The Core for Games with Cooperation Structure

Inés Gallego[1](✉), Michel Grabisch[2,3], Andrés Jiménez-Losada[1], and Alexandre Skoda[3]

[1] Universidad de Sevilla, Sevilla, Spain
inesgal@us.es
[2] Paris School of Economics, Paris, France
[3] Université de Paris I, Paris, France

Abstract. A cooperative game consists of a set of players and a characteristic function that determines the maximal profit or minimal cost that each subset of players can get when they decide to cooperate, regardless of the actions of the rest of the players. The relationships among the players can modify their bargaining and therefore their payoffs. The model of cooperation structures in a game introduces a graph on the set of players setting their relations and in which its components indicate the groups of players that are initially formed. In this paper we define the core and the Weber set and the notion of convexity for this family of games.

Keywords: Cooperative game · A priori unions · Core · Weber set · Cooperation structure · Convexity

1 Introduction

Cooperative game theory studies situations in which a set of agents (called players) negotiate a fair allocation of a common profit resulting from collaboration, i.e., a vector in which each coordinate represents the payoff that each player receives. In order to set these payoffs we have the information given by the characteristic function of the game. It is a mapping that assigns to each subset of players (named coalitions) a number that represents the profit obtained. The Shapley value [14] is the point solution for cooperative games mostly used and studied. It is a function that returns a payoff vector for each game, satisfying a set of reasonable conditions (axioms) which allow us to decide if this value is or not an appropriate solution for the problem. Several variations of the Shapley value have been defined for situations where some extra information among the agents is known. Aumann and Drèze [2] introduced coalition structures. A coalition structure is a partition of the set of players representing the different coalitions obtained when the game ends. Hence there should be no side payments among them. This model has been generalized by Myerson [9] considering communication structures. A communication structure is a graph representing the feasible bilateral communication among the agents. Here the final coalition structure is the set of connected components in the graph but we can also use the

N.T. Nguyen et al. (Eds.): TCCI XXIII, LNCS 9760, pp. 172–188, 2016.
DOI: 10.1007/978-3-662-52886-0_12

information given by the internal structure of these coalitions. Owen [11] proposed a different model to Aumann and Drèze's based on another interpretation of the coalition structure. He considered that the coalition structure is a partition of the set of players in a priori unions that take into account the relations among the agents. However these unions are not seen as a final structure but as a starting point for further negotiations. Thus, as in the original Shapley model, the grand coalition is the final structure. So, a coalition of players forms a union if they have the similar interests in the game. Owen obtained a Shapley-type solution (the Owen value) in this model that fairly allocates the profit of the grand coalition. Later Casajus [3] proposed a modification of the Owen model in the Myerson sense named cooperation structure. That is, we have an a priori union structure but we also know how these unions are formed by means of a connected graph in each group.

This paper is related to the latter model. In a cooperative game the core (Gillies [6]) and the Weber set (Weber [16,17]) are set solutions, i.e., they select a set of reasonable payoff vectors under different perceptions. In particular, the Shapley value is always the center of mass of the Weber set but it need not be included in the core. Between these two sets there exists a relationship of inclusion, the core is always included in the Weber set. There is a property about the characteristic function, convexity, that ensures that both sets are equal. Pulido and Sánchez-Soriano [13] defined the concepts of core, Weber set and convexity for games with a priori union structure. In this article we show that we can use these concepts for games with cooperation structure by modifying them in an appropriate manner. Thus, our results generalize those of Pulido and Sánchez-Soriano.

Next section is dedicated to preliminaries about cooperative games and a priori unions, including the main results of Pulido and Sánchez-Soriano [13]. Section 3 presents in detail the model of Casajus [3], cooperation structures, i.e., a priori unions with communication structure. We introduce the cooperation core in Sect. 4 and prove that it is a restricted core in the sense of Faigle [4]. In Sect. 5 we find an axiomatization of the cooperation core. Finally in Sect. 6 we define the cooperation Weber set and find a condition of convexity that guarantees the inclusion of the cooperation Weber set into the cooperation core.

2 Preliminaries

2.1 Cooperative TU-Games

A *cooperative game with transferable utility*, game for short, is a pair (N, v) where N is a finite set of elements and $v : 2^N \to \mathbb{R}$ is a mapping on the power set of N satisfying that $v(\emptyset) = 0$. The elements of N are named *players*, the subsets of N are called *coalitions*, v is the characteristic function of the game and $v(S)$ is the *worth* of S. We denote as G the set of games. If $(N, v) \in G$ and $S \subseteq N$ then $(S, v) = (S, v_S) \in G$ is a new game where v_S is the restriction of the characteristic function v to 2^S. An example of game is the unanimity game (N, u_T), with $T \subseteq N$ a non-empty coalition, defined as $u_T(S) = 1$ if $T \subseteq S$ and

$u_T(S) = 0$ otherwise. If we fix a set of players N, the family $\{u_T : T \subseteq N\}$ is a basis of the characteristic functions over N, that is, for every game (N, v) there are coefficients c_T with $T \subseteq N$ such that

$$v = \sum_{\{T \subseteq N : T \neq \emptyset\}} c_T u_T. \tag{1}$$

An important class of games is the class of convex games: a game (N, v) is *convex* if for any $S, T \in 2^N$,

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T).$$

An *allocation rule* for games is a function ψ over G which determines for each game (N, v) a vector $\psi(N, v) \in \mathbb{R}^N$ interpreted as a payoff vector. The most important allocation rule is the *Shapley value* ϕ defined for every $(N, v) \in G$ and $i \in N$ as

$$\phi_i(N, v) = \sum_{\{S \subset N : i \notin S\}} \frac{(|N| - |S| - 1)!|S|!}{|N|!} [v(S \cup i) - v(S)]. \tag{2}$$

A function Γ that selects a set of payoff vectors, i.e., given $(N, v), \Gamma(N, v) \subseteq \mathbb{R}^N$, is called a *set solution*. One of the most important set solutions in the literature is the *core* (Gillies [6]), that for each game gives the set of payoffs that are efficient and coalitionally rational, i.e., the core of a usual game is defined by

$$C(N, v) = \left\{x \in \mathbb{R}^N : x(N) = v(N),\ x(S) \geq v(S), S \in 2^N\right\},$$

where $x(S) = \sum_{i \in S} x_i$ and x_i denotes the payoff associated to player $i \in N$.

