Simple games on closure spaces

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Abstract

Let N be a finite set. By a closure space we mean the family of the closed sets of a closure operator on 2^N satisfying the additional condition $\overline{\emptyset} = \emptyset$. A simple game on a closure space $\mathcal{L}$ is a function $v : \mathcal{L} \to \{0, 1\}$ such that $v(\emptyset) = 0$ and $v(N) = 1$. We assume simple games are monotonic. The coalitions are the closed sets of $\mathcal{L}$ and the players are the elements $i \in N$. We will give results concerning the structure of the core and the Weber set for this type of games. We show that a simple game is supermodular if and only if the game is a unanimity game and the $Core(\mathcal{L}, v)$ is a stable set if and only if the game v is a unanimity game.

Key Words: Core, supermodular game, Shapley value, Weber set
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1 Introduction

Cooperative game theory has been used to analyze the relative power of players in voting schemes. The first such power index was proposed by Shapley and Shubik (1954). An assumption of the classical model is that all coalitions of voters are feasible. However, structural and ideological issues will prevent some coalitions from forming. There have been various investigations into models of cooperative games that also reflect constraints on coalition formation. See, for instance Faigle and Kern (1992) (1998), Derks and Gilles (1995) or Brink and Gilles (1996). We will restrict the feasible coalitions by using *closure spaces*, that is, set systems containing the set of all players and closed under intersection of coalitions. We also need *convex geometries* which are closure spaces with the extension property. The theory of these combinatorial structures is well covered by Edelman and Jamison (1985) and Korte, Lovász and Schrader (1991). The paper is organized as follows. In section 2, closure spaces and convex geometries are defined and some of their properties described. Furthermore, simple games on these set systems are introduced. In sections 3 and 4, we obtain general theorems on the *core* and *Weber set* in our framework.

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2 Games on closure spaces

Let $N = \{1, \dots, n\}$ be a finite set. By a *closure operator* $- : 2^N \to 2^N$ we mean an operator satisfying the axioms:

(C1) $A \subseteq \overline{A}$,

(C2) $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$,

(C3) $\overline{\overline{A}} = \overline{A}$,

which are the Kuratowski closure axioms, with the additional condition

(C4) $\emptyset = \overline{\emptyset}$.

Examples of closure operators are the spanning operator of linear algebra and all convex hull operators. A set $S \in 2^N$ is said to be closed if $\overline{S} = S$. The family of the closed sets of a closure operator is a *closure space.*

Alternatively, a family $\mathcal{L} \subseteq 2^N$ satisfying that $\emptyset,\ N \in \mathcal{L}$, and $S \cap T \in \mathcal{L}$ for all $S, T \in \mathcal{L}$, defines an operator $- : 2^N \to 2^N$ by

$$A \mapsto \overline{A} := \bigcap \{C \in \mathcal{L} : A \subseteq C\}.$$

This operator is a closure operator and the family $\mathcal{L}$ is a closure space. A closure space $\mathcal{L} \subseteq 2^N$ is *atomic* if $\{i\} \in \mathcal{L}$ for all $i \in N$. If $\mathcal{L} \subseteq 2^N$ is a closure space, then it is a complete lattice, ordered by inclusion, in which meet and join operations are defined by $A \wedge B = A \cap B,\ A \vee B = \overline{A \cup B},$ for all $A, B \in \mathcal{L}$.

Definition 2.1. A closure space $\mathcal{L}$ is a convex geometry if it satisfies the extension property: If $A \in \mathcal{L}$ and $A \neq N$, then there exists $j \in N \setminus A$ such that $A \cup \{j\} \in \mathcal{L}$.

The closed sets in a convex geometry are called *convex sets*. For $A \subseteq N$ an element $a \in A$ is an *extreme point* of A if $a \notin \overline{A \setminus a}$. For a closed set $A \in \mathcal{L}$ this is equivalent to $A \setminus a \in \mathcal{L}$. Let $ex(A)$ be the set of all extreme points of A. Moreover, convex geometries are the abstract closure spaces satisfying the finite Minkowski-Krein-Milman property: *Every closed set is the closure of its extreme points.*

A *cooperative game* is a function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. The players are the elements of N and the coalitions are the elements $S \subseteq N$ of the boolean algebra 2^N. In this paper, we will define games on closure spaces, taking into account communication restrictions that may cause deficiencies in the cooperation among some players.

