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Closure spaces and restricted games

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Abstract. We study a cooperation structure in which the feasible coalitions are those belonging to a *partition closure system*. First, we develop the relation between the dividends of Harsanyi in the restricted game and the worth function in the original game. Next, we obtain an method for computing the Shapley and Banzhaf values of the *closure-restricted* games using the dividends formula. Finally, we analyze the dividends in symmetric games by means of the k-th difference operator.

Key words: Shapley and Banzhaf values, closure spaces

1 Closure spaces

A *closure space* is a set N together with a *closure operator*

$$- : 2^N \to 2^N, \quad A \mapsto \bar{A},$$

which satisfies $A \subseteq B$ implies $\bar{A} \subseteq \bar{B}$, $A \subseteq \bar{A}$, $\bar{\bar{A}} = \bar{A}$, for all $A, B \subseteq N$. The family of *closed sets* $\mathscr{C} := \{A \subseteq N \mid A = \bar{A}\}$ is a *closure system* on N, closed under arbitrary intersections, and $N \in \mathscr{C}$. The closure system $\mathscr{C}$, ordered by inclusion, is a complete lattice, where $A \wedge B = A \cap B$, and $A \vee B = \overline{A \cup B}$. If $(N, \mathscr{C})$ is a closure system, then the operator

$$A \mapsto \bar{A} := \bigcap \{B \in \mathscr{C} \mid A \subseteq B\},$$

is a closure operator on N. We suppose that $\emptyset \in \mathscr{C}$, and this will not lead us to

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loss of generality (Libkin [15]). In what follows, we use a simplified notation and write $A \cup b$ to mean $A \cup \{b\}$, and $A \backslash a$ to mean $A \backslash \{a\}$.

Given a closure space $(N, -)$, we obtain the partitioning of 2^N induced by the equivalence relation $A \simeq B \Leftrightarrow \bar{A} = \bar{B}$. For a closed set $S \in \mathscr{C}$ an element of the equivalence class $[S]$ is a *generating set* of S. A minimal element of $[S]$ is a *basis* of S. We note that $A \subseteq S$ is a basis of S if

$$\bar{A} = \bar{S} \quad \text{and} \quad \overline{A \backslash a} \neq \bar{S}, \quad \text{for all } a \in A.$$

For $A \subseteq N$ an element $a \in A$ is an *extreme point* of A if $a \notin \overline{A \backslash a}$. Let $ex(A)$ be the set of all extreme points of A. Extreme points of a set may or may not exist, and $ex(A)$ is contained in each generating set of $\bar{A}$. The general theory of closure spaces can be found in Wild [24].

In their investigations of a combinatorial abstraction of discrete convexity, Edelman and Jamison [8] were lead to closure spaces with the *anti-exchange property*, where for every $A \subseteq N$ and elements $x, y \in N \backslash \bar{A}$, $x \neq y$,

$$y \in \overline{A \cup x} \quad \text{implies} \quad x \notin \overline{A \cup y}.$$

Definition 1. *A* **convex geometry** *is a finite closure space* $(N, -)$ *whose closure operator satisfies the anti-exchange property.*

The convex geometries are the abstract convex spaces satisfying the finite Minkowski-Krein-Milman property: *Every closed set is the closure of its extreme points* [8]. Indeed convex geometries are exactly those closure spaces where all equivalence classes $[A]$, $A \in \mathscr{C}$, have *smallest* elements [24]. Then $ex(A)$ is the unique basis of every $\bar{A}$.

A finite lattice $\mathscr{L}$ is a *lower locally distributive* (LLD) lattice if for every x in $\mathscr{L}$ the interval $[x^-, x]$, where $x^- = \bigwedge\{a \mid a \prec x\}$, is a Boolean algebra. The LLD property is also known as a *meet-distributive* condition. There are in fact a lot of characterizations of LLD lattices (see Monjardet [16]). Edelman obtained the following: *A finite lattice is an LLD lattice if and only if it is isomorphic to some convex geometry*.

