SOME APPLICATIONS OF CONVEX ANALYSIS TO COOPERATIVE GAME THEORY

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Abstract. We review the application of some fundamental concepts of *discrete convex analysis* to cooperative game theory, as well as several representations of n-person cooperative games by convex functions of n variables.

1. Introduction

This paper surveys the fundamentals of discrete convex analysis, as developed by Fujishige [9, 10, 11], Martínez-Legaz and Singer [19] and Murota [20, 21, 22], and discusses some representations of n-person cooperative games by functions of n variables [15, 16, 17, 1, 18].

The first works on discrete convex analysis in the sense we consider in this paper are those by Fujishige [9, 10]. In [9], he addressed optimization problems on distributive lattices, and constructed a conjugation theory for submodular (and supermodular) functions that is applicable to problems with submodular objective functions. In particular, Fujishige proved a Fenchel-type min-max duality theorem, which is actually equivalent to Frank's discrete separation (or sandwich) theorem. The same author defined a subdifferential for submodular functions, in terms of which he obtained a Karush-Kuhn-Tucker type theorem for constrained optimization problems. In a subsequent paper [10], Fujishige studied the geometric structure of that subdifferential and established the relationship between the subdifferential of a submodular function and the convex subdifferential of the Lovász extension [14] of the function.

In [19], Martínez-Legaz and Singer developed an abstract conjugation theory, and considered the special case when the functions are defined on $\{0,1\}^n$; the resulting theory is essentially equivalent to that of Fujishige. They showed that the submodularity assumption under which this author proves that a function coincides with its second conjugate can be removed.

In more recent times, Murota [20, 21, 22] has continued the study of discrete convex analysis, following an approach basically identical to that described above;

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in particular, he has extended the essentials of that theory to the context of valuated matroids in the sense of Dress and Wenzel [3, 4]. For a comprehensive survey of Murota's contributions, the reader is referred to [22].

Several dual representations of cooperative games are proposed in [15, 16, 2]. Chronologically, the first one is the indirect function [15], which is closely related to the conjugation theory of discrete convex analysis. The maximum average value function [16] constitutes another dual representation of a (nontrivial nonnegative) cooperative game by a convex function but, unlike the indirect function, which faithfully represents any game, the maximum average value function requires totally balancedness of the game for the full validity of the duality relations. The least increment function [1], another convex function associated to a cooperative game, shares with the maximum average value function the fact that it only preserves all the information on the game if the game is totally balanced.

In the present paper, we present the basics of discrete convex analysis in the language of cooperative game theory. Section 2 deals with conjugation theory, and Section 3 studies subdifferentials. Sections 4, 5 and 6 present indirect, least increment and maximum average value functions, respectively. Section 7 discusses still another representation of n-person games by functions of n-variables, namely, by their Lovász extensions, which are concave in the case of convex games.

2. Conjugation

In this section, we present a generalization to set functions of the Fenchel conjugation and the associated duality theory. This general duality framework, due to Fujishige [9], Martínez-Legaz and Singer [19], and Murota [21], motivates us to investigate solution concepts of cooperative games from the view point of convex analysis.

Let $N = \{1, \ldots, n\}$ be a finite set and $\mathcal{F} \subseteq 2^N$ be a nonempty family. A game on $\mathcal{F}$ is a function $v : \mathcal{F} \to \mathbb{R}$ satisfying the condition $v(\emptyset) = 0$. In view of economic applications, it is convenient to distinguish between profit games and cost games. Since the definitions of most concepts relative to games depend on whether the game in question is regarded as a profit or a cost game, one should make that distinction explicit at a formal level. To this aim one can redefine, in a more rigorous way, a game as a pair (v, σ) consisting of a function $v : \mathcal{F} \to \mathbb{R}$ and a sign $\sigma \in \{1, -1\}$ indicating whether the game is a profit ($\sigma = 1$) or a cost ($\sigma = -1$) game Even though we adopt this point of view, we shall always identify the game (v, σ) with v, and shall distinguish between profit and cost games by explicitly mentioning the type. When the type will not be mentioned, we shall be referring to games that may be of either type.

In the following two definitions, as in the sequel, for $x = (x_1, \ldots, x_n)$ one denotes $x(S) = \sum_{i \in S} x_i$.

Definition 2.1. *The (concave) conjugate function of a profit game* $v : \mathcal{F} \to \mathbb{R}$ *is* $v^\circ : \mathbb{R}^n \to \mathbb{R}$, *given by*

$$v^\circ(x) = \min\{x(S) - v(S) : S \in \mathcal{F}\}.$$

Definition 2.2. *The (convex) conjugate function of a cost game* $c : \mathcal{F} \to \mathbb{R}$ *is* $c^\bullet : \mathbb{R}^n \to \mathbb{R}$, *given by*

$$c^\bullet(x) = \max\{x(S) - c(S) : S \in \mathcal{F}\}.$$

For the functions $v^{\circ} : \mathbb{R}^n \to \mathbb{R}$ and $c^{\bullet} : \mathbb{R}^n \to \mathbb{R}$, we consider their concave and convex conjugate functions, respectively [23]. These conjugate functions $(v^{\circ})^{\circ} : \mathbb{R}^n \to \mathbb{R}$ and $(c^{\bullet})^{\bullet} : \mathbb{R}^n \to \mathbb{R}$ are given by

$$\begin{aligned} v^{\circ\circ}(y) &= \inf\{\langle y, x\rangle - v^{\circ}(x) : x \in \mathbb{R}^n\} \\ c^{\bullet\bullet}(y) &= \sup\{\langle y, x\rangle - c^{\bullet}(x) : x \in \mathbb{R}^n\}. \end{aligned}$$

To every coalition $S \subseteq N$ is associated its *characteristic vector* $\mathbf{1}_S \in \{0,1\}^n$, where $\mathbf{1}_S(i) = 1$ if $i \in S$, and $\mathbf{1}_S(i) = 0$ if $i \notin S$. We denote by $\widehat{\mathcal{F}} = \operatorname{conv}(\mathcal{F})$ the convex hull of the collection of characteristic vectors $\{\mathbf{1}_S : S \in \mathcal{F}\} \subseteq \mathbb{R}^n$, that is,

$$\begin{aligned} \widehat{\mathcal{F}} &= \left\{ y \in \mathbb{R}^n : y = \sum_{S \in \mathcal{F}} \lambda_S \mathbf{1}_S,\ \lambda \in \Lambda_{\mathcal{F}} \right\}, \text{ where} \\ \Lambda_{\mathcal{F}} &= \left\{ \lambda \in \mathbb{R}^{\mathcal{F}} : \sum_{S \in \mathcal{F}} \lambda_S = 1,\ \lambda_S \geq 0 \text{ for all } S \in \mathcal{F} \right\}. \end{aligned}$$

Since $\mathcal{F}$ is finite, $v^{\circ}(x) = \min\{\langle \mathbf{1}_S, x\rangle - v(S) : S \in \mathcal{F}\}$ is a polyhedral concave function (minimum of a finite number of affine functions) and $c^{\bullet}$ is a polyhedral convex function (maximum of a finite number of affine functions).

Stoer and Witzgall [26, Section 4.8] proved that the conjugates of a polyhedral function satisfy the following relations:

$$v^{\circ\circ}(y) = \begin{cases} \max\left\{\sum_{S\in\mathcal{F}} \lambda_S\, v(S) : y = \sum_{S\in\mathcal{F}} \lambda_S \mathbf{1}_S,\ \lambda \in \Lambda_{\mathcal{F}}\right\} & \text{if } y \in \widehat{\mathcal{F}} \\ -\infty & \text{if } y \notin \widehat{\mathcal{F}}. \end{cases} \quad (1)$$

$$c^{\bullet\bullet}(y) = \begin{cases} \min\left\{\sum_{S\in\mathcal{F}} \lambda_S\, c(S) : y = \sum_{S\in\mathcal{F}} \lambda_S \mathbf{1}_S,\ \lambda \in \Lambda_{\mathcal{F}}\right\} & \text{if } y \in \widehat{\mathcal{F}} \\ +\infty & \text{if } y \notin \widehat{\mathcal{F}}. \end{cases} \quad (2)$$

The following proposition shows that $v^{\circ\circ}$ (or $c^{\bullet\bullet}$) is an extension of v (or c), therefore the correspondence between a set function v (or c) and its concave (or convex) conjugate v° (or $c^{\bullet}$) is one to one.

