \mathscr{D}(\mathscr{M})$ a probability distribution such that for every basic sequence $\pi = (A_1, A_2, \ldots, A_k) \in \Pi(\mathscr{M})$ we have*

$$D(\pi) = P_{\mathscr{M}}(A_1) \prod_{m=2}^{k} P_{\mathscr{M} \setminus A_1 \setminus \cdots \setminus A_{m-1}}(A_m).$$

Then there exists a unique value $Sh^D = (Sh_i^D)_{i \in N}$ on $\Gamma(\mathscr{M})$ which satisfies the axioms 1, 2, 3 and 4. Moreover, for every game $v \in \Gamma(\mathscr{M})$ the dynamic Shapley value $Sh_i^D(v)$ of player i is given by, either $Sh_i^D(v) = Sh_i^{P_{\mathscr{M}}}(v)$ whenever i is an isthmus player of $\mathscr{M}$, or otherwise

$$Sh_i^D(v) = Sh_i^{P_{\mathscr{M}}}(v) + \sum_{B \in \mathscr{B}(\mathscr{M} \setminus i)} P_{\mathscr{M}}(B) Sh_i^{D_{\mathscr{M} \setminus B}}(v_{\mathscr{M} \setminus B}). \tag{8}$$

Proof. To prove the existence and uniqueness part, we use that the set of the unanimity games $\{u_S : S \in \mathcal{M}, S \neq \emptyset\}$ forms a basis of the vectorial space $\Gamma(\mathcal{M})$. Let $\Psi = (\Psi_i)_{i \in N}$ be a value satisfying the axioms 1, 2, 3 and 4, and let $S \in \mathcal{M}$ be a nonempty feasible coalition. We consider its associated unanimity game $u_S \in \Gamma(\mathcal{M})$. Obviously, any player $i \in N \backslash S$ is a dummy in the unanimity game u_S, and so, by the dummy player property for Ψ, it follows that $\Psi_i(u_S) = u_S(\{i\}) = 0$ for all $i \in N \backslash S$. Further, the symmetry property for Ψ yields $\Psi_i(u_S) = \Psi_j(u_S)$ for all $i, j \in S$. On the one hand, we obtain that

$$\sum_{j \in N} \Psi_j(u_S) = \sum_{j \in S} \Psi_j(u_S) = \Psi_i(u_S)|S|$$

for all $i \in S$. On the other hand, the dynamic efficiency for Ψ applied to the unanimity game u_S implies

$$\begin{aligned}\sum_{j \in N} \Psi_j(u_S) &= \sum_{\{\pi \in \Pi(\mathcal{M}) : \pi = (A_1, \dots, A_k)\}} D(\pi)(u_S(A_1) + \cdots + u_S(A_m)) \\ &= \sum_{\{T \in \mathcal{M} : S \subseteq T\}} \sum_{\{\pi \in \Pi(\mathcal{M}) : \pi = (A_1, \dots, T, \dots, A_k)\}} D(\pi) \\ &= w^D(S),\end{aligned}$$

where the last but one equality follows from the following property: every sum $u_S(A_1) + \cdots + u_S(A_k)$ is either zero or one (since a basic sequence yields a partition of N). Moreover, the sum $u_S(A_1) + \cdots + u_S(A_k) = 1$ if and only if $S \subseteq A_m$ for exactly one $m \in \{1, \dots, k\}$. From both reasonings, we conclude that

$$\Psi_i(u_S) = \begin{cases} \dfrac{w^D(S)}{|S|}, & \text{if } i \in S, \\ 0, & \text{if } i \notin S. \end{cases} \tag{9}$$

This formula determines Ψ for every unanimity game and the linearity property implies uniqueness. We denote this unique value by $Sh^D = (Sh_i^D)_{i \in N}$.

To prove the formulas which include the static Shapley value it is sufficient to work on the unanimity games. We know that for any $P_{\mathcal{M}} \in \mathcal{P}(\mathcal{M})$ and $S \in \mathcal{M}$, the static Shapley value $Sh^{P_{\mathcal{M}}}$ verifies (7).

If $i \in S$ is an isthmus player then S is an isthmus coalition too, and so

$$Sh_i^D(u_S) = \frac{w^D(S)}{|S|} = \frac{w^{P_{\mathcal{M}}}(S)}{|S|} = Sh_i^{P_{\mathcal{M}}}(u_S),$$

using the property (9) and the formula (6) in Theorem 2.2.