Example. A *convexity* on a connected graph G is a closure system $\mathscr{C}$ on $V(G)$, such that all elements of $\mathscr{C}$, called *convex sets*, induce a connected subgraph of G. There are several kinds of graphs convexities. Polat [19] considers two that seem the most natural: the *geodesic* convexity [12] and the *minimal path* convexity. In the first, a subset C of $V(G)$ is geodesically convex if C contains every vertex on every geodesic (shortest path) joining two vertices in C. The second is defined for chordless paths.

Example. The concept of convexity is essential for pattern recognition and image processing. One of the basic models for a bit-map-display is the *finite metric space*. Hertel [14] defines the following concepts in a finite metric space (X, d):

1. The point $z \in X$ is *between* x and y if

$$d(x, z) + d(z, y) = d(x, y).$$

2. The *segment* xy is the set of all points z between x and y.
3. The subset A of X is called *d-convex* if for each pair of points $x, y \in A$ the whole segment xy belongs to A.

The family of all d-convex sets in X is a closure system on X.

Example. A subset S of a poset $(P, \leq)$ is *convex* whenever $a \in S$, $b \in S$ and $a \leq b$ imply that the interval $[a, b] = \{x \in P \mid a \leq x \leq b\} \subseteq S$. The convex subsets of any poset P form a closure system $Co(P)$. If P (or, equivalently $Co(P)$) is finite, then each element is between a maximal and a minimal one. If $C \in Co(P)$ then $ex(C)$ is the union of the maximal and minimal elements of C. Moreover, $Co(P)$ is a convex geometry (Birkhoff and Bennett [2, Thm. 3]).

Example. A graph $G = (N, E)$ is a *block graph* if every block is a complete graph (see Harary [11, p. 30]). For instance, if G is a disjoint union of trees, then G is a block graph. Jamison [8, Thm. 3.7] showed: *G is a connected block graph if and only if the family $\mathscr{C}$ is a convex geometry.*

2 Closure-restricted games

A cooperative game with transferable utility is a pair (N, v), where N is a finite set and v is a set function $v : 2^N \to \mathbb{R}$, such that $v(\emptyset) = 0$. The elements of $N = \{1, 2, \ldots, n\}$ are called *players*, the subsets $S \in 2^N$ *coalitions* and v is the *characteristic function* of the game (N, v). By Γ^N we denote the set of all games (N, v).

A *game with cooperation structure* is a game (N, v) together with a cooperation structure $\mathscr{P}$ which associates with each coalition $S \subseteq N$ a partition $\mathscr{P}(S)$ of S. The *restricted game* $(N, v^{\mathscr{P}})$ (Weber [23]) is defined by

$$v^{\mathscr{P}}(S) := \sum_{S_i \in \mathscr{P}(S)} v(S_i), \quad S \subseteq N.$$

We study a model of cooperation structure in which the feasible coalitions are those belonging to a special closure system. Grötschel, Lovász and Schrijver [10, Chapter 10] introduced the following concept.

Definition 2. *Let N be a finite set. A family $\mathscr{C}$ of subsets of N is called an* **intersecting family** *if S, $T \in \mathscr{C}$, $S \cap T \neq \emptyset$ imply $S \cap T \in \mathscr{C}$, $S \cup T \in \mathscr{C}$.*

A family $\mathscr{C}$ of subsets of N is *atomic* if $\{i\} \in \mathscr{C}$ for all $i \in N$. The maximal subsets of $S \subseteq N$ belonging to the family $\mathscr{C}$ are called *components* of S and we denote this collection by Π_S.

Proposition 1. *Let $\mathscr{C} \subseteq 2^N$ be an intersecting family which is atomic. Then the components of every $S \subseteq N$, denoted Π_S, form a partition of S.*

Proof. If $S \subseteq N$ then $S = \bigcup_{i \in S} \{i\}$, where $\{i\} \in \mathscr{C}$. Let $\Pi_S = \{C_1, \ldots, C_q\}$ be the collection of components of S. Then we have $S = \bigcup \{C_i \mid C_i \in \Pi_S\}$. If Π_S is not a partition of S, then $C_i \cap C_j \neq \emptyset$, and hence $C_i \cup C_j \in \mathscr{C}$, which contradicts the maximality of C_i and C_j. □

We are now ready to define the cooperation structure and the corresponding restricted game.