Definition 2.2. A game on a closure space $\mathcal{L} \subseteq 2^N$ is a real function $v : \mathcal{L} \to \mathbb{R}$ such that $v(\emptyset) = 0$. The coalitions are the elements of the family $\mathcal{L}$ and the players are the elements $i \in N$. By $\Gamma(\mathcal{L})$ we denote the vector space of all games on $\mathcal{L}$.

We will introduce several examples in the context of cooperative games with restricted cooperation.

Example. A *communication situation* is a triple (N, G, v), where (N, v) is a game and $G = (N, E)$ is a graph. This concept was first introduced in Myerson (1977) and investigated in Borm, Owen and Tijs (1992). If $G = (N, E)$ is a connected block graph, then the family of all coalitions of N that induce connected subgraphs, denoted by

$$\mathcal{L} = \{S \subseteq N \ : \ (S, E(S)) \text{ is connected}\},$$

is a closure space, and also a convex geometry.

Example. Let $(P, \leq)$ be a finite partially ordered set (poset). For any $X \subseteq P$,

$$X \mapsto \overline{X} := \{y \in P : y \leq x \text{ for some } x \in X\},$$

defines a closure operator on P. Its closed sets are the *order ideals* of P, and we denote this lattice $J(P)$. Since the union and intersection of order ideals is again an order ideal, it follows that $J(P)$ is a sublattice of 2^P. Then $J(P)$ is a distributive lattice and so, $J(P)$ is a convex geometry closed under set-union and $ex(S)$ is the set of all maximal points $Max(S)$ of the subposet $S \in J(P)$. There is a 1–1 correspondence between antichains of P and order ideals.

For this reason the games $(\mathcal{L}, v)$ and $(\mathcal{A}, c)$ of Faigle and Kern (1992) (1998), where $\mathcal{L} = J(P)$ is the family of order ideals of P and $\mathcal{A}$ is the set of antichains of a rooted forest, are games on distributive lattices.

Example. We consider the following political game. Let $N = \{1, 2, \ldots, n\}$ be voters and $\mathbb{R}^d$ be an d-dimensional issue space. We will denote by x_i the ideal point of the voter i in this space. Let $u : \mathbb{R}^d \to \mathbb{R}$ be a convex function such that $u(x) = u(-x)$. If $y \in \mathbb{R}^d$ we will let $u_i(y) = u(y - x_i)$ be the utility of the outcome y to voter i. Assume that each voter i will vote in favor of an outcome y if $u_i(y) > \epsilon_y$. Suppose that every voter $i \in A \subseteq N$ will vote in favor of y and that for some $j \notin A$ we have that x_j is in the convex hull of $\{x_i : i \in A\}$. It follows from the convexity of the function u that $u_j(y) > \epsilon_y$ and hence j will vote in favor of y as well. Thus coalitions $A \subseteq N$ that form in this model have the property that x_j in the convex hull of $\{x_i : i \in A\}$ implies that $j \in A$. Since we do not know very much about the function u or the threshold values ϵ_y, it would be reasonable to assume that all coalitions with this closure property might form. This collection of subsets is a *Euclidean convex geometry* (see Example 1 in Edelman and Jamison, 1985).

We introduce two types of supermodularity for games defined on a closure space $\mathcal{L} \subseteq 2^N$. In the first one we use the binary operations join $\vee$ and meet $\wedge$.

Definition 2.3. A game $v \in \Gamma(\mathcal{L})$ is said to be supermodular if

$$v(S \vee T) + v(S \wedge T) \geq v(S) + v(T) \text{ for all } S, T \in \mathcal{L}.$$

Definition 2.4. A game $v \in \Gamma(\mathcal{L})$ is said to be quasi-supermodular if

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T) \text{ for all } S, T \in \mathcal{L} \text{ with } \mathrm{S} \cup T \in \mathcal{L}.$$

Note that every supermodular game is quasi-supermodular.