Proposition 2.1. *For a profit game $v : \mathcal{F} \to \mathbb{R}$ and a cost game $c : \mathcal{F} \to \mathbb{R}$ we have:*

1. $v^{\circ\circ}(\mathbf{1}_S) = v(S)$ *for all* $S \in \mathcal{F}$.
2. $c^{\bullet\bullet}(\mathbf{1}_S) = c(S)$ *for all* $S \in \mathcal{F}$.
3. $\max\left\{v^{\circ\circ}(y) : y \in \widehat{\mathcal{F}}\right\} = \max\{v(S) : S \in \mathcal{F}\}$.
4. $\min\left\{c^{\bullet\bullet}(y) : y \in \widehat{\mathcal{F}}\right\} = \min\{c(S) : S \in \mathcal{F}\}$.
5. $\min\left\{v^{\circ\circ}(y) : y \in \widehat{\mathcal{F}}\right\} = \min\{v(S) : S \in \mathcal{F}\}$.
6. $\max\left\{c^{\bullet\bullet}(y) : y \in \widehat{\mathcal{F}}\right\} = \max\{c(S) : S \in \mathcal{F}\}$.

Proof. Assertions 1 and 2 are immediate consequences of (1) and (2), taking into account that the maxima and minima of a linear function over a convex polytope are attained at some vertices. Assertion 3 follows from 1 together with (1); assertion 4 can be proved in an analogous way. The minimum of concave functions and the maximum of convex functions on $\widehat{\mathcal{F}} = \mathrm{conv}(\mathcal{F})$ are attained on $\mathcal{F}$; hence relations 1 and 2 imply 5 and 6, respectively. □

Note that assertions 1 and 2 of this proposition show that $v : \mathcal{F} \to \mathbb{R}$ and $c : \mathcal{F} \to \mathbb{R}$ can be recovered from their conjugates, namely, for any $S \in \mathcal{F}$ one has

$$\begin{aligned} v(S) &= \min\{x(S) - v^{\circ}(x) : x \in \mathbb{R}^n\}, \\ c(S) &= \max\{x(S) - c^{\bullet}(x) : x \in \mathbb{R}^n\} \end{aligned} \tag{3}$$

(the minimum and the maximum in the preceding expressions are indeed attained, since the functions v° and $c^{\bullet}$ are polyhedral).

Let $\mathcal{F}$ and $\mathcal{G}$ be nonempty families of subsets of N. We shall introduce a Fenchel type theorem into a form of primal and dual program. Given a cost game $c : \mathcal{F} \to \mathbb{R}$, a profit game $v : \mathcal{G} \to \mathbb{R}$, and their conjugate functions $c^{\bullet}$ and v°, respectively, we consider

$$\begin{array}{lll} \text{Primal problem:} & \textit{Find} & \min\{c(S) - v(S) : S \in \mathcal{F} \cap \mathcal{G}\}, \\ \text{Dual problem:} & \textit{Find} & \max\{v^{\circ}(x) - c^{\bullet}(x) : x \in \mathbb{R}^n\}. \end{array}$$

Murota [21] showed the following weak duality result by using the Fenchel's duality theorem of standard convex analysis.

Proposition 2.2. *For a cost game $c : \mathcal{F} \to \mathbb{R}$, a profit game $v : \mathcal{G} \to \mathbb{R}$, and their conjugate functions $c^{\bullet}$ and v°, we have*

$$\begin{aligned} \min\{c(S) - v(S) : S \in \mathcal{F} \cap \mathcal{G}\} &\geq \min\left\{c^{\bullet\bullet}(y) - v^{\circ\circ}(y) : y \in \widehat{\mathcal{F}} \cap \widehat{\mathcal{G}}\right\} \\ &= \max\{v^{\circ}(x) - c^{\bullet}(x) : x \in \mathbb{R}^n\} \\ &\geq \max\{v^{\circ}(x) - c^{\bullet}(x) : x \in \mathbb{Z}^n\}. \end{aligned}$$

Proof. Proposition 2.1 implies that

$$c(S) - v(S) = c^{\bullet\bullet}(\mathbf{1}_S) - v^{\circ\circ}(\mathbf{1}_S)$$

for all $S \in \mathcal{F} \cap \mathcal{G}$ and this gives the first inequality. The next equality is the duality theorem of Fenchel [26, Section 5.1] and the last inequality is obvious. □

Shapley [24] introduced convex (profit) games as follows:

Definition 2.3. *A profit game $v : 2^N \to \mathbb{R}$ is called convex if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subseteq N$. A cost game $c : 2^N \to \mathbb{R}$ is concave if the reverse inequality holds.*

Definition 2.4. *The vector rank function of a concave cost game $c : 2^N \to \mathbb{R}$ is $r_c : \mathbb{R}^n \to \mathbb{R}$, given by*

$$r_c(u) = \min\{c(S) + u(N \setminus S) : S \subseteq N\}.$$

For a cost game $c : 2^N \to \mathbb{R}$ and a profit game $v : 2^N \to \mathbb{R}$ we define the polyhedra

$$\begin{aligned} P(c) &= \{x \in \mathbb{R}^n : x(S) \leq c(S) \text{ for all } S \subseteq N\}, \\ P(v) &= \{x \in \mathbb{R}^n : x(S) \geq v(S) \text{ for all } S \subseteq N\}, \\ C(c) &= \{x \in P(c) : x(N) = c(N)\}, \\ C(v) &= \{x \in P(v) : x(N) = v(N)\}. \end{aligned}$$

The polyhedra $C(c)$ and $C(v)$ are called the cores of the respective games. All these sets can be easily expressed in terms of conjugates:

$$\begin{aligned} P(c) &= \{x \in \mathbb{R}^n : c^{\bullet}(x) = 0\}, \\ P(v) &= \{x \in \mathbb{R}^n : v^{\circ}(x) = 0\}, \\ C(c) &= \{x \in \mathbb{R}^n : c^{\bullet}(x) = 0,\ x(N) = c(N)\}, \\ C(v) &= \{x \in \mathbb{R}^n : v^{\circ}(x) = 0,\ x(N) = v(N)\}. \end{aligned} \tag{4}$$

The function r_c is related to $P(c)$ by the min-max equation

$$r_c(u) = \max\{x(N) : x \in P(c),\ x \leq u\} \qquad (u \in \mathbb{R}^n), \tag{5}$$

which is an immediate consequence of the following generalization of the intersection theorem for polymatroids due to Edmonds [6]:

Theorem 2.3. *For distributive lattices $\mathcal{F}_1, \mathcal{F}_2 \subseteq 2^N$, let $c_1 : \mathcal{F}_1 \to \mathbb{R}$, $c_2 : \mathcal{F}_2 \to \mathbb{R}$ be concave cost games. If there exists a set $S \in \mathcal{F}_1$ such that $N \setminus S \in \mathcal{F}_2$, then we have*

$$\begin{aligned} &\min\{c_1(S) + c_2(N \setminus S) \,:\, S \in \mathcal{F}_1,\ N \setminus S \in \mathcal{F}_2\} \\ &\qquad = \max\{x(N) : x \in P(c_1) \cap P(c_2)\}. \end{aligned}$$

Moreover, if c_1 and c_2 are integer valued, then the maximum is attained by an integral vector.