Definition 3. *A* **partition closure system** *is a closure system which is atomic and intersecting.*

The family $\mathscr{C} \subseteq 2^N$ is a partition closure system if and only if satisfies the following properties: $\emptyset \in \mathscr{C}$, $N \in \mathscr{C}$, $\{i\} \in \mathscr{C}$ for all $i \in N$, $\mathscr{C}$ is $\cap$-stable and $\cup$-stable for closed sets with nonempty intersection. If the closure system is a convex geometry, atomic and intersecting, we obtain a *partition convex geometry*.

Example. A *communication situation* is a triple (N, G, v), where (N, v) is a game and $G = (N, E)$ is a graph. This concept was studied first by Myerson [17] and investigated by Owen [18] and Borm, Owen and Tijs [3]. If the graph $G = (N, E)$ is a block graph then the family

$$\mathscr{C} = \{S \subseteq N \mid (S, E(S)) \text{ is a connected subgraph of } G\},$$

is a partition convex geometry.

Example. In a *sequencing situation* there is a queue, consisting of n customers waiting to be served at a counter. Curiel, Pederzoli and Tijs [5] introduced *sequencing games* (N, v) defined by $v(S) := \sum_{T \in \Pi_S} v(T)$, where the components of Π_S are the maximal intervals of S in a total order on N. Then the collection

$$\mathscr{C} = \{T \subseteq N \mid T \text{ is an interval of } N\},$$

is a partition convex geometry.

Definition 4. *For* $(N, v) \in \Gamma^N$ *and a partition closure system* $(N, \mathscr{C})$*, the* $\mathscr{C}$**-restricted game** *is defined by*

$$v^{\mathscr{C}}(S) := \sum_{T \in \Pi_S} v(T), \tag{1}$$

where Π_S *is the collection of components of* $S \subseteq N$.

We note that $S \in \mathscr{C}$ implies that $v^{\mathscr{C}}(S) = v(S)$ and the mapping

$$L_{\mathscr{C}} : \Gamma^N \to \Gamma^N, \quad v \mapsto v^{\mathscr{C}},$$

is a linear operator.

Example. If $\mathscr{C}$ is the partition system of the connected subgraphs of a block graph, then the game $v^{\mathscr{C}}$ is a Γ-component additive game which are studied by Potters and Reijnierse [20].

Example. Let $N = \{1, \ldots, n\}$ be the player set and let σ be a permutation of N that defines an order on N. Curiel, Hamers, Potters and Tijs [6] defines

σ-component additive games w.r.t. the permutation σ. These games are restricted games for the partition convex geometry $Co(\{\sigma(1) < \cdots < \sigma(n)\})$.

3 The dividends in the $\mathscr{C}$-restricted game

For any $T \subseteq N$, $T \neq \emptyset$, the *unanimity game* $u_T \in \Gamma^N$ is defined by

$$u_T(S) = \begin{cases} 1, & \text{if } T \subseteq S \\ 0, & \text{otherwise.} \end{cases}$$

Every game is a linear combination of games of the form u_T (see Shapley [21, lemma 3]). Then, we have the representation

$$v = \sum_{T \subseteq N} \Delta_v(T) u_T, \quad \text{where} \quad \Delta_v(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T). \tag{2}$$

Following Harsanyi [13] we shall call $\Delta v(T)$ the *dividends* of T, in the game v. It is easy to obtain the restricted game $v^{\mathscr{C}}$, in terms of the images of the unanimity games $u_T^{\mathscr{C}}$,

$$\begin{aligned} v^{\mathscr{C}} &= L_{\mathscr{C}}[v] = L_{\mathscr{C}}\left[\sum_{T \subseteq N} \Delta_v(T) u_T\right] \\ &= \sum_{T \subseteq N} \Delta_v(T) u_T^{\mathscr{C}}. \end{aligned}$$

The unanimity games form a basis of Γ^N. We prove that $\{u_C \mid C \in \mathscr{C}, C \neq \emptyset\}$ form a basis of the image $L_{\mathscr{C}}(\Gamma^N) = \{v^{\mathscr{C}} \mid v \in \Gamma^N\}$.