A game $v \in \Gamma(\mathcal{L})$ is *simple* if $v(S) \in \{0, 1\}$ for each coalition $S \in \mathcal{L}$ and $v(N) = 1$. We assume simple games are *monotonic*, i.e., for any $S, T \in \mathcal{L}$ such that $S \subseteq T$, $v(S) \leq v(T)$. A coalition $S \in \mathcal{L}$ is winning if $v(S) = 1$, and it is called a minimal winning coalition when $v(T) = 0$ for any $T \in \mathcal{L}$ such that $T \subsetneqq S$. For a simple game we denote by $\mathcal{W}$ the family of its minimal winning coalitions.

A player $i \in N$ is called a *veto player* in a simple game $v \in \Gamma(\mathcal{L})$ if i belongs to each winning coalition $S \in \mathcal{L}$. We denote by $\mathcal{V}$ the set of veto players in the game $v \in \Gamma(\mathcal{L})$, i.e.,

$$\mathcal{V} = \bigcap_{\{S \in \mathcal{L} : v(S) = 1\}} S = \bigcap_{S \in \mathcal{W}} S.$$

A simple game is called *weak* if it has at least one veto player, that is $\mathcal{V} \neq \emptyset$. Note that if there exist $i, j \in N, i \neq j$ such that $v(\{i\}) = v(\{j\}) = 1$, then $\mathcal{V} = \emptyset$.

We consider the following simple games. For any $T \in \mathcal{L}$, $T \neq \emptyset$, the unanimity game $\zeta_T : \mathcal{L} \to \mathbb{R}$, is defined by

$$\zeta_T(S) = \begin{cases} 1, & \text{if } T \subseteq S \\ 0, & \text{otherwise.} \end{cases}$$

Notice that the game ζ_T is the unique simple game in which a coalition is winning if it includes all members of T, i.e., coalition T is decisive. Gilboa and Lehrer (1991) obtained the following property for real-valued functions $f : L \to \mathbb{R}$ on a lattice L. The family $\{g_a : L \to \mathbb{R} : a \in L\}$ defined by $g_a(b) = 1$ if $b \geq a$, and $g_a(b) = 0$, otherwise, form a basis of the vector space of all functions $\mathbb{R}^L$.

3 The Weber set and the core in simple games

For each $S \in \mathcal{L}$ and $x \in \mathbb{R}^n$, we define $x(S) = \sum_{i \in S} x_i$, $x(\emptyset) = 0$. The *imputation set* and the *core* of a game $v \in \Gamma(\mathcal{L})$ are given by the following polyhedrons

$$\begin{aligned} I(\mathcal{L}, v) &= \{x \in \mathbb{R}^n : x(N) = v(N),\ x(\{i\}) \geq v(\{i\}) \text{ for all } \{i\} \in \mathcal{L}\} \\ Core(\mathcal{L}, v) &= \{x \in \mathbb{R}^n : x(N) = v(N),\ x(S) \geq v(S) \text{ for all } S \in \mathcal{L}\}. \end{aligned}$$

If we denote by $\{e_i\}_{i=1}^n$ the vectors of the canonical basis of $\mathbb{R}^n$, then there are three possibilities for the imputation set $I(\mathcal{L}, v)$ of a simple game on an atomic closure space $\mathcal{L} \subseteq 2^N$.

- $I(\mathcal{L}, v) = \emptyset$, if there are two winning coalitions of cardinality one.
- $I(\mathcal{L}, v) = \{e_k\}$, if there exists $k \in N$, with $v(\{k\}) = 1$, and $v(\{i\}) = 0$ for any $i \in N$ such that $i \neq k$.
- $I(\mathcal{L}, v) = \text{conv}\{e_1, \ldots, e_n\}$, if $v(\{i\}) = 0$ for all $i \in N$.

We consider now some results concerning the structure of the core for this type of games.

Theorem 3.1. *Let* $\mathcal{L} \subseteq 2^N$ *be an atomic closure space, and* $v \in \Gamma(\mathcal{L})$ *a simple game. Then* v *is weak if and only if* $Core(\mathcal{L}, v) \neq \emptyset$*. Furthermore,*

$$Core(\mathcal{L}, v) = \{x \in \mathbb{R}^n : x \geq 0,\ x(N) = x(\mathcal{V}) = 1\}.$$

Proof. Suppose that $\mathcal{V} \neq \emptyset$, and let $e_i \in \mathbb{R}^n$ where $i \in \mathcal{V}$. Then, $e_i(N) = 1 = v(N)$ and $e_i(S) \geq v(S)$ for all $S \in \mathcal{L}$, and therefore $e_i \in Core(\mathcal{L}, v)$.