If $i \in S$ is not an isthmus player, then the recursive procedure in the dynamical model will end when i becomes an isthmus player in the last deletion minor. Thus, if $\mathcal{M}'$ is the last deletion minor then

$$Sh_i^{D_{\mathcal{M}'}}((u_S)_{\mathcal{M}'}) = \frac{w^{D_{\mathcal{M}'}}(S)}{|S|}.$$

Let us suppose that the above equality is true for all deletion minors $\mathscr{M}\backslash B$ with $B \in \mathscr{B}(\mathscr{M}\backslash i)$, i.e.

$$Sh_i^{D_{\mathscr{M}\backslash B}}((u_S)_{\mathscr{M}\backslash B}) = \frac{w^{D_{\mathscr{M}\backslash B}}(S)}{|S|},$$

and we prove the recursive formula for the matroid $\mathscr{M}$.

$$\begin{aligned} Sh_i^D(u_S) = \frac{w^D(S)}{|S|} &= \frac{1}{|S|}\left(w^{P_{\mathscr{M}}}(S) + \sum_{B \in \mathscr{B}(\mathscr{M}\backslash S)} P_{\mathscr{M}}(B) w^{D_{\mathscr{M}\backslash B}}(S)\right) \\ &= \frac{w^{P_{\mathscr{M}}}(S)}{|S|} + \sum_{B \in \mathscr{B}(\mathscr{M}\backslash S)} P_{\mathscr{M}}(B) \frac{w^{D_{\mathscr{M}\backslash B}}(S)}{|S|} \\ &= Sh_i^{P_{\mathscr{M}}}(u_S) + \sum_{B \in \mathscr{B}(\mathscr{M}\backslash i)} P_{\mathscr{M}}(B) Sh_i^{D_{\mathscr{M}\backslash B}}((u_S)_{\mathscr{M}\backslash B}), \end{aligned}$$

using the property (9) and the formula (6). We now consider $i \notin S$. If i is an isthmus then $Sh_i^D(u_S) = Sh_i^{P_{\mathscr{M}}}(u_S) = 0$. Otherwise, we suppose that $Sh_i^{D_{\mathscr{M}\backslash B}}((u_S)_{\mathscr{M}\backslash B}) = 0$ for all $B \in \mathscr{B}(\mathscr{M}\backslash i)$. Hence,

$$Sh_i^D(u_S) = Sh_i^{P_{\mathscr{M}}}(u_S) + \sum_{B \in \mathscr{B}(\mathscr{M}\backslash i)} P_{\mathscr{M}}(B) Sh_i^{D_{\mathscr{M}\backslash B}}((u_S)_{\mathscr{M}\backslash B}) = 0. \quad \square$$

The static Shapley value is given by

$$Sh_i^{P_{\mathscr{M}}}(v) = \sum_{B \in \mathscr{B}_i(\mathscr{M})} P_{\mathscr{M}}(B) Sh_i^B(v_B),$$

where $Sh_i^B(v_B)$ is the classical Shapley value on the cooperative game $v_B : 2^B \to \mathbb{R}$. For each player i we have that the set of basic coalitions which contain player i is $\mathscr{B}_i(\mathscr{M}) = \{B \cup i : B \in \mathscr{B}(\mathscr{M}/i)\}$ and the set of the basic coalitions not containing player i is $\mathscr{B}(\mathscr{M}\backslash i)$. The dynamic Shapley value for the distribution D can be written using the classical operations of matroids in the formulas proved in Theorem 3.1 as follows.

If i is an isthmus player of $\mathscr{M}$, then

$$Sh_i^D(v) = \sum_{B \in \mathscr{B}(\mathscr{M})} P_{\mathscr{M}}(B) Sh_i^B(v_B),$$

and otherwise, $Sh_i^D(v)$ is equal to

$$\sum_{B \in \mathscr{B}(\mathscr{M}/i)} P_{\mathscr{M}}(B \cup i) Sh_i^{B \cup i}(v) + \sum_{B \in \mathscr{B}(\mathscr{M}\backslash i)} P_{\mathscr{M}}(B) Sh_i^{D_{\mathscr{M}\backslash B}}(v_{\mathscr{M}\backslash B}).$$

The i-coordinate of the Shapley value for D is the sum of a static contraction term on $\mathscr{M}$, and a recurrence dynamic deletion term on $\mathscr{M}\backslash i$. In the last step of the recurrence player i is a dummy player in the games $v_{\mathscr{M}\backslash B}$ and hence the dynamic term is zero.

Definition 3.1. *A game on a matroid $v \in \Gamma(\mathcal{M})$ is weakly superadditive if $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \in \mathcal{M}$ with $S \cap T = \emptyset$ and $S \cup T \in \mathcal{M}$.*

In the next proposition we show that our dynamic Shapley value for this class of games verifies the *individual rationality principle*, i.e. the payments are at least the individual worths.