It is well known that if the game is convex, its core is nonempty (Shapley [15]).

When we are interested only in coalitional rationality for a determined family of coalitions, $\mathcal{F} \subseteq 2^N$, we can use the *restricted core* of a game given by Faigle [4] which is defined by

$$C(N, v, \mathcal{F}) = \left\{x \in \mathbb{R}^N : x(N) = v(N), x(S) \geq v(S), S \in \mathcal{F}\right\},$$

where $\mathcal{F} \subseteq 2^N$.

A *permutation* π on N is a bijective mapping $\pi : N \to N$. Given a finite set N, Π_N will denote the set of all permutations on N. Let (N, v) be a game and $\pi \in \Pi_N$, the *marginal worth vector* of v with respect to the ordering $\pi, m^{\pi,v} \in \mathbb{R}^N$, is defined as

$$m^{\pi,v}_{\pi(i)} = v(\{\pi(1), \ldots, \pi(i)\}) - v(\{\pi(1), \ldots, \pi(i-1)\}).$$

The *Weber set* of v is the convex hull of all $n!$ marginal vectors:

$$W(N, v) = conv(m^{v,\pi} | \pi \in \Pi_N).$$

Weber proved in [16] and [17] that the Weber set contains the core. Shapley in [15] proved that these two sets coincide when v is convex, and in Ichiishi [8] the other implication is proven.

2.2 A Priori Unions

A game with *a priori unions* is a triple $(N, v, \mathcal{P})$ where $(N, v) \in G$ is a game and $\mathcal{P} = \{b_1, ..., b_m\}$ a partition of N. For each k, players in b_k have similar interests in the game and then they negotiate together to get payoffs. The *Owen value* ω is an allocation rule over the class of games with a priori unions. It is supposed that players are interested in the grand coalition N but considering the a priori unions as bargaining elements. Given $(N, v, \mathcal{P})$, the *quotient game* is a game $(M, v^{\mathcal{P}})$ with set of players $M = \{1, ..., m\}$ defined as

$$v^{\mathcal{P}}(Q) = v\left(\bigcup_{q \in Q} b_q\right), \forall Q \subseteq M. \tag{3}$$

Let $k \in M$. For each $S \subset b_k$ the partition $\mathcal{P}_S$ of $N \setminus (b_k \setminus S)$ consists in replacing b_k with S in $\mathcal{P}$, i.e.,

$$\mathcal{P}_S = \{b_1, \ldots, \overset{k)}{S}, \ldots, b_m\}.$$

We define the game (b_k, v_k) as $v_k(S) = \phi_k\left(M, v^{\mathcal{P}_S}\right), \forall S \subseteq b_k$. Finally in every group the game is solved using also the Shapley value. So, for each player $i \in N$ if $k(i)$ is such that $i \in b_{k(i)}$ then the Owen value is

$$\omega_i(N, v, \mathcal{P}) = \phi_i\left(b_{k(i)}, v_{k(i)}\right). \tag{4}$$

Owen [11] gave an axiomatization of this value over the class of games with a priori unions.

Following Owen's philosophy, Pulido and Sánchez-Soriano [13] constructed a concept of core on the class of games with a priori unions, the *coalitional core*. They also defined the *coalitional Weber set* and found a condition of convexity that is equivalent to the inclusion of the coalitional Weber set into the coalitional core.

In the first step, i.e., in the negotiation among the a priori unions, they distribute $v(N)$ according to a core allocation of the quotient game $y \in C(M, v^{\mathcal{P}})$, so each a priori union receives y_k in the first step. Then in the second step the members of each union b_k reach an agreement based on a core allocation of game (b_k, v_k^y), where

$$v_k^y(S) = \begin{cases} y_k, & \text{if } S = b_k \\ \max_{R \subseteq M \setminus k}\left[v\left(\bigcup_{r \in R} b_r \cup S\right) - y(R)\right], & \text{if } S \subset b_k. \end{cases}$$

Then, the coalitional core is defined by

$$C_c(N, v, \mathcal{P}) = \bigcup_{y \in C(M, v^{\mathcal{P}})} \prod_{k=1}^{m} C\left(b_k, v_k^y\right).$$

Pulido and Sánchez-Soriano [13] proved that the coalitional core is a restricted core in the sense of Faigle [4]. Given $(N, v, \mathcal{P})$ a game with a priori union structure, they defined the concept of $\mathcal{P}$*-coalition* as any set $R \subseteq N$ s.t. $R = S \cup Q$ with $S \subseteq b_p$ for some $p \in M$ and $Q = \cup_{k \in K} b_k$ for some subset $K \subseteq M \setminus p$.

Then, if $\Omega_{\mathcal{P}}$ is the set of $\mathcal{P}$-coalitions in $(N, v, \mathcal{P})$, it holds

$$C_c(N, v, \mathcal{P}) = C(N, v, \Omega_{\mathcal{P}})$$

To define the coalitional Weber set, they only consider a subset of permutations of Π_N, which they called $\mathcal{P}$-consistent permutations. A permutation $\pi \in \Pi_N$ is *$\mathcal{P}$-consistent* if the members of each union are kept together in it. So the coalitional Weber set of $(N, v, \mathcal{P})$ is defined by

$$W_c(N, v, \mathcal{P}) = conv(m^{\pi,v} | \pi \in \Pi_{\mathcal{P}}),$$

where $\Pi_{\mathcal{P}}$ is the set of all $\mathcal{P}$-consistent permutations in $(N, v, \mathcal{P})$.

In [13] there is an example that proves that the inclusion of the core in the Weber set cannot be extended to games with a priori unions, but the authors found a weaker condition than convexity implying the reverse inclusion. It is convexity restricted to sets in the family $\Omega_{\mathcal{P}}$. An even weaker condition of convexity, named coalitional convexity, that assures this fact was found.