Conversely, if $Core(\mathcal{L}, v)$ is nonempty, then there exists $x \neq 0$, and the atomicity of $\mathcal{L}$ implies that $x \geq 0$. Further, $\sum_{i \in S} x_i = 1$, for all $S \in \mathcal{W}$. We show, by contradiction, that $\mathcal{V} \neq \emptyset$. Assume that the set of veto players is empty. Then we can take the sum of the equations $\sum_{i \in S} x_i = 1$ for all $S \in \mathcal{W}$, and we obtain $\alpha_1 x_1 + \cdots + \alpha_n x_n = |\mathcal{W}|$, with $\alpha_j < |\mathcal{W}|$ for all $1 \leq j \leq n$. Therefore, $(|\mathcal{W}| - \alpha_1) x_1 + \cdots + (|\mathcal{W}| - \alpha_n) x_n = 0$, and this gives a contradiction.

Finally, if $i \notin \mathcal{V}$ then $x_i = 0$, for all $x \in Core(\mathcal{L}, v)$. □

Corollary 3.1. *Let* $\mathcal{L} \subseteq 2^N$ *be an atomic closure space, and* $v \in \Gamma(\mathcal{L})$ *a simple game. If* v *is weak, then*

$$Core(\mathcal{L}, v) = \operatorname{conv}\{e_i : i \in \mathcal{V}\}.$$

In the framework of games on convex geometries, the formation of the grand coalition N can be modeled as a sequential process by using the extension property. Each one of these sequential orders can to be identified with the following concept.

A *compatible ordering* of the convex geometry $\mathcal{L}$ is a total ordering of the elements of N, $i_1 < i_2 < \cdots < i_n$ such that the sets

$$\{i_1, i_2, \ldots, i_j\} \in \mathcal{L}, \quad \text{for all } 1 \leq j \leq n.$$

A compatible ordering of $\mathcal{L}$ corresponds exactly to a maximal chain in $\mathcal{L}$. We will denote the set of all the maximal chains of $\mathcal{L}$ by $\mathcal{C}(\mathcal{L})$. Given an element $i \in N$ and a maximal chain C, the set

$$C(i) = \{j \in N : j \leq i \text{ in the chain } C\},$$

is the coalition of $\mathcal{L}$ constructed by the player i and his predecessors in the chain C. Note that $i \in ex(C(i))$ by using the definition of extreme point in the previous section.

Definition 3.1. Let $\mathcal{L} \subseteq 2^N$ be a convex geometry. The marginal worth vector $a^C \in \mathbb{R}^n$ with respect to the chain $C \in \mathcal{C}(\mathcal{L})$ in the game $v \in \Gamma(\mathcal{L})$ is given by

$$a_i^C = v(C(i)) - v(C(i) \setminus i) \text{ for all } i \in N.$$

The *Weber set* of a game v is the convex hull of the marginal worth vectors,

$$Weber(\mathcal{L}, v) = \text{conv}\left\{a^C : C \in \mathcal{C}(\mathcal{L})\right\}.$$

It is easy to prove that $a^C(S) = v(S)$ for all $S \in C$.

Proposition 3.1. *Let $\mathcal{L} \subseteq 2^N$ be a convex geometry and let $v \in \Gamma(\mathcal{L})$ be a simple game. If $\{S_1, \ldots, S_r\}$ is the set of its minimal winning coalitions, then*

$$Weber(\mathcal{L}, v) = \text{conv}\left\{e_i : i \in \bigcup_{k=1}^{r} ex(S_k)\right\}.$$