Proof. See Fujishige [11, Theorem 4.9]. □

Fujishige [11, Theorem 4.12] proved the following Frank's discrete separation theorem [8] by using Theorem 2.3.

Theorem 2.4. *For distributive lattices $\mathcal{F}, \mathcal{G} \subseteq 2^N$ with $\emptyset, N \in \mathcal{F} \cap \mathcal{G}$, let $c : \mathcal{F} \to \mathbb{R}$ be a concave cost game and let $v : \mathcal{G} \to \mathbb{R}$ be a convex profit game. If $v(S) \leq c(S)$ for all $S \in \mathcal{F} \cap \mathcal{G}$, there exists $x \in \mathbb{R}^n$ such that $v(S) \leq x(S)$ for all $S \in \mathcal{G}$ and $x(S) \leq c(S)$ for all $S \in \mathcal{F}$.*

Proof. We consider the concave cost game defined by $v^D(N \setminus S) = v(N) - v(S)$ for all $S \in \mathcal{G}$, called the dual game of v. >From Theorem 2.3, we deduce that if $v \leq c$ then

$$\begin{aligned} \max\left\{x(N) : x \in P(c) \cap P\left(v^D\right)\right\} &= \min\left\{c(S) + v^D(N \setminus S) \,:\, S \in \mathcal{F} \cap \mathcal{G}\right\} \\ &= \min\{c(S) + v(N) - v(S) \,:\, S \in \mathcal{F} \cap \mathcal{G}\} \\ &= v(N). \end{aligned}$$

Then there exists $x \in P(c) \cap P\left(v^D\right)$ such that $x(N) = v(N)$ and hence

$$x \in P(c) \cap C\left(v^D\right) = P(c) \cap C(v).$$

It follows from the definitions that x satisfies the required conditions. □

We end this section with a Fenchel-type min-max duality theorem proved by Fujishige [9, Theorem 6.3].

Theorem 2.5. *For distributive lattices $\mathcal{F}, \mathcal{G} \subseteq 2^N$ with $\emptyset, N \in \mathcal{F} \cap \mathcal{G}$, let $c : \mathcal{F} \to \mathbb{R}$ be a concave cost game and let $v : \mathcal{G} \to \mathbb{R}$ be a convex profit game. Then we have*

$$\min\{c(S) - v(S) : S \in \mathcal{F} \cap \mathcal{G}\} = \max\{v^{\circ}(x) - c^{\bullet}(x) : x \in \mathbb{R}^n\}. \tag{6}$$

Moreover, if c and v are integer valued, then the maximum is attained at a vector $x^* \in \mathbb{Z}^n$.

Proof. Let v^D be dual of v defined in the proof of the preceding theorem. We observe that the min-max equation (6) is equivalent to

$$\min\left\{c(S)+v^D(N\setminus S):S\in\mathcal{F}\cap\mathcal{G}\right\}=\max\left\{v^\circ(x)+v(N)-c^\bullet(x):x\in\mathbb{R}^n\right\}.$$

If r_c and r_{v^D} are the vector rank functions of the concave cost games c and v^D, respectively, then for all $x\in\mathbb{R}^n$,

$$\begin{aligned}
c^\bullet(x) &= \max\{x(S)-c(S):S\in\mathcal{F}\}=-\min\{c(S)-x(S):S\in\mathcal{F}\}\\
&= x(N)-\min\{c(S)+x(N\setminus S):S\in\mathcal{F}\}=x(N)-r_c(x).\\
v^\circ(x) &= \min\{x(S)+v(N)-v(S):S\in\mathcal{G}\}-v(N)\\
&= \min\left\{v^D(N\setminus S)+x(S):S\in\mathcal{G}\right\}-v(N)=r_{v^D}(x)-v(N).
\end{aligned}$$

It therefore follows that the min-max equation (6) is equivalent to

$$\min\left\{c(S)+v^D(N\setminus S):S\in\mathcal{F}\cap\mathcal{G}\right\}=\max\left\{r_c(x)+r_{v^D}(x)-x(N):x\in\mathbb{R}^n\right\}.$$

Let $x\in\mathbb{R}^n$. Then, equation (5) implies that

$$\begin{aligned}
r_c(x) &= \max\{y(N):y\in P(c),\ y\le x\},\\
r_{v^D}(x) &= \max\left\{z(N):z\in P(v^D),\ z\le x\right\},
\end{aligned}$$

and there exist vectors $y,z\in\mathbb{R}^n$ such that

$$\begin{aligned}
r_c(x) &= y(N),\ y\in P(c),\ y\le x,\\
r_{v^D}(x) &= z(N),\ z\in P(v^D),\ z\le x.
\end{aligned}$$

By taking $u=\min\{y,z\}$, we deduce that $u\in P(c)\cap P\left(v^D\right)$. From the above relations we obtain

$$\begin{aligned}
r_c(x)+r_{v^D}(x)-x(N) &= y(N)+z(N)-x(N)\\
&= u(N)+\max\{y,z\}(N)-x(N)\\
&\le u(N)=r_c(u)+r_{v^D}(u)-u(N).
\end{aligned}$$

Therefore, the min-max equation (6) is equivalent to

$$\min\left\{c(S)+v^D(N\setminus S):S\in\mathcal{F}\cap\mathcal{G}\right\}=\max\left\{u(N):u\in P(c)\cap P\left(v^D\right)\right\},$$

which is true due to Theorem 2.3. We may apply this theorem and the above equation to obtain the integrality property. □

3. Subdifferentials

Fujishige [9, 10] introduced the concepts of subgradient and subdifferential for submodular functions and Murota [21] applied these concepts to set functions. In the context of cooperative games, the corresponding definitions are as follows:

Definition 3.1. *Let $\mathcal{F}\subseteq 2^N$, $v:\mathcal{F}\to\mathbb{R}$ be a profit game and $c:\mathcal{F}\to\mathbb{R}$ be a cost game. The vector $x\in\mathbb{R}^n$ is a subgradient of v at $S\in\mathcal{F}$ if $v(T)-v(S)\le x(T)-x(S)$ for all $T\in\mathcal{F}$. The vector $x\in\mathbb{R}^n$ is a subgradient of c at $S\in\mathcal{F}$ if $c(T)-c(S)\ge x(T)-x(S)$ for all $T\in\mathcal{F}$.*

Definition 3.2. *The subdifferential $\partial g(S)$ of the game $g:\mathcal{F}\to\mathbb{R}$ at $S\in\mathcal{F}$ is the set of all the subgradients of g at S.*

Note that $\mathbb{R}^n$ is divided into $|\mathcal{F}|$ unbounded polyhedra $\partial g(S)$, for $S \in \mathcal{F}$. Moreover, if $S, T \in \mathcal{F}$ and $S \neq T$ then the subdifferentials $\partial g(S)$ and $\partial g(T)$ may have common faces but not common interior vectors.

Definition 3.3. *The dual of a profit game $v : 2^N \to \mathbb{R}$ is the cost game $v^D : 2^N \to \mathbb{R}$ given by $v^D(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$. The dual of a cost game $c : 2^N \to \mathbb{R}$ is the profit game $c^D : 2^N \to \mathbb{R}$ given by $c^D(S) = c(N) - c(N \setminus S)$ for all $S \subseteq N$.*

Proposition 3.1. *Let $v : 2^N \to \mathbb{R}$ be a profit game and $c : 2^N \to \mathbb{R}$ be a cost game with dual games v^D and c^D, respectively. Then*

1. $\partial v(\emptyset) = P(v)$ *and* $\partial v(N) = P\left(v^D\right)$,
2. $\partial c(\emptyset) = P(c)$ *and* $\partial c(N) = P\left(c^D\right)$,
3. $\partial v(\emptyset) \cap \partial v(N) = C(v) = C\left(v^D\right)$,
4. $\partial c(\emptyset) \cap \partial c(N) = C(c) = C\left(c^D\right)$.