Let $(N, v, \mathcal{P})$ be a game with a priori unions. This game is *coalitional convex* if the following conditions hold:

1. For every $i \in M$, $K \subseteq M \setminus i$ and $S, T \subseteq b_i$,

$$v\left((S \cup T) \cup \bigcup_{k \in K} b_k\right) + v\left((S \cap T) \cup \bigcup_{k \in K} b_k\right) \geq v\left(S \cup \bigcup_{k \in K} b_k\right) + v\left(T \cup \bigcup_{k \in K} b_k\right)$$

2. For every $i \in M$, $K_1 \subset K_2 \subseteq M \setminus i$ and $S \subseteq b_i$

$$v\left(S \cup \bigcup_{k \in K_2} b_k\right) - v\left(\bigcup_{k \in K_2} b_k\right) \geq v\left(S \cup \bigcup_{k \in K_1} b_k\right) - v\left(\bigcup_{k \in K_1} b_k\right)$$

3. For every $i \in M$, $K_1 \subset K_2 \subseteq M \setminus i$ and $S \subseteq b_i$

$$v\left(b_i \cup \bigcup_{k \in K_2} b_k\right) - v\left(S \cup \bigcup_{k \in K_2} b_k\right) \geq v\left(b_i \cup \bigcup_{k \in K_1} b_k\right) - v\left(S \cup \bigcup_{k \in K_1} b_k\right)$$

They proved this equivalence: $(N, v, \mathcal{P})$ is a coalitional convex game if and only if $W_c(N, v, \mathcal{P}) \subseteq C_c(N, v, \mathcal{P})$.

3 Cooperation Structures

Myerson [9] considered that there were real situations in which not all coalitions were feasible, so he introduced communication structures in games. Given the set of all possible (unordered) pairs of N, $LN = \{\{i, j\} : i, j \in N \text{ and } i \neq j\}$, a *communication structure* over N, (N, L), is represented by a graph, where the set of vertices N is the set of players and the set of feasible communications among them is the set of links, $L \subseteq LN$. Then a game with communication structure is a triple (N, v, L), where $(N, v) \in G$ and L is a communication structure over N. A coalition $S \subseteq N$ is *connected* in L if for every pair $\{i, j\}$ of players in S there is a path in L_S linking i and j where L_S denotes the restricted graph. We denote by N/L the set of *connected components* of the graph L that is, the set of maximal connected coalitions of L, and by S/L the set of connected components of L_S. Myerson introduced the graph restricted game as a way of incorporating the information given by the graph. It is defined as

$$v/L(S) = \sum_{T \in S/L} v(T),\ S \in 2^N.$$

Then the *Myerson value* μ is defined as

$$\mu(N, v, L) = \phi(N, v/L).$$

In the Myerson value, the coalition structure that results at the end of the game is N/L, then two distinct connected components cannot cooperate, and inside each one, the benefits are shared according to their feasible communications. Casajus [3] proposed the Owen perspective of a communication structure, named *cooperation structure.* Now, the connected components are the a priori unions and the graph represents the communication structure that forms each union. He also proposed an allocation rule ξ for games with cooperation structure. It is obtained by following Owen's two-step procedure. If $N/L = \{N_1, \ldots, N_m\}$ and $M = \{1, \ldots, m\}$, in the first step players distribute the profit among the connected components using the Shapley value and the quotient game, i.e., for each $k \in M$,

$$v_k(S) = \phi_k\left(M, v^{(N/L)_S}\right),\ \forall S \subseteq N_k,$$

where $v^{(N/L)}$ is defined by (3) and $v^{(N/L)_S}$ corresponds to the quotient game with respect to the partition $(N/L)_S$. In the second step players allocate the profit inside each union taking into account the communication structure

$$\xi_i(N, v, L) = \mu_i\left(N_{k(i)}, v_{k(i)}, L_{N_{k(i)}}\right),$$

where $k(i)$ is s.t. $i \in N_{k(i)}$ and $L_{N_{k(i)}}$ denotes the restricted graph.

This value, which is named *Myerson-Owen* value in [5], is a generalization of other well-known values

1. If the graph L is connected then the Myerson-Owen value ξ coincides with the Myerson value μ.
2. If each component of L is a complete subgraph, we take the complete components as a priori unions and then the Myerson-Owen value ξ coincides with the Owen value ω
3. If L is the complete graph then the Myerson-Owen value ξ coincides with the Shapley value ϕ.

In this paper we introduce the core for games with cooperation structure following Casajus [3]. This approach is a generalization of the core for games with a priori unions given in Pulido and Sánchez-Soriano [13].

4 The Cooperation Core

Pulido and Sánchez-Soriano [13] studied the core and the Weber set of games with a priori unions, which they named coalitional core and coalitional Weber set. They defined the coalitional core in a two-step procedure following Owen [11]. They also gave a necessary and sufficient condition of coalitional convexity over v to guarantee that the coalitional Weber set is a subset or equal to the coalitional core. We are going to generalize their coalitional core (and Weber set) to cooperation situations, in the sense that they will coincide with the case of a priori unions when the components of the underlying graph are complete.

Definition 1. *Let $y \in \mathbb{R}^M$. For all $k \in M$, we define the game (N_k, v_k^y) as*

$$v_k^y(S) = \begin{cases} y_k & if\ S = N_k, \\ \max_{Q \subseteq M \setminus k} [v(N_Q \cup S) - y(Q)] & if\ S \subset N_k\ connected, \end{cases} \tag{5}$$

where $N_Q = \bigcup_{k \in Q} N_k$.

Definition 2. *Let (N, v, L) be a game with cooperation structure. Let $M = \{1, \ldots, m\}$ with $m = |N/L|$. Then, the cooperation core is defined by*

$$C(N, v, L) = \bigcup_{y \in C(M, v^{N/L})} \prod_{k \in M} C(N_k, v_k^y / L_{N_k}).$$

The only relevant coalitions in the cooperation core are the L-feasible coalitions as we prove in Theorem 1.

Definition 3. *Let N be a finite set of players and L a graph over N. A coalition $R \subseteq N$ is said to be L-feasible if $R = S \cup N_Q$, where $S \subseteq N_k$ is connected and $N_Q = \bigcup_{q \in Q} N_q$ with $Q \subseteq M \setminus k$. We will denote the set of L-feasible coalitions as $\mathcal{F}_L$.*

Theorem 1. *Let (N, v, L) be a game with cooperation structure. Then*

$$C(N, v, L) = \left\{x \in \mathbb{R}^N : x(N) = v(N), x(S) \geq v(S), \forall S \in \mathcal{F}_L\right\}.$$

Proof. We mimic the proof in Pulido and Sánchez-Soriano [13].