Proof. Let $C \in \mathcal{C}(\mathcal{L})$ be a maximal chain and we consider the marginal worth vector a^C. For all $i \in N$ we obtain $a_i^C = v(C(i)) - v(C(i) \setminus i) \geq 0$ since v is a monotonic game. Moreover $a^C(N) = v(N) = 1$. Hence, $a^C \in \mathbb{R}^n$ is a vector with only one of its components equal to 1 and the rest of them equal to 0. Assume that the component j is equal to 1, i.e., $v(C(j)) = 1$ and $v(C(j) \setminus j) = 0$. Then $a^C = e_j$. On the other hand, $C(j)$ is a winning coalition and therefore there exists a minimal winning coalition S_k such that $S_k \subseteq C(j)$. Note that $j \in S_k$ since if $j \notin S_k$ then $S_k \subseteq C(j) \setminus j$ and so $C(j) \setminus j$ is a winning coalition but $v(C(j) \setminus j) = 0$. Moreover, by the definition of extreme point, we have that $j \in ex(S_k)$ since $S_k \setminus j = (C(j) \setminus j) \cap S_k \in \mathcal{L}$.

In order to prove the reverse inclusion, if $i \in \bigcup_{k=1}^{r} ex(S_k)$, then there exists a minimal winning coalition S_k such that $i \in ex(S_k)$. Let $C \in \mathcal{C}(\mathcal{L})$ be the maximal chain such that $S_k = C(i)$. The marginal worth vector for this chain satisfies $a^C = e_i$. □

Bilbao, Lebrón and Jiménez (1999) showed the following result: *A game $v \in \Gamma(\mathcal{L})$ on a convex geometry $\mathcal{L}$ is quasi-supermodular if and only if $Weber(\mathcal{L}, v) \subseteq Core(\mathcal{L}, v)$.*

Proposition 3.2. *Let $\mathcal{L} \subseteq 2^N$ be a convex geometry and let $v \in \Gamma(\mathcal{L})$ be a simple game. The following statements are equivalent:*

1. *The game v is quasi-supermodular.*
2. *The game v is a unanimity game.*
3. *The game v is supermodular.*

Proof. (2) $\Rightarrow$ (1) We show that unanimity games are quasi-supermodular games. For $T \in \mathcal{L}$, $T \neq \emptyset$, if we consider the game ζ_T, we distinguish the following cases:

(a) Let $A, B \in \mathcal{L}$ such that $A \cup B \in \mathcal{L}$, $A \supseteq T$, $B \supseteq T$. In this case, we have $\zeta_T(A) = 1$, $\zeta_T(B) = 1$, $\zeta_T(A \cup B) = 1$ and $\zeta_T(A \cap B) = 1$.

(b) If $A, B \in \mathcal{L}$ such that $A \cup B \in \mathcal{L}$, with $A \supseteq T$, $B \not\supseteq T$, then we have $\zeta_T(A) = 1$, $\zeta_T(B) = 0$, $\zeta_T(A \cup B) = 1$ and $\zeta_T(A \cap B) = 0$.

(c) If $A, B \in \mathcal{L}$ such that $A \cup B \in \mathcal{L}$, with $A \not\supseteq T$, $B \not\supseteq T$, then we have $\zeta_T(A) = 0$, $\zeta_T(B) = 0$, $\zeta_T(A \cup B) \geq 0$ and $\zeta_T(A \cap B) = 0$.

(3) $\Rightarrow$ (2) Let $v \in \Gamma(\mathcal{L})$ be a supermodular game. Let $T \in \mathcal{L}$ such that

$$|T| = \min\{|S| : v(S) = 1\} = \alpha.$$

This coalition T is unique because if $A, T \in \mathcal{L}$, $A \neq T$ satisfy $v(T) = v(A) = 1$ and $|T| = |A| = \alpha$, then $v(T) + v(A) > v\left(\overline{T \cup A}\right) + v(T \cap A)$. But it is impossible since the game v is supermodular. Thus, there is only one coalition for which the minimum is attained. It is clear that $v(A) = 1$ for all $A \in \mathcal{L}$ such that $A \supseteq T$. Let $A \in \mathcal{L}$ such that $A \not\supseteq T$. Then

$$v(A) + v(T) \leq v\left(\overline{A \cup T}\right) + v(A \cap T) = 1.$$

Hence $v(A) = 0$ and $v = \zeta_T$.

(1) $\Rightarrow$ (3) It follows from theorem 6 in Bilbao *et al.* (1999) that every monotonic quasi-supermodular game is a supermodular game. □

Remark 3.1. When $\mathcal{L}$ is a closure space, but not a convex geometry, only the equivalence (2) $\Longleftrightarrow$ (3) is true.