Proof. 1. Notice that $v(\emptyset) = x(\emptyset) = 0$ imply that

$$\partial v(\emptyset) = \{x \in \mathbb{R}^n : v(T) \leq x(T) \text{ for all } T \subseteq N\} = P(v).$$

Thus

$$\begin{aligned} \partial v(N) &= \{x \in \mathbb{R}^n : v(T) - v(N) \leq x(T) - x(N) \text{ for all } T \subseteq N\} \\ &= \{x \in \mathbb{R}^n : x(N \setminus T) \leq v(N) - v(T) \text{ for all } T \subseteq N\} \\ &= \left\{x \in \mathbb{R}^n : x(S) \leq v^D(S) \text{ for all } S \subseteq N\right\} = P\left(v^D\right). \end{aligned}$$

2. The proof is similar to that of 1 above.

Properties 1, 2 together give 3 and 4. □

Fujishige [11] proved the following characterization of the extreme points of the subdifferentials for concave cost games.

Theorem 3.2. *Let $c : \mathcal{F} \to \mathbb{R}$ be a concave cost game on a distributive lattice $\mathcal{F} \subseteq 2^N$. For each $S \in \mathcal{F}$, a vector $x \in \mathbb{R}^n$ is an extreme point of $\partial c(S)$ if and only if there exists an ordering of $N = \{e_1, \ldots, e_n\}$ such that the components of x are $x(e_i) = c(S_i) - c(S_{i-1})$, where $S_i = \{e_1, \ldots, e_i\}$ for $1 \leq i \leq n$, $S_0 = \emptyset$ and $S = S_j$ for some $1 \leq j \leq n$.*

Proposition 3.3. *Let $v : \mathcal{F} \to \mathbb{R}$ be a profit game on $\mathcal{F} \subseteq 2^N$, let $x \in \mathbb{R}^n$ and let $S \in \mathcal{F}$. Then the following statements are equivalent:*

1. $x \in \partial v(S)$,
2. $\max\{v(T) - x(T) : T \in \mathcal{F}\} = v(S) - x(S)$,
3. $v(S) + v^{\circ}(x) = x(S)$.

Proof. The result follows from the definitions. □

Proposition 3.4. *Let $c : \mathcal{F} \to \mathbb{R}$ be a cost game on $\mathcal{F} \subseteq 2^N$, let $x \in \mathbb{R}^n$ and let $S \in \mathcal{F}$. Then the following statements are equivalent:*

1. $x \in \partial c(S)$,
2. $\min\{c(T) - x(T) : T \in \mathcal{F}\} = c(S) - x(S)$,
3. $c(S) + c^{\bullet}(x) = x(S)$.

Proof. The result follows from the definitions. □

Propositions 3.3 and 3.4 imply the following optimization property:

Proposition 3.5. *Let $v : \mathcal{F} \to \mathbb{R}$ be a profit game and let $c : \mathcal{F} \to \mathbb{R}$ be a cost game. Then*

$$\begin{aligned} \max\{v(T) : T \in \mathcal{F}\} &= v(S) \text{ if and only if the vector } \mathbf{0} \in \partial v(S), \\ \min\{c(T) : T \in \mathcal{F}\} &= c(S) \text{ if and only if the vector } \mathbf{0} \in \partial c(S). \end{aligned}$$

For the concave conjugate function $v^\circ : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$ we define

$$\partial_2 v^\circ(x) = \{S \in \mathcal{F} : v^\circ(u) - v^\circ(x) \leq u(S) - x(S) \text{ for all } u \in \mathbb{R}^n\}.$$

For the convex conjugate function $c^\bullet : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$ we define

$$\partial_2 c^\bullet(x) = \{S \in \mathcal{F} : c^\bullet(u) - c^\bullet(x) \geq u(S) - x(S) \text{ for all } u \in \mathbb{R}^n\}.$$

The collections of coalitions $\partial_2 v^\circ(x) \subseteq \mathcal{F}$ and $\partial_2 c^\bullet(x) \subseteq \mathcal{F}$ are called the binary subdifferentials of v° and $c^\bullet$ at x.

Proposition 3.6. *Let $v : \mathcal{F} \to \mathbb{R}$ be a profit game and let $c : \mathcal{F} \to \mathbb{R}$ be a cost game. Then*

1. *The vector $x \in \partial v(S)$ if and only if the set $S \in \partial_2 v^\circ(x)$.*
2. *The vector $x \in \partial c(S)$ if and only if the set $S \in \partial_2 c^\bullet(x)$.*

Proof. 1. From Proposition 2.1.1 we obtain for all $S \in \mathcal{F}$,

$$v(S) = v^{\circ\circ}(\mathbf{1}_S) = \min\{u(S) - v^\circ(u) : u \in \mathbb{R}^n\}. \tag{7}$$

Proposition 3.3, equation (7) and the respective definitions together give

$$\begin{aligned} x \in \partial v(S) &\iff x(S) - v^\circ(x) = v(S) \\ &\iff x(S) - v^\circ(x) = \min\{u(S) - v^\circ(u) : u \in \mathbb{R}^n\} \\ &\iff x(S) - v^\circ(x) \leq u(S) - v^\circ(u) \text{ for all } u \in \mathbb{R}^n \\ &\iff S \in \partial_2 v^\circ(x). \end{aligned}$$

The statement 2 follows similarly from Propositions 2.1.2 and 3.4. □

We recall that $v^\circ : \mathbb{R}^n \to \mathbb{R}$ and $v^{\circ\circ} : \overline{\mathcal{F}} \to \mathbb{R}$ are concave functions, so we can consider the following subdifferentials in the sense of convex analysis (see Rockafellar [23]):

$$\begin{aligned} \partial v^\circ(x) &= \{y \in \mathbb{R}^n : v^\circ(u) - v^\circ(x) \leq \langle y, u - x\rangle \text{ for all } u \in \mathbb{R}^n\}, \\ \partial v^{\circ\circ}(y) &= \left\{x \in \mathbb{R}^n : v^{\circ\circ}(z) - v^{\circ\circ}(y) \leq \langle x, z - y\rangle \text{ for all } z \in \widehat{\mathcal{F}}\right\}. \end{aligned}$$

To characterize these subdifferentials we prove the following lemma.

Lemma 3.7. *Let $v : \mathcal{F} \to \mathbb{R}$ be a profit game and let $u \in \mathbb{R}^n$. Then*

1. $(v - u)^\circ(x) = v^\circ(x + u)$ *for all* $x \in \mathbb{R}^n$.
2. $(v - u)^{\circ\circ}(y) = v^{\circ\circ}(y) - \langle y, u\rangle$ *for all* $y \in \widehat{\mathcal{F}}$.
3. $\max\left\{v^{\circ\circ}(y) - \langle y, u\rangle : y \in \widehat{\mathcal{F}}\right\} = \max\{v(S) - u(S) : S \in \mathcal{F}\}.$

Proof. 1. For each $x \in \mathbb{R}^n$ we obtain

$$\begin{aligned} (v - u)^\circ(x) &= \min\{x(S) - (v - u)(S) : S \in \mathcal{F}\} \\ &= \min\{(x + u)(S) - v(S) : S \in \mathcal{F}\} \\ &= v^\circ(x + u). \end{aligned}$$

2. For each $y \in \widehat{\mathcal{F}}$ we have

$$\begin{aligned} (v-u)^{\circ\circ}(y) &= \min\left\{\langle y,x\rangle - (v-u)^{\circ}(x) : x \in \mathbb{R}^n\right\} \\ &= \min\left\{\langle y,x+u\rangle - v^{\circ}(x+u) : x \in \mathbb{R}^n\right\} - \langle y,u\rangle \\ &= v^{\circ\circ}(y) - \langle y,u\rangle . \end{aligned}$$