Let $x \in C(N, v, L)$. By the construction of the cooperation core we have that $x^{N/L} = (x(N_1), \ldots, x(N_m)) \in C(M, v^{N/L})$ and $x_{N_k} \in C\left(N_k, v_k^y / L_{N_k}\right)$, with $y = x^{N/L}$, for each $k \in M$, where x_{N_k} is the restriction of x to N_k. Then, $x(N) = \sum_{k \in M} x(N_k) = v^{N/L}(M) = v(N)$. On the other hand, let $R \in \mathcal{F}_L$. If $R = \cup_{k \in K} N_k$, for some $K \subseteq M$, then $x(R) \geq v(R)$ is clear since $x^{N/L} \in C(M, v^{N/L})$. Now suppose $R = S \cup Q$, where S is connected in N_k for some $k \in M$ and $Q = \cup_{p \in P} N_p$ for some $P \subseteq M \backslash k$. Since $x_{N_k} \in C\left(N_k, v_k^{x^{N/L}} / L_{N_k}\right)$, it follows that $x(R) = x(S) + \sum_{p \in P} x(N_p) \geq v_k^{x^{N/L}} / L_{N_k}(S) + x^{N/L}(P) = \max_{H \subseteq M \backslash k} [v\left(\cup_{h \in H} N_h \cup S\right) - x^{N/L}(H)] + x^{N/L}(P) \geq v(\cup_{p \in P} N_p \cup S) - x^{N/L}(P) + x^{N/L}(P) = v(R)$.

We have proven

$$C(N,v,L) \subseteq C(N,v,\mathcal{F}_L) = \left\{x \in \mathbb{R}^N : x(N) = v(N), x(S) \geq v(S), \forall S \in \mathcal{F}_L\right\}.$$

Let us see the other inclusion. Let $x \in \mathbb{R}^N$ such that $x(N) = v(N)$ and $x(S) \geq v(S), \forall S \in \mathcal{F}_L$. Define $x^{N/L} \in \mathbb{R}^M$ the quotient vector of x as $(x^{N/L})_k = x(N_k), \forall k \in M$. Since the unions of N_k are L-feasible, it follows that $x^{N/L} \in C(M, v^{N/L})$. By the construction of the cooperation core it only remains to prove that $x_{N_k} \in C\left(N_k, v_k^{x^{N/L}}/L_{N_k}\right), \forall k \in M$. Let $k \in M$ and $S \subset N_k$ connected. Then $S \cup (\cup_{p\in P} N_p) \in \mathcal{F}_L$, $\forall P \subseteq M\backslash k$. Consequently, $x\left(S \cup \bigcup_{p\in P} N_p\right) \geq v\left(S \cup \bigcup_{p\in P} N_p\right)$, $\forall P \subseteq M\backslash k$ and therefore $x(S) \geq v\left(S \cup \bigcup_{p\in P} N_p\right) - x^{N/L}(P)$, $\forall P \subseteq M\backslash k$. Then $x(S) \geq v_k^{x^{N/L}}(S)$, and, as S is a connected subset in N_k, we have $v_k^{x^{N/L}}(S) = v_k^{x^{N/L}}/L_{N_k}(S)$ and therefore $x_{N_k}(S) \geq v_k^{x^{N/L}}/L_{N_k}(S)$. On the other hand, $(x^{N/L})_k = x(N_k)$, by definition of quotient vector, so we conclude that $x_{N_k} \in C\left(N_k, v_k^{x^{N/L}}/L_{N_k}\right), \forall k \in M$. □

Remark 1. In particular, following Faigle [4] the cooperation core is a restricted core, i.e., $C(N,v,L) = C(N,v,\mathcal{F}_L)$.

In Grabisch [7] there is a survey of properties of the restricted core. We denote by $C_0(N,\mathcal{F})$ the recession cone of $C(N,v,\mathcal{F})$, defined by

$$C_0(N,\mathcal{F}) = \{x \in \mathbb{R}^N : x(N) = 0, x(S) \geq 0, \ \forall S \in \mathcal{F}\},$$

where $\mathcal{F} \subseteq 2^N$.

We know from the fundamental theory of polyhedra, that due to its definition, the cooperation core is a closed convex polyhedron.

We recall the following lemma from Grabisch [7].

Lemma 1. *For any game v and any family of coalitions $\mathcal{F} \subseteq 2^N$,*

1. *$C(N,v,\mathcal{F})$ has rays (but no line) if and only if $C_0(N,\mathcal{F})$ is a pointed cone different from $\{0\}$.*
2. *$C(N,v,\mathcal{F})$ is pointed, (i.e., has vertices) if and only if $C_0(N,\mathcal{F})$ does not contain a line, or equivalently, if the system $x(S) = 0$, $\forall S \in \mathcal{F}$, has 0 as unique solution.*
3. *$C(N,v,\mathcal{F})$ is bounded if and only if $C_0(N,\mathcal{F}) = \{0\}$.*

In our particular case, when $\mathcal{F} = \mathcal{F}_L$, we have that $\{i : i \in N\} \subseteq \mathcal{F}_L$, so $C_0(N,\mathcal{F}_L) = \{0\}$ and consequently $C(N,v,\mathcal{F}_L)$ is bounded and pointed.

Remark 2. Due to Theorem 2 in Pulido and Sánchez-Soriano [13] and Theorem 1, we have that the core of games with a priori unions $C_c(N,v,\mathcal{P})$ is a subset of $C(N,v,L)$, since the set of inequalities of the latter is contained in the set of inequalities of the first one.

5 An Axiomatization of the Cooperation Core

We can extend the results in Pulido and Sánchez-Soriano [12] to obtain an axiomatization of the cooperation core. First we are going to recall some definitions that are useful to understand the main theorem. We will denote by Ω any collection of subsets of N.