The unanimity games on atomic convex geometries are weak games. If we consider the unanimity game $v = \zeta_T$, $T \neq \emptyset$, $T \in \mathcal{L}$, then we obtain:

- $\mathcal{V} = T$,
- $Core(\mathcal{L}, v) = \text{conv}\{e_i : i \in T\}$,
- $Weber(\mathcal{L}, v) = \text{conv}\{e_i : i \in ex(T)\}$,
- $Core(\mathcal{L}, v) = Weber(\mathcal{L}, v) \Longleftrightarrow T = ex(T) \Longleftrightarrow 2^T \subseteq \mathcal{L}$.

We next define the concept of *dummy player.* We will show that each imputation that satisfies the dummy axiom is an element of the Weber set.

Definition 3.2. The player $i \in N$ is dummy in a game $v \in \Gamma(\mathcal{L})$ if, for every $S \in \mathcal{L}$ such that $i \in ex(S)$, we have

$$v(S) - v(S \setminus i) = \begin{cases} v(\{i\}), & \text{if } \{i\} \in \mathcal{L} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3.3. A nonempty set $J \subseteq \mathbb{R}^n$ satisfies the dummy player property when for all $i \in N$ such that i is a dummy player in a game v and for all $x \in J$, then

$$x_i = \begin{cases} v(\{i\}), & \text{if } \{i\} \in \mathcal{L} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.2. *Let $\mathcal{L} \subseteq 2^N$ be an atomic convex geometry and let $v \in \Gamma(\mathcal{L})$ be a simple game. If $J \subseteq I(\mathcal{L}, v)$ satisfies the dummy player property then $J \subseteq Weber(\mathcal{L}, v)$. The converse is true when $I(\mathcal{L}, v) = \text{conv}\{e_1, \ldots, e_n\}$.*

Proof. First, we prove that if J satisfies the dummy player property, then it is contained in the Weber set. If $I(\mathcal{L}, v) = \{e_i\}$ the result follows easily. Thus, we assume that $I(\mathcal{L}, v) = \text{conv}\{e_1, \ldots, e_n\}$, i.e., there is no winning coalition $T \in \mathcal{L}$ such that $|T| = 1$ and so, $v(\{i\}) = 0$ for all $i \in N$. If $x \in J$, then $x \in \text{conv}\{e_1, \ldots, e_n\}$ and so, $x = \sum_{i=1}^{n} \mu_i e_i$ with $\mu_i \geq 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \mu_i = 1$. In order to prove that $x \in Weber(\mathcal{L}, v)$, we show that $\mu_k = 0$ when $k \in N \setminus \bigcup_{j=1}^{r} ex(S_j)$. Let $k \in N \setminus \bigcup_{j=1}^{r} ex(S_j)$, we have that $v(S) - v(S \setminus k) \geq 0$ for all $S \in \mathcal{L}$ such that $k \in ex(S)$ since the game v is monotonic. Moreover, $v(S) - v(S \setminus k) = 0$ for all $S \in \mathcal{L}$ such that $k \in ex(S)$. Indeed, if $v(S) - v(S \setminus k) = 1$, then $v(S) = 1$ and $v(S \setminus k) = 0$.

In this case, there would be a minimal winning coalition $S_j \subseteq S$ with $k \in S_j$, because if $S_j \subseteq S \setminus k$ we would obtain $v(S \setminus k) = 1$, which is not possible. Therefore, k is a dummy player in v and so, $\mu_k = x_k = v(\{k\}) = 0$.