Theorem 1. *Let $\mathscr{C} \subseteq 2^N$ be a partition closure system and let $v \in \Gamma^N$ be a game. Then, the $\mathscr{C}$-restricted game $v^{\mathscr{C}}$ satisfies*

$$v^{\mathscr{C}} = \sum_{C \in \mathscr{C}} \Delta_{v^{\mathscr{C}}}(C) u_C, \quad \textit{where} \quad \Delta_{v^{\mathscr{C}}}(\emptyset) = 0. \tag{3}$$

Proof. We consider the game $u_T^{\mathscr{C}}$ which satisfies

$$u_T^{\mathscr{C}}(S) = \sum_{C \in \Pi_S} u_T(C) = \begin{cases} 1, & \text{if there exists } C \in \mathscr{C} \text{ such that } T \subseteq C \subseteq S \\ 0, & \text{otherwise.} \end{cases}$$

The following conditions are equivalent for all $S \subseteq N$:

$$u_T^{\mathscr{C}}(S) = 1 \Leftrightarrow \exists C \in \mathscr{C} \text{ such that } T \subseteq C \subseteq S \Leftrightarrow \overline{T} \subseteq S \Leftrightarrow u_{\overline{T}}(S) = 1.$$

We have $u_T^{\mathscr{C}} = u_{\overline{T}}(S)$, and hence

$$L_{\mathscr{C}}[\{u_T \mid T \in 2^N, T \neq \emptyset\}] = \{u_C \mid C \in \mathscr{C}, C \neq \emptyset\}. \qquad \square$$

Let (N, v) be a game, the *Shapley value* [21] for the player $i \in N$ is defined by

$$\Phi_i(N, v) = \sum_{\{S \subseteq N \mid i \in S\}} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S \backslash i)],$$

where $n = |N|$, and $s = |S|$. This value is an average of *marginal contribution* $v(S) - v(S \backslash i)$ of a player i to all possible coalitions $S \in 2^N \backslash \{\emptyset\}$. In this value, the sets S of different size get different weight. Dubey and Shapley, in [7], suggested the following *Banzhaf value*,

$$\beta_i'(N, v) = \sum_{\{S \subseteq N \mid i \in S\}} \frac{1}{2^{n-1}} [v(S) - v(S \backslash i)], \quad i \in N.$$

Definition 5. *Let $(N, \mathscr{C})$ be a partition closure system and let (N, v) be a game. The* **closure Shapley value** *for the player i is $\Phi_i^{\mathscr{C}}(N, v) := \Phi_i(N, v^{\mathscr{C}})$ and the* **closure Banzhaf value** *is $(\beta_i')^{\mathscr{C}}(N, v) := \beta_i'(N, v^{\mathscr{C}})$.*

Remark 1. The direct procedure for computing these values involve some kind of enumeration of 2^{n-1} coalitions and the complexity grow exponentially with n. For some special classes of partition closure systems and for symmetric games we describe methods based on the dividends that works more efficiently.

The Shapley and Banzhaf values are linear mappings w.r.t. the characteristic function and the images of the unanimity games are respectively

$$\Phi_i(N, u_S) = \begin{cases} 1/|S|, & \text{if } i \in S \\ 0, & \text{otherwise,} \end{cases}$$

$$\beta_i'(N, u_S) = \begin{cases} 1/2^{|S \backslash i|}, & \text{if } i \in S \\ 0, & \text{otherwise.} \end{cases}$$

Thus Theorem 1 implies the following formulas

$$\Phi_i(N, v^{\mathscr{C}}) = \sum_{\{S \in \mathscr{C} \mid i \in S\}} \frac{\Delta_{v^{\mathscr{C}}}(S)}{|S|}, \tag{4}$$

$$\beta_i'(N, v^{\mathscr{C}}) = \sum_{\{S \in \mathscr{C} \mid i \in S\}} \frac{\Delta_{v^{\mathscr{C}}}(S)}{2^{|S \backslash i|}}. \tag{5}$$

Let $(N, \mathscr{C})$ be the partition convex geometry of subsets of vertices which induce connected subgraphs of the connected block graph $G = (N, E)$. In this geometry, the closure Shapley value is called *Myerson value* [9]. If G is a tree, then Owen [18] gave the following formula for computing the dividends in the restricted game $v^{\mathscr{C}}$,

$$\Delta_{v^{\mathscr{C}}}(S) = \sum_{\{T \in 2^N \mid \bar{T} = S\}} \Delta_v(T) = \sum_{\{T \mid ex(S) \subseteq T \subseteq S\}} \Delta_v(T).$$

We extend this formula to the case when $\mathscr{C}$ is any partition convex geometry. First, we study the formula for $v^{\mathscr{C}}$, when $\mathscr{C}$ is a partition closure system.