3. From Proposition 2.1.3 and 2 we obtain

$$\begin{aligned} \max\left\{v^{\circ\circ}(y) - \langle y,u\rangle : y \in \widehat{\mathcal{F}}\right\} &= \max\left\{(v-u)^{\circ\circ}(y) : y \in \widehat{\mathcal{F}}\right\} \\ &= \max\left\{(v-u)(S) : S \in \mathcal{F}\right\}. \ \square \end{aligned}$$

Theorem 3.8. *Let $v : \mathcal{F} \to \mathbb{R}$ be a profit game on a nonempty $\mathcal{F} \subseteq 2^N$. Then*

1. $\partial v^{\circ}(x) = \left\{y \in \widehat{\mathcal{F}} : v^{\circ\circ}(y) + v^{\circ}(x) = \langle y,x\rangle\right\}$ *for all* $x \in \mathbb{R}^n$.
2. $\partial v^{\circ\circ}(y) = \left\{x \in \mathbb{R}^n : v^{\circ}(x) + v^{\circ\circ}(y) = \langle x,y\rangle\right\}$ *for all* $y \in \widehat{\mathcal{F}}$.
3. *If* $(x,y) \in \mathbb{R}^n \times \widehat{\mathcal{F}}$ *then* $y \in \partial v^{\circ}(x) \Longleftrightarrow x \in \partial v^{\circ\circ}(y)$.
4. $\partial v^{\circ\circ}(\mathbf{1}_S) = \partial v(S)$ *for all* $S \in \mathcal{F}$.

Proof. 1. For each $x \in \mathbb{R}^n$ we have

$$\begin{aligned} \partial v^{\circ}(x) &= \left\{y \in \mathbb{R}^n : v^{\circ}(u) - \langle y,u\rangle \le v^{\circ}(x) - \langle y,x\rangle \quad \text{for all } u \in \mathbb{R}^n\right\} \\ &= \left\{y \in \mathbb{R}^n : \max\left\{v^{\circ}(u) - \langle y,u\rangle : u \in \mathbb{R}^n\right\} = v^{\circ}(x) - \langle y,x\rangle\right\} \\ &= \left\{y \in \mathbb{R}^n : \min\left\{\langle y,u\rangle - v^{\circ}(u) : u \in \mathbb{R}^n\right\} = \langle y,x\rangle - v^{\circ}(x)\right\} \\ &= \left\{y \in \widehat{\mathcal{F}} : v^{\circ\circ}(y) + v^{\circ}(x) = \langle y,x\rangle\right\}. \end{aligned}$$

2. Let $y \in \widehat{\mathcal{F}}$. Using Lemma 3.7.3, we can write

$$\begin{aligned} \partial v^{\circ\circ}(y) &= \left\{x \in \mathbb{R}^n : v^{\circ\circ}(z) - \langle x,z\rangle \le v^{\circ\circ}(y) - \langle x,y\rangle \quad \text{for all } z \in \widehat{\mathcal{F}}\right\} \\ &= \left\{x \in \mathbb{R}^n : \max\left\{v^{\circ\circ}(z) - \langle x,z\rangle : z \in \widehat{\mathcal{F}}\right\} = v^{\circ\circ}(y) - \langle x,y\rangle\right\} \\ &= \left\{x \in \mathbb{R}^n : \max\left\{v(T) - x(T) : T \in \mathcal{F}\right\} = v^{\circ\circ}(y) - \langle x,y\rangle\right\} \\ &= \left\{x \in \mathbb{R}^n : \min\left\{x(T) - v(T) : T \in \mathcal{F}\right\} = \langle x,y\rangle - v^{\circ\circ}(y)\right\} \\ &= \left\{x \in \mathbb{R}^n : v^{\circ}(x) + v^{\circ\circ}(y) = \langle x,y\rangle\right\}. \end{aligned}$$

3. If $(x,y) \in \mathbb{R}^n \times \widehat{\mathcal{F}}$ then

$$y \in \partial v^{\circ}(x) \Longleftrightarrow v^{\circ\circ}(y) + v^{\circ}(x) = \langle y,x\rangle \Longleftrightarrow x \in \partial v^{\circ\circ}(y).$$

4. From Propositions 2.1.1 and 3.3.3 we obtain, for all $S \in \mathcal{F}$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} x \in \partial v^{\circ\circ}(\mathbf{1}_S) &\Longleftrightarrow v^{\circ\circ}(\mathbf{1}_S) + v^{\circ}(x) = \langle x,\mathbf{1}_S\rangle \\ &\Longleftrightarrow v(S) + v^{\circ}(x) = x(S) \\ &\Longleftrightarrow x \in \partial v(S). \ \square \end{aligned}$$

Remark 3.1. *Similar results to the ones obtained above for the subdifferentials of v° and $v^{\circ\circ}$ can be obtained for those of $c^{\bullet}$ and $c^{\bullet\bullet}$.*

4. The indirect function

In this section we study a representation of n-person cooperative games by functions of n variables. As in [15], where this representation was introduced, we shall restrict the presentation to profit games; of course, a parallel theory can be developed for cost games:

Definition 4.1. *The indirect function of a profit game* $v : 2^N \to \mathbb{R}$, *with* $N = \{1, \ldots, n\}$, *is* $\pi_v : \mathbb{R}^n \to \mathbb{R}$, *given by* $\pi_v(x) = \max\{v(S) - x(S) : S \subseteq N\}$ *for all* $x \in \mathbb{R}^n$.

As an immediate consequence of this definition, it follows that the indirect function is a polyhedral convex function. Indeed, the indirect function of v is strongly related to the concave conjugate of v; namely, for all $x \in \mathbb{R}^n$ it satisfies

$$\pi_v(x) = -\min\{x(S) - v(S) : S \subseteq N\} = -v^{\circ}(x). \tag{8}$$

The indirect function admits an economic interpretation. Let us regard the players of the profit game as workers, and $v(S)$ as the profit (measured in money units) that coalition S yields when its members work together, provided that they have available the resources needed for production. Suppose that an employer, owning these resources, wishes to choose those workers who would provide him with the maximum possible profit. If the subset S is selected then the total amount of money that S will yield is $v(S)$. If $x = (x_1, \ldots, x_n)$ is the vector of (possibly negative) salaries demanded by the workers then $\pi_v(x)$ represents the maximum net profit the employer can obtain under those given salaries.

Equation (3) implies the following result:

Theorem 4.1. *Let* $v : 2^N \to \mathbb{R}$ *be a profit game. Then, for all* $S \subseteq N$, *one has*

$$v(S) = \min\{x(S) + \pi_v(x) : x \in \mathbb{R}^n\}. \tag{9}$$

The importance of the preceding theorem lies in that it shows that the indirect function π_v of a profit game v contains all the information on the game, as it allows to recover v from π_v.

Indirect functions of profit games are characterized in Martínez-Legaz [15] by three properties, two of which are expressed in terms of the convex analytic subdifferential [23], denoted by ∂:

Theorem 4.2. *Let* $\pi : \mathbb{R}^n \to \mathbb{R}$. *There exists a profit game* $v : 2^N \to \mathbb{R}$ *such that* $\pi = \pi_v$ *if and only if* π *satisfies the following properties:*

1. $\partial\pi(x) \bigcap \{0, -1\}^n \neq \emptyset$, *for all* $x \in \mathbb{R}^n$.
2. $\{0, -1\}^n \subset \bigcup_{x \in \mathbb{R}^n} \partial\pi(x)$.
3. $\min\{\pi(x) : x \in \mathbb{R}^n\} = 0$.