Conversely, let v be a simple game such that $I(\mathcal{L}, v) = \operatorname{conv}\{e_1, \ldots, e_n\}$ and let $J \subseteq Weber(\mathcal{L}, v)$. Using the inclusion $Weber(\mathcal{L}, v) \subseteq I(\mathcal{L}, v)$, we have $J \subseteq I(\mathcal{L}, v)$. Next, if $k \in N$ is a dummy player in v and $x \in J$, then $x_k = v(S) - v(S \setminus k)$ for some $S \in \mathcal{L}$ such that $k \in ex(S)$, because $x \in Weber(\mathcal{L}, v)$. Therefore, $x_k = v(\{k\})$. □

4 Other solution concepts in simple games

Faigle and Kern (1992) introduced the *hierarchical strength* $h_S(i)$ in distributive lattices $\mathcal{L}$ as follows.

$$h_S(i) := \frac{|\{C \in \mathcal{C}(\mathcal{L}) \ : \ C(i) \cap S = S\}|}{|\mathcal{C}(\mathcal{L})|}, \quad \text{for all } i \in S, \ S \in \mathcal{L}.$$

The hierarchical strength $h_S(i)$ is the average number of compatible orderings of $\mathcal{L}$ in which i is the last member of S in the ordering. Note that $h_S(i)$ satisfies $h_S(i) \neq 0$ if and only if $i \in ex(S)$.

For games on distributive lattices, Faigle and Kern (1992) showed that the Shapley value is the unique operator

$$\Phi : \Gamma(\mathcal{L}) \to \mathbb{R}^n, \quad v \mapsto (\Phi_1(v), \ldots, \Phi_n(v)),$$

which satisfies the following axioms:

(A1) *(Linearity)* For all $\alpha, \beta \in \mathbb{R}$, and $v, w \in \Gamma(\mathcal{L})$ we have

$$\Phi(\alpha v + \beta w) = \alpha \Phi(v) + \beta \Phi(w).$$

(A2) *(Carrier)* If $U \in \mathcal{L}$ is a carrier of $v \in \Gamma(\mathcal{L})$, that is, $v(S) = v(S \cap U)$ for all $S \in \mathcal{L}$ then

$$\sum_{i \in U} \Phi_i(v) = v(U).$$

(A3) *(Hierarchical strength)* For any $S \in \mathcal{L}$ and $i, j \in S$,

$$h_S(i)\Phi_j(\zeta_S) = h_S(j)\Phi_i(\zeta_S).$$

Moreover, they obtained the formula

$$\Phi_i(v) = \frac{1}{c(N)} \sum_{C \in \mathcal{C}(\mathcal{L})} [v(C(i)) - v(C(i) \setminus i)], \tag{4.1}$$

for all $i \in N$, where $c(N)$ is the number of maximal chains in $\mathcal{L}$.

Bilbao and Edelman (2000) extended (4.1) for games on convex geometries. We observe next that

$$\Phi(v) = \frac{1}{c(N)} \sum_{C \in \mathcal{C}(\mathcal{L})} a^C,$$

and hence $\Phi(v) \in Weber(\mathcal{L}, v)$ for all $v \in \Gamma(\mathcal{L})$.

Theorem 4.1. *Let $\mathcal{L} \subseteq 2^N$ be an atomic convex geometry. A simple game $v \in \Gamma(\mathcal{L})$ is a unanimity game if and only if $\Phi(v) \in Core(\mathcal{L}, v)$.*

Proof. Suppose that $\Phi(v) \in Core(\mathcal{L}, v) = \operatorname{conv}\left\{e_i : i \in \bigcap_{j=1}^r S_j\right\}$. Then $\Phi_i(v) = 0$ for all $i \in N \setminus \bigcap_{j=1}^r S_j$. Moreover, (4.1) implies that $\Phi_i(v) \neq 0$ for all $i \in \bigcup_{j=1}^r ex(S_j)$. We show that v is quasi-supermodular. Assume not, then $Weber(\mathcal{L}, v) \nsubseteq Core(\mathcal{L}, v)$. It follows from corollary (3.1) and proposition (3.1) that there exists a player

$$i \in \bigcup_{j=1}^r ex(S_j) \setminus \bigcap_{j=1}^r S_j.$$

For this i, we have $\Phi_i(v) = 0$ and $\Phi_i(v) \neq 0$ but it is impossible.

The converse follows from $\Phi(v) \in Weber(\mathcal{L}, v) \subseteq Core(\mathcal{L}, v)$, where the inclusion holds for quasi-supermodular games. □

Now we introduce the notion of *dominance* between imputations and the concept of *stable set.*

Definition 4.1. Let $v \in \Gamma(\mathcal{L})$ and $x, y \in I(\mathcal{L}, v)$. The imputation x dominates the imputation y (we denote $x \operatorname{dom} y$) if there exists a nonempty $S \in \mathcal{L}$ such that $x_i > y_i$ for all $i \in S$, and $x(S) \leq v(S)$.