Proposition 3.2. *Let $\mathcal{M}$ be a matroid and $D \in \mathcal{D}(\mathcal{M})$. If $v \in \Gamma(\mathcal{M})$ is weakly superadditive then $Sh_i^D(v) \geq v(\{i\})$ for all $i \in N$.*

Proof. Let $P_{\mathcal{M}} \in \mathcal{P}(\mathcal{M})$. First, note that the classical Shapley value verifies the individual rationality principle on the class of superadditive games. Then the components of the static Shapley value satisfy

$$\begin{aligned} Sh_i^{P_{\mathcal{M}}}(v) &= \sum_{B \in \mathcal{B}_i(\mathcal{M})} P_{\mathcal{M}}(B) Sh_i^B(v_B) \\ &\geq \left(\sum_{B \in \mathcal{B}_i(\mathcal{M})} P_{\mathcal{M}}(B) \right) v(\{i\}) \\ &= w^{P_{\mathcal{M}}}(\{i\}) v(\{i\}). \end{aligned}$$

If i is an isthmus player then $Sh_i^D(v) = Sh_i^{P_{\mathcal{M}}}(v) \geq v(\{i\})$, because for any isthmus player $w^{P_{\mathcal{M}}}(\{i\}) = 1$. When i is not isthmus in $\mathcal{M}$ the property is true in the last step of the recurrence that defines the dynamic Shapley value. We suppose that it is true for all deletion minors different of $\mathcal{M}$ and we will prove this property for $\mathcal{M}$,

$$\begin{aligned} Sh_i^D(v) &= Sh_i^{P_{\mathcal{M}}}(v) + \sum_{B \in \mathcal{B}(\mathcal{M} \setminus i)} P_{\mathcal{M}}(B) Sh_i^{D_{\mathcal{M} \setminus B}}(v_{\mathcal{M} \setminus B}) \\ &\geq w^{P_{\mathcal{M}}}(\{i\}) v(\{i\}) + \sum_{B \in \mathcal{B}(\mathcal{M} \setminus i)} P_{\mathcal{M}}(B) v(\{i\}) \\ &= \left(\sum_{B \in \mathcal{B}_i(\mathcal{M})} P_{\mathcal{M}}(B) + \sum_{B \in \mathcal{B}(\mathcal{M} \setminus i)} P_{\mathcal{M}}(B) \right) v(\{i\}) \\ &= v(\{i\}). \quad \square \end{aligned}$$

In [1] we introduced the following concept of individual value for games in $\Gamma(\mathcal{M})$. If $\lambda_i \in [0, 1]$, an individual value $\psi_i : \Gamma(\mathcal{M}) \to \mathbb{R}$ for player $i \in N$, is a *λ_i-quasi-probabilistic value* if there exists a collection of positive numbers $\{p_S^i \geq 0 : S \in \mathcal{M}/i\}$ with $\sum_{S \in \mathcal{M}/i} p_S^i = \lambda_i$ such that

$$\Psi_i(v) = \sum_{S \in \mathcal{M}/i} p_S^i [v(S \cup i) - v(S)].$$

In particular, the *probabilistic values* (Weber [7]) are the individual values ψ_i such that $\lambda_i = 1$. Bilbao et al. proved in [1, Theorem 3.1] that this set of values is defined by the following axioms: *linearity, λ_i-dummy player property* (if i is a dummy player for $v \in \Gamma(\mathcal{M})$ then $\Psi_i(v) = \lambda_i v(\{i\})$) and *monotonicity* (if $v \in \Gamma(\mathcal{M})$ verifies $v(S) \leq v(T)$ for all $S, T \in \mathcal{M}$ with $S \subseteq T$, then $\Psi_i(v) \geq 0$). Furthermore, [1, Theorem 4.2] shows that the static Shapley value $Sh^{P_{\mathcal{M}}}$ for the distribution $P_{\mathcal{M}}$ is a value such that its components $Sh_i^{P_{\mathcal{M}}}$ are $w^P(\{i\})$-quasi-probabilistic. We now prove that the components of the dynamic Shapley value are probabilistic values in the above sense.

Proposition 3.3. *Let $\mathcal{M}$ be a matroid and $D \in \mathcal{D}(\mathcal{M})$. Then, the components of the dynamic Shapley value are probabilistic values.*

Proof. Theorem 3.1 implies that the components of the dynamic Shapley value for D verify linearity and dummy player. Then, by using [1, Theorem 3.1] we only need to prove that they satisfy monotonicity. Let $v \in \Gamma(\mathcal{M})$ be a monotonic game and $i \in N$. We have to see that $Sh_i^D(v) \geq 0$. Using the recurrence (8) of $Sh_i^D(v)$ and the fact that the static values verify this property we obtain the property. □

We remark that [1, Theorem 3.2] implies that there exists a probability distribution P^i on $\mathcal{B}(\mathcal{M})$ such that $Sh_i^D(v) = \sum_{B \in \mathcal{B}_i(\mathcal{M})} P^i(B) Sh_i^B(v)$. Finally, our new value can be written like a linear combination of the marginal contributions following [1, Theorem 3.1].