Theorem 2. *Let $(N, \mathscr{C})$ be a partition closure system and let (N, v) be a game. The dividend of $S \in \mathscr{C}$ in the restricted game $v^{\mathscr{C}}$ satisfies*

$$\Delta_{v^{\mathscr{C}}}(S) = \sum_{\{T \in 2^N \mid \overline{T} = S\}} \Delta_v(T) = \sum_{T \in \bigcup\{[B,S] \mid B \in \mathscr{B}(S)\}} \Delta_v(T),$$

where $\mathscr{B}(S)$ is the family of all bases of S.

Proof. We know that $u_T^{\mathscr{C}} = u_{\overline{T}}$ for every nonempty $T \subseteq N$, and $u_T^{\mathscr{C}}$ has a unique representation in the basis $\{u_S \mid S \in 2^N, S \neq \emptyset\}$. Then, we have

$$u_T^{\mathscr{C}} = \sum_{S \subseteq N} \Delta_{u_T^{\mathscr{C}}}(S) u_S = u_{\overline{T}} \Leftrightarrow \Delta_{u_T^{\mathscr{C}}}(S) = \begin{cases} 1, & \text{if } S = \overline{T} \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 1, the restricted game $v^{\mathscr{C}}$ satisfies

$$\begin{aligned} v^{\mathscr{C}} &= \sum_{T \subseteq N} \Delta_v(T) u_T^{\mathscr{C}} = \sum_{T \subseteq N} \Delta_v(T) \left[\sum_{S \in \mathscr{C}} \Delta_{u_T^{\mathscr{C}}}(S) u_S \right] \\ &= \sum_{S \in \mathscr{C}} \left[\sum_{T \subseteq N} \Delta_v(T) \Delta_{u_T^{\mathscr{C}}}(S) \right] u_S. \end{aligned}$$

The coefficients are equal and so

$$\Delta_{v^{\mathscr{C}}}(S) = \sum_{T \in 2^N} \Delta_v(T) \Delta_{u_T^{\mathscr{C}}}(S) = \sum_{\{T \in 2^N \mid \overline{T} = S\}} \Delta_v(T).$$

Finally, we prove the equality

$$\{T \subseteq N \mid \overline{T} = S\} = \bigcup_{B \in \mathscr{B}(S)} \{T \subseteq N \mid B \subseteq T \subseteq S\},$$

where B is a basis (minimal generating set) of S. Let T be such that $\overline{T} = S$. Then T is a generating set of $S \in \mathscr{C}$ and there exists $B \in \mathscr{B}(S)$ with $B \subseteq T \subseteq S$.

Conversely, if $B \subseteq T \subseteq S$, the properties of the closure operator imply that $S = \overline{B} \subseteq \overline{T} \subseteq \overline{S} = S$. □

Corollary 1. *Let $(N, \mathscr{C})$ be a partition convex geometry and let (N, v) be a game. The dividend of $S \in \mathscr{C}$ in the restricted game $v^{\mathscr{C}}$ satisfies*

$$\Delta_{v^{\mathscr{C}}}(S) = \sum_{\{T \in 2^N \mid \overline{T} = S\}} \Delta_v(T) = \sum_{\{T \mid ex(S) \subseteq T \subseteq S\}} \Delta_v(T).$$

We study the relation between the dividends of Harsanyi in the restricted game $v^{\mathscr{C}}$ and the characteristic function v in the next results. For that we consider the *Möbius function* μ of the finite poset $(P, \leq)$.