Definition 4. *A structure is a mapping $\mathcal{E}$ assigning to each finite set N a collection of collections of sets $\mathcal{E}(N) \subseteq 2^{2^N}$ such that:*

1. *$\emptyset, N \in \Omega$, for each $\Omega \in \mathcal{E}(N)$.*
2. *For any permutation π on N, $\pi(\mathcal{E}(N)) = \mathcal{E}(\pi(N))$, where*

$$\pi(\mathcal{E}(N)) = \{\pi(\Omega) : \Omega \in \mathcal{E}(N)\}$$

with $\pi(\Omega) = \{\pi(S) : S \in \Omega\}$.

The structure $\mathcal{E}$ is called complete if $C_0(N, \Omega) = \{0\}$, $\forall \Omega \in \mathcal{E}(N)$, and it is called consistent if for all finite set N and all $\Omega \in \mathcal{E}(N)$ it holds that $\Omega_S \in \mathcal{E}(S)$, for all $S \subset N, S \neq \emptyset$, where $\Omega_S = \{S \cap P : P \in \Omega\}$.

Definition 5. *The cooperation structure $\mathcal{E}_{co}$ is defined by*

$$\mathcal{E}_{co}(N) = \{\mathcal{F}_L : L \, is \, a \, graph \, on \, N\}.$$

Proposition 1. *$\mathcal{E}_{co}$ is a complete and consistent structure.*

Proof. It is clear that $\mathcal{E}_{co}$ is a complete structure since $\{i : i \in N\} \subseteq \mathcal{F}_L$, $\forall$ L graph over N.

Now we see the consistency. Take N, fix L and consider $S \subset N$. The aim is to find L' on S s.t. $\mathcal{F}_{L'}$ and $\mathcal{F}_L(S)$, where $\mathcal{F}_L(S) := \{R \cap S; R \in \mathcal{F}_L\}$ are the same. Suppose that the components of N/L are $N_1, \ldots, N_k$. We denote by $S_1, \ldots, S_k$ the connected components of L_S, (i.e., $S_1 = N_1 \cap S, \ldots, S_k = N_k \cap S$, where some may be empty). Take the subgraph of L restricted by S (denoted by L_S) and define L' by putting additional links in L_S as follows: if $i, j \in S$ are not linked, (i.e., $ij \notin L_S$), create link ij iff there exists $K \subseteq N \setminus S$ s.t. $K \cup ij$ is connected in L. Then it is easy to check that L' serves our purpose:

- Take $T \in \mathcal{F}_L$ and consider $T \cap S$. If T is a union of connected components in L, then $T \cap S$ is a union of connected components in L_S. If T is a connected subset in L, by construction of L', $T \cap S$ is also connected in S. If T is a union of components and a connected subset, $T \cap S$ too, following the previous reasoning.
- Take $T \in \mathcal{F}'_L$. If it is a union of components in L_S, then just take the corresponding components in N. If T is a connected subset of a component, one can find $K \subseteq N \setminus S$ s.t. $T \cup K$ is connected in N, by construction of L'. If T is a union of components and a connected subset, proceed the same way as before with both parts of T. □

Remark 3. The difference between the communication and the cooperation structure can be easily seen in the consistence condition, since the graphs L' and L_S are not equal.

We denote by $\mathcal{G}^{\mathcal{E}}$ the set of games (N, v, Ω) such that $\Omega \in \mathcal{E}(N)$ for all finite N.

Definition 6. *Let $\mathcal{E}$ be a structure and consider a set of games $\mathcal{A} \subseteq \mathcal{G}^{\mathcal{E}}$. A solution on $\mathcal{A}$ is a mapping σ which associates to each game $v \in \mathcal{A}$ a set of payoff vectors $x \in \mathbb{R}^N$ such that $x(N) \leq v(N)$.*

We need two more definitions to introduce some properties of the solutions.

Definition 7. *Let (N, v, Ω) be a game with restricted cooperation and $x \in \mathbb{R}^N$ s.t. $x(N) \leq v(N)$. The reduced game w.r.t. x and $S \subseteq N$ is a game on S defined by*

$$v_{S,x}(T) = \begin{cases} v(N) - x(N \backslash S) & \text{if } T = S, \\ \max_{R \subseteq N \backslash S, T \cup R \in \Omega}(v(T \cup R) - x(R)) & \text{if } \emptyset \neq T \subset S, \\ 0 & \text{if } T = \emptyset. \end{cases}$$

We see now some desirable properties for a solution. Let $\mathcal{A} \subseteq \mathcal{G}^{\mathcal{E}}$.

EFFICIENCY. A solution σ on $\mathcal{A}$ is efficient if for all $(N, v, \Omega) \in \mathcal{A}$ and any $x \in \sigma(N, v, \Omega)$ it holds $x(N) = v(N)$.

INDIVIDUAL RATIONALITY. A solution σ on $\mathcal{A}$ is individually rational if for any game $(N, v, \Omega) \in \mathcal{A}$ and any $x \in \sigma(N, v, \Omega)$ it holds $x_i \geq v(\{i\})$ for all $\{i\} \in \Omega$.

NONEMPTINESS. A solution σ on $\mathcal{A}$ satisfies nonemptiness if for any $(N, v, \Omega) \in \mathcal{A}, \sigma(N, v, \Omega) \neq \emptyset$.

We need to present the concept of separable players to introduce next axiom.

Definition 8. *The set of separable players in (N, v, Ω) is defined by*

$$\mathcal{S}(N, \Omega) = \{\{i, j\} \subset N : \exists T \in \Omega \text{ with } i \in T, j \notin T\}.$$

WEAK REDUCED GAME PROPERTY. A solution σ on $\mathcal{A}$ satisfies the weak reduced game property if for every $S \in \mathcal{S}(N, \Omega)$ and $x \in \sigma(N, v, \Omega)$, we have $(S, v_{S,x}, \Omega_S) \in \mathcal{A}$ and $x_S \in \sigma(S, v_{S,x}, \Omega_S)$.

MONOTONICITY. A solution σ on $\mathcal{A}$ satisfies the monotonicity property if for every pair of games $(N, v, \Omega), (N, v', \Omega) \in \mathcal{A}$ such that $v(N) = v'(N)$ and $v \leq v'$ it holds $\sigma(N, v', \Omega) \subseteq \sigma(N, v, \Omega)$.