The following alternative characterization of indirect functions in terms of gradients, instead of subdifferentials, was given by Martínez-Legaz [16]:

Theorem 4.3. *Let* $\pi : \mathbb{R}^n \to \mathbb{R}$ *and let* $\nabla\pi$ *denote the gradient mapping of* π. *Then there exists a profit game* $v : 2^N \to \mathbb{R}$ *such that* $\pi = \pi_v$ *if and only if* π *satisfies the following properties:*

1. π *is convex.*
2. *The range of* $\nabla\pi$ *is* $\{0, -1\}^n$.
3. $\min\{\pi(x) : x \in \mathbb{R}^n\} = 0$.

Many concepts in the theory of cooperative profit games can be easily expressed in terms of indirect functions; we refer the reader for details to [15]. In particular, totally balanced (profit) games, an important class of games that will be dealt with in the subsequent sections, are characterized by their indirect functions in [17]. Just to give an example, we end this section by showing how the core of a profit game can be expressed in terms of its indirect function.

Proposition 4.4. *Let $v : 2^N \to \mathbb{R}$ be a profit game. Then*

$$C(v) = \{x \in \mathbb{R}^n : \mathbf{0}, -\mathbf{1}_N \in \partial\pi(x)\}.$$

Proof. Let $x \in \mathbb{R}^n$. According to (4) and (8), one has $x \in C(v)$ if and only if $\pi(x) = 0$ and $x(N) = v(N)$. By Theorem 4.3, the first condition means that x is a minimum point of π, which is equivalent to $\mathbf{0} \in \partial\pi(x)$. On the other hand, for such a point the second condition can be equivalently written as $\langle \mathbf{1}_N, x\rangle = v^{\circ\circ}(\mathbf{1}_N) + v^{\circ}(x)$, which, by Theorem 3.8, means that $\mathbf{1}_N \in \partial v^{\circ}(x) = -\partial\pi(x)$. □

5. The least increment function

This section is devoted to a different way of representing profit games by convex functions, namely, by the so-called *least increment functions* [2, 1]. The interested reader can adapt this representation to the case of cost games.

Definition 5.1. *The least increment function of a profit game $v : 2^N \to \mathbb{R}$, with $N = \{1, \ldots, n\}$, is $\epsilon_v : \mathbb{R}^n \to \mathbb{R}$, given by*

$$\epsilon_v(x) = \min\{y(N) - x(N) : y \in P(v),\ y \geq x\} \qquad \text{for all } x \in \mathbb{R}^n.$$

The least increment function admits the following interpretation. Suppose that a payoff vector $x \in \mathbb{R}^n$ is offered to the players. Then $\epsilon_v(x)$ is the least amount $y(N) - x(N)$ by which the total payoff $x(N)$ should be incremented to make the resulting payoff vector acceptable by all coalitions ($y(S) \geq v(S)$ for all $S \subseteq N$) and preferred to the initial one by all players ($y \geq x$).

According to the preceding definition, the least increment function is the optimum value function of parametric linear programming problem. Using the duality theorem of linear programming, it easily follows that, like the indirect function, the least increment function is a polyhedral convex function. The following proposition compares both functions:

Proposition 5.1. *If $v : 2^N \to \mathbb{R}$ is a profit game then, for all $x \in \mathbb{R}^n$,*

$$\epsilon_v(x) \geq \pi_v(x).$$

Proof. For each $S \subseteq N$ and for each $y \in P(v)$ such that $y \geq x$, we have $y(N) = y(S) + y(N \setminus S) \geq v(S) + x(N \setminus S)$, and therefore

$$\min\{y(N) - x(N) : y \in P(v),\ y \geq x\} \geq \max\{v(S) - x(S) : S \subseteq N\} = \pi_v(x). \ \square$$

We shall next give a sufficient condition for the inequality in the preceding proposition to hold with the equal sign:

Theorem 5.2. *If $v : 2^N \to \mathbb{R}$ is a convex profit game then $\epsilon_v = \pi_v$.*

Proof. The result follows from (5) applied to the concave cost game $-v$:

$$\begin{aligned}
\epsilon_v(x) &= \min\{y(N) - x(N) : y \in P(v),\ y \geq x\} \\
&= \min\{y(N) : y \in P(v),\ y \geq x\} - x(N) \\
&= \min\{y(N) : -y \in P(-v),\ y \geq x\} - x(N) \\
&= -\max\{z(N) : z \in P(-v),\ z \leq -x\} - x(N) \\
&= -\min\{-v(S) - x(N \setminus S) : S \subseteq N\} - x(N) \\
&= \max\{v(S) + x(N \setminus S) : S \subseteq N\} - x(N) \\
&= \max\{v(S) - x(S) : S \subseteq N\} \\
&= \pi_v(x). \ \square
\end{aligned}$$

A natural question to ask is whether the convexity assumption can be removed in the preceding theorem. In other words, do the indirect function and the least increment function of any profit game coincide? If this were the case, according to (9) the expression $\min\{x(S) + \epsilon_v(x) : x \in \mathbb{R}^n\}$ would coincide with $v(S)$ for any profit game and any coalition S. However, from the next theorem it follows that the equality $v(S) = \min\{x(S) + \epsilon_v(x) : x \in \mathbb{R}^n\}$ is characteristic of totally balanced profit games. To give the definition of this class of games, we first need to recall the notion of x-balanced collection:

Definition 5.2. *For $x \in \mathbb{R}^n_+$, $\{\lambda_T\}_{T\subseteq N}$ is an x-balanced collection if $\lambda_T \geq 0$ for all $T \subseteq N$ and $\sum_{T\subseteq N} \lambda_T \mathbf{1}_T = x$.*

Definition 5.3. *A profit game $v : 2^N \to \mathbb{R}$ is totally balanced if for all $S \in 2^N$ and all $\mathbf{1}_S$-balanced collection $\{\lambda_T\}_{T\subseteq N}$ it satisfies $\sum_{T\subseteq N} \lambda_T v(T) \leq v(S)$.*

The class of totally balanced profit games is closed under pointwise infimum, that is, if $\{v_i\}_{i\in I}$ is an arbitrary nonempty family of totally balanced profit games then the profit game $v : 2^N \to \mathbb{R}$ defined by $v(S) = \inf_{i\in I} v_i(S)$ for all $S \in 2^N$ is totally balanced, too. Besides, any profit game $v : 2^N \to \mathbb{R}$ admits a totally balanced majorant, i.e., a totally balanced profit game $w : 2^N \to \mathbb{R}$ satisfying $w(S) \geq v(S)$ for all $S \in 2^N$ Indeed, one can take, e.g., the (additive) game defined by $w(S) = k\,|S|$ for all $S \in 2^N$, with $k \geq \max\left\{\frac{v(S)}{|S|} : S \in 2^N \setminus \{\emptyset\}\right\}$. In view of these properties, the following concept is well defined:

Definition 5.4. *The totally balanced cover of a profit game $v : 2^N \to \mathbb{R}$ is the profit game $\widetilde{v} : 2^N \to \mathbb{R}$ defined by*

$$\widetilde{v}(S) = \inf\left\{w(S) : w : 2^N \to \mathbb{R} \text{ is a totally balanced majorant of } v\right\}.$$

>From this definition it immediate follows:

Proposition 5.3. *The totally balanced cover $\widetilde{v} : 2^N \to \mathbb{R}$ of $v : 2^N \to \mathbb{R}$ is the smallest (in the pointwise sense) totally balanced majorant of v. Therefore, v is totally balanced if and only if $\widetilde{v} = v$.*

We are now in a position to state the theorem announced above:

Theorem 5.4. *For any profit game $v : 2^N \to \mathbb{R}$, one has*

$$\widetilde{v}(S) = \min\{x(S) + \epsilon_v(x) : x \in \mathbb{R}^n\} \quad \text{for all } S \in 2^N.$$

In view of the preceding theorem, one can say that the least increment function provides a dual representation of totally balanced profit games. Indeed, unlike the indirect function, the least increment function of an arbitrary profit game does not contain all the information on the game, but only on its totally balanced cover, since one can prove that profit games having the same totally balanced cover have also the same least increment function.