4. Summary and concluding remarks

Firstly, the paper dealt with a dynamic approach to a given matroid $\mathcal{M}$ such that, in a recursive manner, an arbitrarily chosen basic coalition is removed from the (resulting deletion) matroid, until the "empty-set" matroid arises. We are concerned with the set $\Pi(\mathcal{M})$ of all such basic sequences on the matroid $\mathcal{M}$ as well as the associated set $\mathcal{D}(\mathcal{M})$ of probability distributions on $\Pi(\mathcal{M})$. Secondly, with respect to a given probability distribution D on $\Pi(\mathcal{M})$, we introduced the so-called dynamic Shapley value $Sh^D(v)$ for games v on the matroid $\mathcal{M}$ as the unique value satisfying the following four axioms: (1) Linearity on the game space. (2) Dummy player property. (3) Symmetry property applied to unanimity games. (4) Dynamic efficiency which is formulated as an expected payoff in terms of the set $\Pi(\mathcal{M})$ of all basic sequences on $\mathcal{M}$ and the fixed probability distribution D as well.

Concerning the unanimity game u_S, associated with the feasible coalition $S \in \mathcal{M}$, its dynamic Shapley value $Sh^D(u_S)$ allocates nothing to the non-members of S (since they are treated as dummies), whereas members of S are paid the average of the so-called dynamic influence of coalition S with respect to the given probability distribution D (denoted by $w^D(S)$). In fact, the dynamic influence of S w.r.t. D represents the overall payoff according to the dynamic Shapley value for the unanimity game u_S.

A first main result states that, for any feasible coalition S, its dynamic influence $w^D(S)$ can be related, in a recursive manner, to some static influence $w^{P_{\mathcal{M}}}(S)$ (where D is replaced by an induced probability distribution $P_{\mathcal{M}}$ on the set $\mathcal{B}(\mathcal{M})$ of basic coalitions of the matroid $\mathcal{M}$). To be more precise, the dynamic and static influence do agree for isthmus coalitions which intersect every basic coalition of the matroid $\mathscr{M}$. For any non-isthmus coalition S, the dynamic influence differs from the static influence by some amount which is composed as the sum, over all basic coalitions B of the deletion matroid $\mathscr{M}\backslash S$, of the product between its probability $P_{\mathscr{M}}(B)$ and the dynamic influence of coalition S with reference to some induced probability distribution $D_{\mathscr{M}\backslash B}$ on the set of all basic sequences on the deletion matroid $\mathscr{M}\backslash B$.

Consequently, the second main result states that the dynamic Shapley value $Sh_i^D(v)$ of a player $i \in N$ in an arbitrary game v on the matroid $\mathscr{M}$ can be related, in a recursive manner, to the static Shapley value $Sh_i^{P_{\mathscr{M}}}(v)$ as introduced in Bilbao, Driessen, Jiménez-Losada and Lebrón [1]. More exactly, the dynamic and static Shapley values do agree for isthmus players who belong to every basic coalition of the matroid $\mathscr{M}$. For any non-isthmus player i, the dynamic Shapley value differs from the static Shapley value by some amount which is composed as the sum, over all basic coalitions B of the deletion matroid $\mathscr{M}\backslash i$, of the product between its probability $P_{\mathscr{M}}(B)$ and the dynamic Shapley value of player i with reference to some induced probability distribution $D_{\mathscr{M}\backslash B}$ on the set of all basic sequences on the deletion matroid $\mathscr{M}\backslash B$.

Acknowledgments. We would like to express our gratitude to an anonymous referee for interesting comments and for their careful reading of the manuscript.

References

[1] Bilbao JM, Driessen TSH, Jiménez-Losada A, Lebrón E (2001) The Shapley value for games on matroids: The static model. Math. Meth. Oper. Res. 53:333–348

[2] Driessen TSH (1988) Cooperative games, solutions and applications. Kluwer Academic Publishers, Dordrecht, The Netherlands

[3] Faigle U (1989) Cores of games with restricted cooperation. ZOR–Meth. Mod. Oper. Res. 33:405–422

[4] Korte B, Lovász L, Schrader R (1991) Greedoids. Springer-Verlag, Berlin

[5] Nagamochi H, Zeng D-Z, Kabutoya N, Ibaraki T (1997) Complexity of the minimum base game on matroids. Math. of Oper. Res. 22:146–164

[6] Shapley LS (1953) A value for *n*-person games. In: Kuhn HW, Tucker AW (eds) Contributions to the theory of games II. Princeton University Press, Princeton NJ, pp. 307–317

[7] Weber RJ (1988) Probabilistic values for games. In: Roth AE (ed) The Shapley value: essays in honor of LS Shapley. Cambridge University Press, Cambridge UK, pp. 101–119

[8] Welsh DJA (1976) Matroid theory. Academic Press, London