Next theorem provides a characterization of the core in our structure. Let $\mathcal{G}_B^{\mathcal{E}}$ be the set of balanced games in $\mathcal{G}^{\mathcal{E}}$.

Theorem 2 (Pulido and Sánchez-Soriano, 2006). *Let $\mathcal{E}$ be a consistent and complete structure. The core is the unique solution on $\mathcal{G}_B^{\mathcal{E}}$ satisfying individual rationality, efficiency, the weak reduced game property, nonemptiness, and monotonicity.*

As we proved in Proposition 1, $\mathcal{E}_{co}$ is a consistent and complete structure, so the following result is straightforward.

Corollary 1. *The cooperation core is the unique solution on* $\mathcal{G}_B^{\mathcal{E}_{co}}$ *satisfying individual rationality, efficiency, the weak reduced game property, nonemptiness, and monotonicity.*

6 The Cooperation Weber Set

The marginal vectors are the more reasonable vectors when the ordering of the players is fixed. It is then convenient to find when these vectors are contained in the core. In this section we will construct the cooperation Weber set and we will see under which conditions we can guarantee that it is contained in the cooperation core. We are now going to define the cooperation Weber set using a particular kind of permutations of the player set.

Definition 9. *Let* L *be a communication structure on* N*. A permutation* $\pi \in \Pi_N$ *is named* L*-consistent if*

1. *All members of the same connected component are kept together.*
2. *All sets* $\{\pi(1), \ldots, \pi(i)\}$ *are* L*-feasible.*

We will denote by Π_L the set of L-consistent permutations on N.

Example 1. Given the graph L_1 of Fig. 1, permutation $\pi = (1, 3, 2, 6, 4, 5)$ is L_1-consistent since both conditions in Definition 9 hold. Nevertheless π is not L_2-consistent because condition 2 does not hold. In fact L_1 is a cooperation structure that coincides with the situation of the priori union structure $\mathcal{P} = \{123, 45, 6\}$, since the connected components of L_1 are complete subgraphs. That is the reason why the condition of L_1-consistency of a permutation here is equivalent to maintain all members of the same component together.

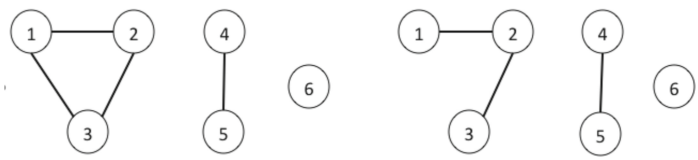

Fig. 1. Graphs L_1 (left) and L_2 (right).

Definition 10. *Let* (N, v, L) *be a game with cooperation structure. The cooperation Weber set is given by*

$$W(N, v, L) = conv\,(m^{\pi,v} | \pi \in \Pi_L).$$

Note that every L-consistent permutation can be determined in two steps:

1. Order the connected components: Let θ in Π_M and consider the marginal vector of $v^{N/L}$ with respect to θ. We will denote this vector by m^θ in order to simplify the notation.
2. Then we order the members inside each component, i.e., take a permutation $\mu_k \in \Pi_{L_{N_k}}$ (such that only connected sets are formed in L_{N_k}).

Then consider the marginal vector of the game w_k^θ with respect to μ_k, which we will denote by $m_\theta^{\mu_k}$, where for each $k \in M$, (N_k, w_k^θ) is the game defined by

$$w_k^\theta(S) = \begin{cases} m_k^\theta, & \text{if } S = N_k \\ v\left(\bigcup_{r \in P_\theta(k)} N_r \cup S\right) - m^\theta(P_\theta(k)) & \text{if } S \subset N_k, \end{cases}$$

where $m^\theta(P_\theta(k)) = \sum_{r \in P_\theta(k)} m_r^\theta$ and $P_\theta(k)$ denotes the set of predecessors of k w.r.t. θ.

Then, the cooperation Weber set can be obtained equivalently as

$$W(N, v, L) = conv\left(\bigcup_{\theta \in \Pi_M, \left\{\mu_k \in \Pi_{L_{N_k}}\right\}_{k \in M}} \prod_{k=1}^{m} m_\theta^{\mu_k}\right).$$

Remark 4. w_k^θ can be written in a simpler way

$$w_k^\theta(S) = v\left(\bigcup_{r \in P_\theta(k)} N_r \cup S\right) - v\left(\bigcup_{r \in P_\theta(k)} N_r\right), \ \forall S \subseteq N_k.$$

In Pulido and Sánchez-Soriano [13], Example 7 shows that in general the coalitional core is not included in the coalitional Weber set. This example also shows that in general the cooperation core is not included in the cooperation Weber set, since a communication structure is more general than an a priori union system. This fact is not surprising because when we introduce a communication situation the core enlarges and the Weber set gets smaller.

We are looking for a kind of convexity that guarantees the inclusion of the cooperation Weber set into the cooperation core. First, we extend the supermodularity condition to games with cooperation structure.

Definition 11. *Let (N, v, L) be a game with cooperation structure. It is said to be convex if*

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T),$$

for all $S, T \in \mathcal{F}_L$ such that $S \cup T, S \cap T \in \mathcal{F}_L$.

We will see later that in this class of games, the cooperation Weber set is included in the cooperation core, but this class of games is not characterized by that condition, as it is shown in Example 9 in Pulido and Sánchez-Soriano [13]. To obtain a class that characterizes that property, it is necessary to relax the definition of convexity.

Definition 12. *Let (N, v, L) be a game with cooperation structure. It is said to be weakly convex if all the following conditions hold:*

1. *For every $k \in M$, $Q \subseteq M \backslash k$ and $S, T, S \cup T$ connected in N_k*

$$v((S \cup T) \cup N_Q) + v((S \cap T) \cup N_Q) \geq v(S \cup N_Q) + v(T \cup N_Q)$$

2. *For every $k \in M$, $Q_1 \subseteq Q_2 \subseteq M \backslash k$ and S connected in N_k*

$$v(S \cup N_{Q_2}) - v(N_{Q_2}) \geq v(S \cup N_{Q_1}) - v(N_{Q_1})$$

3. *For every $k \in M$, $Q_1 \subseteq Q_2 \subseteq M \backslash k$ and S connected in N_k*

$$v(N_k \cup N_{Q_2}) - v(S \cup N_{Q_2}) \geq v(N_k \cup N_{Q_1}) - v(S \cup N_{Q_1})$$

where $N_Q = \cup_{q \in Q} N_q$, $\forall Q \subseteq M$.