An easy way to compute $\mu : P \times P \to \mathbb{Z}$, is from the recurrence

$$\mu(x,y) = -\sum_{x \le z < y} \mu(x,z),$$

for $x < y$, with the initial condition $\mu(x,x) = 1$ (see Aigner [1]).

We will need the following theorem of Crapo [4] concerning the computation of the Möbius function of the quotient lattice $(\mathscr{C}, \subseteq)$ with respect the closure operator defined in the lattice $(2^N, \subseteq)$.

If P is a finite lattice with closure $x \to \bar{x}$, and $\bar{P}$ is the quotient lattice, consisting of the closed elements of P, then for all elements $x \in P$, and elements y closed in P, $x \le y$, the sum

$$\sum_{\{l \in P \mid x \le l \le \bar{l} = y\}} \mu(x,l) = \begin{cases} \mu_{\bar{P}}(x,y), & \text{if } x \text{ is closed} \\ 0, & \text{otherwise.} \end{cases}$$

The formulas for obtaining the dividends of the closed coalitions in the restricted game follow from the next theorems.

Theorem 3. *Let $(N, \mathscr{C})$ be a partition closure system and let v be a game. If $v^{\mathscr{C}}$ is the restricted game associated to v, then*

$$\Delta_{v^{\mathscr{C}}}(S) = \sum_{\{T \in \mathscr{C} \mid T \subseteq S\}} \left[\sum_{L \in \bigcup\{[B,S] \mid B \in \mathscr{B}(S)\}} (-1)^{|L|-|T|} \right] v(T), \quad S \in \mathscr{C}. \tag{6}$$

Proof. It follows from Theorem 1 that for all $S \in \mathscr{C}$,

$$v(S) = v^{\mathscr{C}}(S) = \sum_{T \in \mathscr{C}} \Delta_{v^{\mathscr{C}}}(T) u_T(S) = \sum_{\{T \in \mathscr{C} \mid T \subseteq S\}} \Delta_{v^{\mathscr{C}}}(T).$$

The closure system $\mathscr{C}$, ordered by inclusion, is the quotient lattice of 2^N by the closure operator. The Möbius inversion formula (Stanley [22, p. 116]) of $\mathscr{C}$ implies that

$$\Delta_{v^{\mathscr{C}}}(S) = \sum_{\{T \in \mathscr{C} \mid T \subseteq S\}} \mu_{\mathscr{C}}(T,S) v(T).$$

We use the above result of Crapo and we obtain

$$\begin{aligned} \mu_{\mathscr{C}}(T,S) &= \sum_{\{L \in 2^N \mid T \subseteq L \subseteq \bar{L} = S\}} \mu(T,L) \\ &= \sum_{\{L \in 2^N \mid \bar{L} = S\}} \mu(T,L) \\ &= \sum_{L \in \bigcup\{[B,S] \mid B \in \mathscr{B}(S)\}} (-1)^{|L|-|T|}. \end{aligned}$$

□

Theorem 4. *Let $(N, \mathscr{C})$ be a partition convex geometry. Then*

$$\Delta_{v^{\mathscr{C}}}(S) = \sum_{T \in [S \setminus ex(S), S]} (-1)^{|S|-|T|} v(T), \tag{7}$$

where $ex(S)$ is the set of all extreme points of $S \in \mathscr{C}$.

Proof. If $\mathscr{C}$ is a convex geometry, then the family $\mathscr{B}(S) = \{ex(S)\}$, for all $S \in \mathscr{C}$. Thus the closure operator satisfies $\bar{L} = S \Leftrightarrow ex(S) \subseteq L$, and so

$$\begin{aligned} \mu_{\mathscr{C}}(T, S) &= \sum_{L \in [T \cup ex(S), S]} (-1)^{|L|-|T|} \\ &= (-1)^{|T \cup ex(S)| - |T|} \sum_{L \in [T \cup ex(S), S]} (-1)^{|L| - |T \cup ex(S)|} \\ &= (-1)^{|T \cup ex(S)| - |T|} \sum_{L \in \mathscr{P}} \mu_{\mathscr{P}}(\hat{0}, L), \end{aligned}$$

where $\mathscr{P}$ is a finite lattice isomorphic to interval $[T \cup ex(S), S]$.