Based on the last theorem, a plausible conjecture is that, even though indirect functions and least increment functions do not generally coincide, they do in the case of totally balanced profit games. However, an example of a 4-players totally balanced (but not convex, of course) profit game presented in [2] shows that this conjecture is wrong.

6. The maximum average value function

This section contains still another representation of profit games by convex functions, namely, by the so-called *maximum average value* (*MAV*, for short) *function* [16]. As in the preceding two sections, here we shall not consider cost games, although they admit of course an obvious corresponding representation.

Unlike in the cases of the indirect and the least increment functions, the MAV function is defined only for nonnegative profit games:

Definition 6.1. *The maximum average value function of a nontrivial (i.e., not identically zero) nonnegative profit game* $v : 2^N \to \mathbb{R}_+$, *with* $N = \{1, \dots, n\}$, *is* $\mu_v : \mathbb{R}^n_+ \setminus \{0\} \longrightarrow \mathbb{R}_{++} \cup \{+\infty\}$, *given by*

$$\mu_v(x) = \max\left\{\frac{v(S)}{x(S)} : S \subseteq N\right\} \qquad \text{for all } x \in \mathbb{R}^n$$

(with the conventions $\frac{\alpha}{0} = +\infty$ *for any* $\alpha > 0$ *and* $\frac{0}{0} = 0$*).*

This function admits an economic interpretation similar to the one we have given for the indirect function. Thus we again regard the players as workers and $v(S)$ as the output that coalition S yields when its members are hired by an employer owning the resources needed for production. In the case of the maximum average value function, we assume that this employer seeks to maximize the amount of output per unit of money spent rather than the net profit. If $x = (x_1, \dots, x_n)$ is the vector of salaries demanded by the workers then $\mu_v(x)$ represents the maximum output per unit of money spent that the employer can obtain under those given salaries.

Similarly to the case of least increment functions, the totally balanced cover of a profit game can be recovered from its MAV function:

Theorem 6.1. . *For any nontrivial nonnegative profit game* $v : 2^N \to \mathbb{R}_+$, *one has*

$$\tilde{v}(S) = \min\left\{\mu_v(x)\, x(S) : x \in \mathbb{R}^n_+ \setminus \{0\}\right\} \quad \text{for all} \quad S \in 2^N$$

(with the convention $(+\infty) \cdot 0 = +\infty$ *).*

One can also prove that profit games having the same totally balanced cover have the same MAV function, too, and so the totally balanced cover of a profit game is all the information on the game that its MAV function allows one to recover. In view of this, one can say that the MAV function provides a dual representation of totally balanced profit games. In other words, the mapping assigning to each totally balanced nontrivial nonnegative profit game its MAV function is one-to-one. The range of this mapping is identified in the following theorem:

Theorem 6.2. *Let $\mu : \mathbb{R}^n_+ \backslash \{0\} \to \mathbb{R}_{++} \cup \{+\infty\}$. There exists a nontrivial profit game $v : 2^N \to \mathbb{R}$ such that $\mu = \mu_v$ if and only if μ is convex, continuous, positively homogeneous of degree -1, finite valued on $\mathbb{R}^n_{++}$, and such that, at each point where the gradient exists, all of its nonzero components are the same.*

In [18], it is shown how the core of a nontrivial nonnegative totally balanced game can be easily computed by using the MAV function of the game.

7. The Lovász extension

Let $N = \{1, 2, \ldots, n\}$ be a finite set of players and $v : 2^N \to \mathbb{R}$ be a profit game. Through the identification of coalitions with their characteristic functions, an n-person game can be regarded as a function $v : \{0,1\}^n \to \mathbb{R}$, with $v(\mathbf{0}) = 0$. Adopting this point of view, any profit game has a unique extension $f(v)$ as a multilinear (more precisely, multiaffine) polynomial in n variables (Hammer and Rudeanu [12]); it is defined, for all $x = (x_1, \ldots, x_n) \in [0,1]^n$, by

$$f(v)(x) = \sum_{S \subseteq N} d_S(v) \prod_{i \in S} x_i,$$

where the coefficient $d_S(v) \in \mathbb{R}$ is the dividend of coalition S in v. This set of coefficients is defined by $v = \sum_{S \subseteq N} d_S(v) u_S$, $u_S : 2^N \to \mathbb{R}$ being the S-unanimity (profit) game $u_S : 2^N \to \mathbb{R}$ given by $u_S(T) = 1$ if $T \supseteq S$, and $u_S(T) = 0$ otherwise (since the family of unanimity games $\{u_S\}_{S \subseteq N,\, S \neq \emptyset}$ is a basis of the real vector space of all profit games $v : \{0,1\}^n \to \mathbb{R}$, the dividends $\{d_S(v)\}_{S \subseteq N,\, S \neq \emptyset}$ are uniquely determined as the coefficients of v in that basis). One can compute the dividends of a profit game v either directly, using

$$d_S(v) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T),$$

or recursively, by

$$d_S(v) = \begin{cases} 0, & \text{if } S = \emptyset \\ v(S) - \sum_{T \subset S} d_T(v), & \text{if } S \neq \emptyset. \end{cases}$$

We now consider a different extension of a profit game $v : 2^N \to \mathbb{R}$ to a function on $\mathbb{R}^n$. We say that a non-negative function $f : 2^N \to \mathbb{R}_+$ is a *weighted chain* if the family $\mathcal{F} = \{S \subseteq N : f(S) > 0\}$ is a chain, i.e. $S \subseteq T$ or $T \subseteq S$, for every pair $S, T \in \mathcal{F}$. To every weighted chain f, we associate a non-negative vector

$$x = \sum_{S \subseteq N} f(S) \mathbf{1}_S \in \mathbb{R}^n_+,$$

called the *depth vector* of f. The correspondence: assigning $x \in \mathbb{R}^n_+$ to $f : 2^N \to \mathbb{R}_+$ is one-to-one. Let $0 \leq x^1 < x^2 < \cdots < x^k$ be the components of x with different values and let $S_p = \{i \in N : x_i \geq x^p\}$. Then we define $f_x : 2^N \to \mathbb{R}_+$ by

$$f_x(S) = \begin{cases} x^p - x^{p-1}, & \text{if } S = S_p \\ 0, & \text{otherwise,} \end{cases}$$

where $x^0 = 0$. Obviously f_x is the only weighted chain whose depth vector is x.

There is a natural way of extending a profit game $v : 2^N \to \mathbb{R}$ to all non-negative vectors:

Definition 7.1. *The Lovász extension of the profit game* $v : 2^N \to \mathbb{R}$ *is the function* $\widehat{v} : \mathbb{R}^n_+ \to R$ *given by* $\widehat{v}(x) = \sum_{S \subseteq N} f_x(S)\, v(S)$.

The function $\widehat{v}$ is an extension of v because $\widehat{v}\,(\mathbf{1}_S) = v(S)$, for all $S \in 2^N$ and it has the following properties:

1. $\widehat{v}$ is positively homogeneous, i.e., $\widehat{v}(\lambda x) = \lambda \widehat{v}(x)$ for all $\lambda \geq 0$.
2. $\widehat{v_1 + v_2} = \widehat{v}_1 + \widehat{v}_2$.
3. $\widehat{\lambda v} = \lambda \widehat{v}$ for all $\lambda \in \mathbb{R}$.