An immediate consequence is

Proposition 2. *Convexity implies weak convexity.*

The following lemma will serve to prove the main theorem.

Lemma 2. *Let (N, v, L) be a weakly convex game. It holds*

1. *The quotient game $(M, v^{N/L})$ is convex.*
2. *For every $\theta \in \Pi_M$ and each $k \in M$, the game (N_k, w_k^θ) is convex.*
3. *For every $\theta \in \Pi_M$ and each $k \in M$, $w_k^\theta = v_k^{m^\theta}$ where $v_k^{m^\theta}$ is defined by (5).*

Proof. If we take condition 2 in the definition of weak convexity and $S = N_k$ we obtain (1). From Remark 4, we have (2) using condition 1 in the definition of weak convexity, and (3) follows from Lemma 15 in Pulido and Sánchez-Soriano [13], replacing b_i with N_i and v^P with $v^{N/L}$. □

Theorem 3. *Let (N, v, L) be a weakly convex game. Then*

$$W(N, v, L) \subseteq C(N, v, L).$$

Proof. Since $W(N, v, L)$ is the convex hull of the marginal vectors corresponding to L-consistent orders and $C(N, v, L)$ is a convex set, we only need to prove that all these marginal vectors belong to the cooperation core. Let π be an L-consistent order. Then there exist $\theta \in \Pi_M$ and $\left\{\mu_k \in \Pi_{L_{N_k}}, k \in M\right\}$ such that

$$m^{\pi,v} = m^{\theta,\{\mu_k\},v}$$

where $m^{\theta,\{\mu_k\},v} = (m_\theta^{\mu_1}, \ldots, m_\theta^{\mu_m})$. We are going to prove that $m^{\pi,v}$ belongs to the cooperation core, following its two-step construction. By definition, m^θ is the quotient vector of $m^{\pi,v}$. In Lemma 2 we saw that $v^{N/L}$ is a convex game, so $m^\theta \in C(M, v^{N/L})$. Now in the second step we are going to prove that for all $k \in M$, $m_\theta^{\mu_k} \in C\left(N_k, v_k^{m^\theta}\right)$. Let $k \in M$. Following the notation of the Weber set, $m_\theta^{\mu_k}$ is a marginal vector of w_k^θ. Following Lemma 2, w_k^θ coincides with $v_k^{m^\theta}$ and it is convex, thus $m_\theta^{\mu_k} \in C(N_k, v_k^{m^\theta})$. □

Next theorem states that the other implication is also true.

Theorem 4. *Let (N, v, L) be a game with cooperation structure such that $W(N, v, L) \subseteq C(N, v, L)$. Then (N, v, L) is a weakly convex game.*

Proof. Analogous to that of Theorem 17 in Pulido and Sánchez-Soriano [13] but considering instead of the set $\left\{\mu_j \in \Theta_{b_j}\right\}_{j\in M}$, the set $\left\{\mu_j \in \Pi_{L_{N_j}}\right\}_{j\in M}$.

Remark 5. If we have a complete graph, then the cooperation core and the cooperation Weber set coincide with the usual core and Weber set respectively. Moreover, weak convexity in this case is traditional convexity. The latter statement is also true when all players are isolated.

Remark 6. The coalitional Weber set defined in Pulido and Sánchez-Soriano [13] contains the cooperation Weber set, since the latter is the convex hull of a subset of points of the first one, those corresponding to marginal vectors with respect to L-consistent orders (which are included in the set of marginal vectors with respect to $\mathcal{P}$-consistent orders).

Remark 7. Following Pulido and Sánchez-Soriano [13], if v is coalitional convex we have the following chain of inclusions

$$W(N, v, L) \subseteq W_c(N, v, \mathcal{P}) \subseteq C_c(N, v, \mathcal{P}) \subseteq C(N, v, L).$$

Consequently $W(N, v, L) \subseteq C(N, v, L)$. But as we have seen in Theorem 3 we can relax the condition and obtain the same inclusion (since coalitional convexity implies weak convexity).

Remark 8. It is clear that if v is weakly convex then $C(N, v, L) \neq \emptyset$, since $W(N, v, L) \subseteq C(N, v, L)$.

Otherwise the construction of the coalitional core following Pulido and Sánchez-Soriano [13] could be used to define other set solutions connected with the Weber set. We propose the following definition similar to the one of cooperation core.

Definition 13. *Let (N, v, L) be a game with cooperation structure. We define the set solution $A(N, v, L)$ by*

$$A(N, v, L) = \bigcup_{y\in W(N, v^{N/L})} \prod_{k\in M} W\left(N_k, v_k^y / L_{N_k}\right).$$

It holds that $C(N, v, L) \subseteq A(N, v, L)$. Moreover, for certain graphs the convexity of (N, v) implies the equality of both sets.

Definition 14. *A graph L is cycle-complete if for every cycle K in L, the restricted subgraph L_K is complete.*

Proposition 3. *Let (N, v, L) be a game with cooperation structure with (N, v) convex and L a cycle-complete graph. Then*

$$C(N, v, L) = A(N, v, L).$$

Proof. As (N,v) is convex, Proposition 2 and Lemma 2 imply that $\left(M, v^{N/L}\right)$ is convex. We get then $C\left(M, v^{N/L}\right) = W\left(M, v^{N/L}\right)$.