If $T \cup ex(S) \neq S$, then $\mathscr{P}$ contains at least two elements and the Möbius function satisfies $\sum_{L \in \mathscr{P}} (\hat{0}, L) = 0$ (Aigner [1]). Otherwise, $T \cup ex(S) = S$ and the sum is $\mu_{\mathscr{C}}(T, S) = \mu(T, S) = (-1)^{|S|-|T|}$. Thus we obtain

$$\mu_{\mathscr{C}}(T, S) = \begin{cases} (-1)^{|S|-|T|}, & \text{if } S = T \cup ex(S) \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $S = T \cup ex(S)$ if and only if $S \setminus ex(S) \subseteq T$. □

Example. The family of the convex sets $\mathscr{C} = Co(P)$ of the chain

$$P = \{1 < 2 < \cdots < n\}$$

is a partition convex geometry. Note that the feasible coalitions are the empty set and the intervals $[i, j] = \{i, i+1, \ldots, j-1, j\}$, where $i \leq j$. The set of the extreme points $ex([i, j]) = \{i, j\}$ and Theorem 4 implies that the dividends for the restricted game $v^{\mathscr{C}}$ are $\Delta_{v^{\mathscr{C}}}(\{i\}) = v(\{i\})$, for $i \in N$, and

$$\Delta_{v^{\mathscr{C}}}(S) = v(S) - v(S \setminus i) - v(S \setminus j) + v(S \setminus \{i, j\}),$$

for $S = [i, j]$, $i < j$.

4 Symmetric games

Definition 6. *A* **symmetric game** *(N, v) is characterized by a sequence of $|N| + 1$ numbers, $\{0, v_1, \ldots, v_{|N|}\}$, where for any $S \in 2^N$, the characteristic function is $v(S) = v_{|S|}$.*

We inductively define the k-th difference operator, denoted δ^k, for integers $k \geq 1$ by

$$\delta^0(y_i) := y_i, \quad \delta^k(y_i) := \delta^{k-1}(y_{i+1}) - \delta^{k-1}(y_i),$$

where $(y_i)_{i \geq 0}$ is a sequence.

Proposition 2. *Let $(N, \mathscr{C})$ be a partition convex geometry and let (N, v) be a symmetric game with $\{0, v_1, \ldots, v_{|N|}\}$. The dividend of $S \in \mathscr{C}$ in $v^{\mathscr{C}}$ satisfies*

$$\Delta_{v^{\mathscr{C}}}(S) = \begin{cases} v_1, & \text{if } |S| = 1 \\ (-1)^k \delta^k f_s(0), & \text{if } |ex(S)| = k, \end{cases}$$

where $s = |S|$, $f_s(i) = v_{s-i}$, $0 \leq i \leq s$, and δ^k is the k-th difference operator.

Proof. If $|S| = 1$, then $S \setminus ex(S) = \emptyset$, and so $\Delta_{v^{\mathscr{C}}}(S) = v_1$. Let $S \in \mathscr{C}$ be such that $|ex(S)| = k$. Since the interval $[S \setminus ex(S), S]$ is a Boolean algebra, we have

$$\begin{aligned} \Delta_{v^{\mathscr{C}}}(S) &= \sum_{T \in [S \setminus ex(S), S]} (-1)^{|S|-|T|} v(T) \\ &= \sum_{S \setminus T \subseteq ex(S)} (-1)^{|S \setminus T|} v(T) \\ &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} v_{s-i}. \end{aligned}$$

The k-th difference of f_s at 0 satisfies the formula (see Stanley [22, p. 37])

$$\delta^k f_s(0) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f_s(i).$$

By definition $f_s(i) = v_{s-i}$, and hence

$$\begin{aligned} \Delta_{v^{\mathscr{C}}}(S) &= (-1)^k \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} v_{s-i} \\ &= (-1)^k \delta^k f_s(0). \end{aligned}$$