Definition 4.2. Let $I(\mathcal{L}, v) \neq \emptyset$. A set $E \subseteq I(\mathcal{L}, v)$ is stable if the imputations in E are undominated by other elements of E, and every imputation not in E is dominated by an imputation in E.

Theorem 4.2. *Let $\mathcal{L} \subseteq 2^N$ be an atomic convex geometry and let $v \in \Gamma(\mathcal{L})$ be a simple game. The $Core(\mathcal{L}, v)$ is a stable set if and only if the game v is a unanimity game.*

Proof. If $Core(\mathcal{L}, v)$ is a stable set, then it must be nonempty and we can now apply corollary 3.1 to obtain

$$Core(\mathcal{L}, v) = \operatorname{conv}\left\{ e_i \,:\, i \in \bigcap_{j=1}^{r} S_j \right\},$$

where $\{S_1, \ldots, S_r\}$ is the set of minimal winning coalition with the ordering $|S_1| \leq |S_2| \leq \ldots \leq |S_r|$. We next consider two cases:

(1) If $|S_1| = 1$ or $|S_1| = n$, then $v = \zeta_{S_1}$.

(2) If $1 < |S_1| < n$, and assume that $Weber(\mathcal{L}, v) \nsubseteq Core(\mathcal{L}, v)$, i.e. v is not a unanimity game, then there exists $k \in \bigcup_{j=1}^{r} ex(S_j) \setminus \bigcap_{j=1}^{r} S_j$ and so $e_k \notin Core(\mathcal{L}, v)$. As $Core(\mathcal{L}, v)$ is a stable set, there exists $x \in Core(\mathcal{L}, v)$ such that $x \operatorname{dom} e_k$ using a coalition $T \in \mathcal{L}$. Since $x(T) = v(T)$, $x_i > 0$ for all $i \in T$, $i \neq k$, and $x_k > 1$ if $k \in T$, there must be $v(T) = 1$ and $k \notin T$. Thus, there exists a minimal winning coalition $S_{j^*} \in \mathcal{L}$ such that $S_{j^*} \subseteq T$ and $x(S_{j^*}) = v(S_{j^*}) = 1$. Then, we infer that $x_i > 0$ for all $i \in S_{j^*}$ and since $x(N \setminus S_{j^*}) = 0$, it implies that $x(S_p \setminus S_{j^*}) = 0$ for every minimal winning coalition $S_p \in \mathcal{L}$, $S_p \neq S_{j^*}$. However, we have that $x \in Core(\mathcal{L}, v)$, $x(S_p) = 1$ holds and hence $x(S_p \cap S_{j^*}) = 1$ for every minimal winning coalition $S_p \in \mathcal{L}$. Then, we assert that $S_{j^*} \subseteq \bigcap_{j=1}^{r} S_j$. Assume not, there exists a minimal winning coalition $S_p \in \mathcal{L}$ such that $S_{j^*} \nsubseteq S_p$ and we should have that

$$x(N) = x(S_{j^*} \cap S_p) + x(S_{j^*} \setminus S_p) + \ldots > 1,$$

but it is impossible. Therefore, $\bigcap_{j=1}^{r} S_j = S_{j^*}$ and hence $v = \zeta_{S_{j^*}}$.

Conversely, let $v = \zeta_T$ be a unanimity game. When $|T| = 1$ or $|T| = n$, the $Core(\mathcal{L}, v) = I(\mathcal{L}, v)$ and so it is stable. Assume that $1 < |T| < n$, and $Core(\mathcal{L}, v) \neq I(\mathcal{L}, v)$. For every $x \in I(\mathcal{L}, v) \setminus Core(\mathcal{L}, v)$ we can take $y \in Core(\mathcal{L}, v)$, defined by

$$y_i = \begin{cases} 0, & \text{if } i \notin T \\ x_i + (1 - x(T)) \,/\, |\, T \,|, & \text{if } i \in T. \end{cases}$$

Thus $y \operatorname{dom} x$ using coalition T, and the core is a stable set. □

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