The Lovász extension of a convex profit game satisfies (see Fujishige [11, Section 6.3]) the following optimization property:

$$\widehat{v}(x) = \min \{\langle x, y\rangle : y \in P(v)\}. \tag{10}$$

The following theorem is the supermodular version of a result given by Lovász [14] for submodular functions (see also [11, Theorem 6.13]).

Theorem 7.1. *A profit game is convex if and only if its Lovász extension is concave.*

Let us assume a total ordering $i_1 < i_2 < \cdots < i_n$ of the elements of N. Given the previous ordering C, consider the following chain of coalitions,

$$C_0 \subset C_1 \subset \cdots \subset C_{n-1} \subset C_n,$$

where $C_0 = \emptyset$ and $C_k = \{i_1, i_2, \ldots, i_k\}$, $k = 1, \ldots, n$.

Definition 7.2. *The marginal worth vector with respect to the ordering* C *in the profit game* v *is* $a^C(v) \in \mathbb{R}^n$, *given by* $a^C_{i_k}(v) = v(C_k) - v(C_{k-1})$, $k = 1, \ldots, n$.

The Lovász extension is strongly related to the greedy algorithm. We say that an ordering $i_1 < i_2 < \cdots < i_n$ is *compatible* with the vector $x \in \mathbb{R}^n_+$, or x-compatible, if

$$x_{i_1} \geq x_{i_2} \geq \cdots \geq x_{i_n} \geq 0.$$

Theorem 7.2. *Let* $v : 2^N \to \mathbb{R}$ *be a convex profit game,* $x \in \mathbb{R}^n_+$, *and* $a^C(v)$ *be the marginal worth vector with respect to an* x*-compatible ordering* C. *Then*

$$\widehat{v}(x) = \min \{\langle x, y\rangle : y \in C(v)\} = \left\langle x, a^C(v)\right\rangle.$$

Proof. If v is a convex profit game then $a^C(v) \in C(v) \subset P(v)$ and hence we obtain the result if we prove $\langle x, y\rangle \geq \left\langle x, a^C(v)\right\rangle$, for all $y \in P(v)$. Let $y \in P(v)$, the summation by parts implies that

$$\begin{aligned}\langle x, y\rangle &= \sum_{k=1}^{n} x_{i_k} y_{i_k} \\ &= \sum_{k=1}^{n-1}\left[\left(x_{i_k} - x_{i_{k+1}}\right)\sum_{j=1}^{k} y_{i_j}\right] + x_{i_n}\sum_{j=1}^{n} y_{i_j} \\ &\geq \sum_{k=1}^{n-1}\left(x_{i_k} - x_{i_{k+1}}\right) v\left(C_k\right) + x_{i_n} v\left(C_n\right) \\ &= \sum_{k=1}^{n} x_{i_k}\left[v\left(C_k\right) - v\left(C_{k-1}\right)\right] \\ &= \sum_{k=1}^{n} x_{i_k} a_{i_k}^{C}(v) = \left\langle x, a^{C}(v)\right\rangle. \quad \square\end{aligned}$$

The following expression for the Lovász extension in terms of dividends was obtained by Driessen and Rafels [5]:

Theorem 7.3. *Let $v : 2^N \to \mathbb{R}$ be a profit game with dividends $\{d_S(v)\}_{S\subseteq N,\, S\neq\emptyset}$. Then $\widehat{v}(x) = \sum_{S\subseteq N} d_S(v)\min_{i\in S} x_i$ for all $x \in \mathbb{R}^n_+$.*

Proof. Properties 2 and 3 of the Lovász extension imply that $\widehat{v} = \sum_{S\subseteq N} d_S(v)\widehat{u_S}$. Every unanimity game u_S is convex and we can use the optimization property showed in Theorem 7.2. Let $\{e_1, \ldots, e_n\}$ be the natural basis in $\mathbb{R}^n$. Thus,

$$\widehat{u_S}(x) = \min\left\{\langle x, y\rangle : y \in C\left(u_S\right)\right\} = \min\left\{\langle x, e_i\rangle : i \in S\right\} = \min_{i\in S} x_i,$$

where the second equation follows from the characterization of the core for unanimity games (see Einy and Wettstein [7]): $C\left(u_S\right) = \operatorname{conv}\left\{e_i : i \in S\right\}$. $\square$

We will now consider a game model of a pure exchange economy. Let N be a set of n traders and let us suppose that they participate in a market encompassing trade in m commodities. The space $\mathbb{R}^m_+$ is considered as the commodity space. Every trader $i \in N$ is characterized by means of an initial endowment vector $w^{(i)} \in \mathbb{R}^m_+$ and by a utility function $u_i : \mathbb{R}^m_+ \to \mathbb{R}$ which measures the worth, for him/her, of any bundle of commodities. The individual utility functions u_i are continuous and concave. The quadruple $\mathcal{M} = \left(N, m, \left\{w^{(i)}\right\}_{i\in N}, \{u_i\}_{i\in N}\right)$, is called a *market.* We denote the aggregate endowment of the coalition of traders S by $w(S) = \sum_{i\in S} w^{(i)} \in \mathbb{R}^m_+$.

Then, $w(S)$ can be reallocated as a collection $\{a^{(i)} : i \in S\}$ of bundles such that each $a^{(i)} \in \mathbb{R}^m_+$ and $a(S) = \sum_{i\in S} a^{(i)} = w(S)$. We denote the set of these collections by $A(S)$.

Since the individual utility functions are continuous and $A(S)$ is a compact set, we can define a profit game $v_{\mathcal{M}} : 2^N \to S$ by

$$v_{\mathcal{M}}(S) = \max\left\{\sum_{i\in S} u_i\left(a^{(i)}\right) \;:\; \{a^{(i)} : i \in S\} \in A(S)\right\},$$

for all $S \subseteq N$. The game $v_{\mathcal{M}}$ is called a *market game* (see Shapley and Shubik [25] and Kannai [13]), and it corresponds to the original market in a natural way.

The following result is due to Shapley and Shubik [25].

Theorem 7.4. *A profit game is a market game if and only if it is totally balanced.*

For any totally balanced profit game $v : 2^N \to \mathbb{R}$ we define a market, called *direct market*, $\mathcal{M}_0 = \left(N, n, \left\{\mathbf{1}_{\{i\}}\right\}_{i \in N}, u\right)$, where u is the same utility function for all traders, defined by

$$u(x) = \max\left\{\sum_{T \subseteq N} \lambda_T \, v(T) \ : \ \{\lambda_T\}_{T \subseteq N} \text{ is an } x\text{-balanced collection}\right\}.$$

Every convex profit game is totally balanced. In this case we obtain the following property of the Lovász extension.

Theorem 7.5. *Let $v : 2^N \to \mathbb{R}$ be a market game such that the traders have a common utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$. Then $v : 2^N \to \mathbb{R}$ is convex if and only if $u = \widehat{v}$.*

Proof. The utility function is the solution of the following linear programming problem, $u(x) = \max\left\{\sum_{T \subseteq N} \lambda_T \, v(T) : A\lambda = x, \ \ \lambda \geq 0\right\}$, where $\lambda = (\lambda_T)_{T \subseteq N}$, $v = (v(T))_{T \subseteq N}$ and the matrix $A = (\mathbf{1}_T(i))_{i \in N, T \subseteq N}$. The duality theorem of linear programming implies that

$$\begin{aligned} u(x) &= \min\left\{\langle y, x \rangle : y(T) \geq v(T) \text{ for all } T \subseteq N\right\} \\ &= \min\left\{\langle y, x \rangle : y \in P(v)\right\}. \end{aligned}$$

If $v : 2^N \to \mathbb{R}$ is convex then (10) implies that $\widehat{v} = u$. Conversely, if the utility function satisfies $u = \widehat{v}$ then the Lovász extension $\widehat{v}$ is concave and Theorem 7.1 implies that $v : 2^N \to \mathbb{R}$ is convex. □

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