□

Example. Let v be a symmetric game defined by $v(S) := |S|(|S| - 1)$ if $|S| \geq 1$, and $v(\emptyset) = 0$ (Owen [18]). In this game, the sequence $\{0, v_1, \ldots, v_n\}$ satisfies $v_i = i(i-1)$. We apply Proposition 2 to the restricted game $v^{\mathscr{C}}$, where $\mathscr{C}$ is a partition convex geometry. We have that $f_s(i) = v_{s-i} = (s-i)(s-i-1)$. The difference table (beginning with $f_s(0)$) looks like

$$\begin{array}{ccccccccc} s^2 - s & & s^2 - 3s + 2 & & s^2 - 5s + 6 & & s^2 - 7s + 12 & & \cdots \\ & -2s + 2 & & -2s + 4 & & -2s + 6 & & \cdots & \\ & & 2 & & 2 & & \cdots & & \\ & & & 0 & & \cdots & & & \\ & & & & \ddots & \cdots & & & \end{array}$$

Then, since $v_1 = 0$ and $|S| > 1$ implies $|ex(S)| = k > 1$, we have

$$\Delta v^{\mathscr{C}}(S) = \begin{cases} 2, & \text{if } |ex(S)| = 2 \\ 0, & \text{otherwise.} \end{cases}$$

For instance, if (N, E) is the tree defined by Owen [18], where $N = \{1, \ldots, 13\}$ and

$$E = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,5\}, \{2,6\}, \{2,7\}, \{3,8\}, \\ \{3,9\}, \{3,10\}, \{4,11\}, \{4,12\}, \{4,13\}\}.$$

The family of subsets of N which induces a connected subgraph is a partition convex geometry. The Myerson value satisfies, for every $i \in N$,

$$\begin{aligned} \Phi_i^{\mathscr{C}}(N, v) &= \sum_{\{S \in \mathscr{C} \mid i \in S\}} \frac{\Delta_{v^{\mathscr{C}}}(S)}{|S|} \\ &= \sum \left\{ \frac{2}{|S|} \mid S \in \mathscr{C}, i \in S, \quad \text{and } |ex(S)| = 2 \right\} \\ &= \sum \left\{ \frac{2}{|S|} \mid (S, E(S)) \text{ is a path containing } i \right\}. \end{aligned}$$

$$\Phi_1^{\mathscr{C}}(N, v) = 2\left(\frac{3}{2} + \frac{12}{3} + \frac{18}{4} + \frac{27}{5}\right) = \frac{154}{5},$$

$$\Phi_2^{\mathscr{C}}(N, v) = \Phi_3^{\mathscr{C}}(N, v) = \Phi_4^{\mathscr{C}}(N, v) = 2\left(\frac{4}{2} + \frac{8}{3} + \frac{12}{4} + \frac{18}{5}\right) = \frac{338}{15},$$

$$\Phi_5^{\mathscr{C}}(N, v) = \cdots = \Phi_{13}^{\mathscr{C}}(N, v) = 2\left(\frac{1}{2} + \frac{3}{3} + \frac{2}{4} + \frac{6}{5}\right) = \frac{32}{5}.$$

The closure Banzhaf value satisfies for $i \in N$,

$$(\beta_i')^{\mathscr{C}}(N,v) = \sum_{\{S \in \mathscr{C} \mid i \in S\}} \frac{\Delta_{v^{\mathscr{C}}}(S)}{2^{|S|-1}}$$

$$= \sum \left\{ \frac{1}{2^{|S|-2}} \,\middle|\, (S, E(S)) \text{ is a path containing } i \right\}.$$

$$(\beta_1')^{\mathscr{C}}(N,v) = 3 + 12\left(\frac{1}{2}\right) + 18\left(\frac{1}{2}\right)^2 + 27\left(\frac{1}{2}\right)^3 = \frac{135}{8},$$

$$(\beta_2')^{\mathscr{C}}(N,v) = (\beta_3')^{\mathscr{C}}(N,v) = (\beta_4')^{\mathscr{C}}(N,v)$$

$$= 4 + 8\left(\frac{1}{2}\right) + 12\left(\frac{1}{2}\right)^2 + 18\left(\frac{1}{2}\right)^3 = \frac{53}{4},$$

$$(\beta_5')^{\mathscr{C}}(N,v) = \cdots = (\beta_{13}')^{\mathscr{C}} = 1 + 3\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right)^3 = \frac{15}{4}.$$

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