Let $y \in C\left(M, v^{N/L}\right)$ and $k \in M$. We prove now that (N_k, v_k^y) is also convex. Let $S, T \subseteq N_k$ (we suppose w.l.o.g. that both sets are different from N_k). We know that there exist $Q_S, Q_T \subseteq M \setminus k$ with

$$v_k^y(S) = v\left(N_{Q_S} \cup S\right) - y\left(Q_S\right) \;\; \textit{and} \;\; v_k^y(T) = v\left(N_{Q_T} \cup T\right) - y\left(Q_T\right).$$

Then we have $v_k^y(S) + v_k^y(T) = v\left(N_{Q_S} \cup S\right) - y\left(Q_S\right) + v\left(N_{Q_T} \cup T\right) - y\left(Q_T\right) \leq v\left(N_{Q_S \cup Q_T} \cup (S \cup T)\right) - y\left(Q_S \cup Q_T\right) + v\left(N_{Q_S \cap Q_T} \cup (S \cap T)\right) - y\left(Q_S \cap Q_T\right) \leq v_k^y(S \cup T) + v_k^y(S \cap T)$. Notice that if $S \cup T = N_k$, then $v_k^y(S \cup T) = y_k$ and since $y \in C\left(M, v^{N/L}\right)$ it holds

$$v^{N/L}(Q \cup k) = v\left(N_Q \cup N_k\right) \leq y(Q) + y_k, \; \forall Q \in M \setminus k.$$

Then

$$v\left(N_{Q_S \cup Q_T} \cup (S \cup T)\right) - y\left(Q_S \cup Q_T\right) \leq v_k^y(S \cup T).$$

Moreover in van den Nouweland and Borm [10] it is proven that if v is convex and L_{N_k} is cycle-complete the graph restricted game is also convex. Then $(N_k, v_k^y/L_{N_k})$ is convex and therefore

$$C\left(N_k, v_k^y/L_{N_k}\right) = W\left(N_k, v_k^y/L_{N_k}\right). \qquad \square$$

The a priori unions structures are, in some sense, cycle-complete graphs. Nevertheless the previous definition cannot be considered as a valid definition of a cooperation Weber set as it violates a vital condition of this set, that in our case would be that the Myerson-Owen value is the center of the set.

Example 2. Let (N,v) be a convex game with $N = \{1,2,3\}$ and L the graph of Fig. 2. In this example $C(N,v,L) = C(N,v)$ and then $W(N,v) = A(N,v,L)$, since L is cycle-complete. The center of $A(N,v,L)$ is the Shapley value $\phi(N,v)$, but it is not difficult to find a convex game s.t. the Owen value in this case is not equal to the Shapley value.

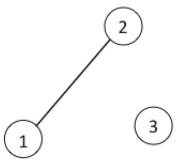

Fig. 2. Graph L.

7 Conclusions

In this work we have seen that the concept of core for games with a priori union structure given by Pulido and Sánchez-Soriano [13] can be extended to cooperation structures. We obtain similar results of characterization and relations with the Weber set. The line we followed to define both sets is different from the classic concepts. In this article we analyze the possibility of defining the Weber set in an analogous way to the cooperation core. Nevertheless, although for certain types of graphs (including a priori unions) the classic relations between the core and the Weber set hold, this concept fails on a primordial characteristic of the Weber set. In these situations the Owen value should be the center of mass of the Weber set but in general it is not. All of this leads us to think that the construction of the Weber set should not be modified and the definition of intermediate sets in this context (for instance in the sense of Adam and Kroupa [1]) could bring both concepts closer.

Acknowledgments. This research has been partially supported by the Spanish Ministry of Economy and Competitiveness ECO2013-40755-P, and by the FQM237 grant of the Andalusian Government.

The second author thanks the Agence Nationale de la Recherche for financial support under contract ANR-13-BSHS1-0010 (DynaMITE).

References

1. Adam, L., Kroupa, T.: The intermediate set and limiting superdifferential for coalition games: between the core and the Weber Set. Preprint submitted to Games and Economic Behavior (2015)
2. Aumann, R.J., Drèze, J.H.: Cooperative games with coalition structures. Int. J. Game Theory **3**, 217–237 (1974)
3. Casajus A.: Beyond basic structures in game theory. Ph.D. thesis. University of Leipzig, Germany (2007)
4. Faigle, U.: Cores of games with restricted cooperation. Methods Models Oper. Res. **33**, 405–422 (1989)
5. Fernández, J.R., Gallego, I., Jiménez-Losada, A., Ordóñez, M.: Cooperation among agents with a proximity relation. Eur. J. Oper. Res. **250**, 555–565 (2015)
6. Gillies, D.B.: Solutions to general non-zero-sum games. In: Tucker, A.W., Luce, R.D. (eds.) Contributions to the Theory of Games IV. Annals of Mathematics, vol. 40, 47th edn, p. 85. Princeton University Press, Princeton (1959)
7. Grabisch, M.: The core of games on ordered structures and graphs. Q. J. Oper. Res. **7**, 207–238 (2009)
8. Ichiishi, T.: Super-modularity: applications to the convex and to the greedy algorithm for LP. J. Econ. Theory **25**, 283–286 (1981)
9. Myerson, R.B.: Graphs and cooperation in games. Math. Oper. Res. **2**(3), 225–229 (1977)
10. van den Nouweland, A., Borm, P.: On the convexity of communication games. Int. J. Game Theory **19**, 421–430 (1991)

11. Owen, G.: Values of games with a priori unions. In: Henn, R., Moeschlin, O. (eds.) Mathematical Economics and Game Theory. Lecture Notes in Economics and Mathematical Systems, vol. 141, pp. 76–88. Springer, Heidelberg (1977)
12. Pulido, M.A., Sánchez-Soriano, J.: Characterization of the core in games with restricted cooperation. Eur. J. Oper. Res. **175**, 860–869 (2006)
13. Pulido, M.A., Sánchez-Soriano, J.: On the core, the Weber set and convexity in games with a priori unions. Eur. J. Oper. Res. **193**, 468–475 (2009)
14. Shapley, L.S.: A value for n-person games. Ann. Math. Stud. **28**, 307–317 (1953)
15. Shapley, L.S.: Cores and convex games. Int. J. Game Theory **1**, 1–26 (1971)
16. Weber, R.J.: Probabilistic values for games. Cowles Foundation Discussion Paper 417R. Yale University, New Haven (1978)
17. Weber, R.J.: Probabilistic values for games. In: Roth, A. (ed.) The Shapley value: Essays in Honor of Lloyd S. Shapley, pp. 101–119. Cambridge University Press, Cambridge (1988)