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A Banzhaf value for games with fuzzy communication structure: Computing the power of the political groups in the European Parliament

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Abstract

In 2013, Jiménez-Losada et al. introduced several extensions of the Myerson value for games with fuzzy communication structure. In a fuzzy communication structure the membership of the players and the relations among them are leveled. Now we study a Banzhaf value for these situations. The Myerson model is followed to define the fuzzy graph Banzhaf value taking as base point the Choquet integral. We propose an axiomatization for this value introducing leveled amalgamation of players. An algorithm to calculate this value is provided and its complexity is studied. Finally we show an applied example computing by this fuzzy value the power of the groups in the European Parliament.

Keywords: Game theory; Fuzzy graphs; Banzhaf value; Power indices; European Parliament

Introduction

A cooperative game with transferable utility over a finite set of players is defined as a function establishing the worth of each coalition (subset of players). The outcome of a game is a payoff vector, namely it is a vector in which each component represents the payment for each player because of their cooperation possibilities. Although payoff vectors are usually efficient with regard to the worth of the great coalition, Owen [16] introduced the (probabilistic) Banzhaf value, based in the Banzhaf power index for simple games, as an interesting non-efficient outcome for each game. The usual payoff vectors suppose that all the communications are feasible. Myerson [12] considered that the communication among the players can be different at any time though the game was thought in a total cooperation situation. He described the communication situation by a graph where the vertices are the players and the links are the feasible bilateral communications among them. This graph is named the communication structure of the game. Hence we will use both, graph or communication structure, alike. So, Myerson proposed as a more realistic outcome for the game a payoff vector for each communication structure. A communication value assigns a payoff vector to each game with a specific communication structure. Owen [17] defined the graph Banzhaf value as a communication value

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which coincides on the complete graph (the total cooperation situation) with the Banzhaf value for each game. Given a communication structure, the graph Banzhaf value calculates the Banzhaf value of the game modified by the graph. So, the worth of a coalition taking into account the graph is the sum of the worths of the maximal connected coalitions in the subgraph restricted by the corresponding set of vertices. Alonso-Meijide and Fiestras-Janeiro [1] characterized this communication value by three axioms: isolation (the payment of an isolated player is the worth of his individual coalition), fairness (the loss of one bilateral communication implies the same loss of payment for the players involved in this link) and pairwise merging (the payoff vector is neutral against pairwise mergers).

Aubin [2] and Butnariu [3] supposed uncertainty about the membership of the players in the coalitions studying games with fuzzy coalitions. To calculate the worth of a fuzzy coalition in a game it is necessary to consider a specific partition by levels of this fuzzy set. One of these partitions was defined by Tsurumi et al. [18] using the Choquet integral. Following this way, the uncertainty about the existence of the communications among the players can be extended to the uncertainty about these communications. Recently, Jiménez-Losada et al. [8] introduced fuzzy graphs to analyze communication among players. Fuzzy graphs allow leveling the links between being feasible or not, and they also allow considering membership levels for the players. The idea of partition by levels was extended to fuzzy communication structures in Jiménez-Losada et al. [9], proposing different extensions of the Myerson value for fuzzy situations. The main goal of this paper is to study the Banzhaf value for games with fuzzy communication structures extending the concept of graph Banzhaf value.

The legislative power in the European Union (EU) focuses on two chambers: the Council and the Parliament. The European Parliament is the elected body that represents the EU's citizens. It exercises political supervision over the EU's activities and takes part in the legislative process. Since 1979, members of the European Parliament have been directly elected, by universal suffrage, every five years. The Parliament takes part in the legislative work of the EU at three levels. The cooperation procedure (introduced in 1987 by the Single European Act): the European Parliament can give its opinion on draft directives and regulations proposed by the European Commission, which is asked to amend its proposals taking into account the Parliament's position. The assent procedure (also from 1987): the European Parliament must give its assent to international agreements negotiated by the Commission and to any proposed enlargement of the EU. The co-decision procedure (introduced in 1992 by the Treaty of Maastricht): the European Parliament is put on an equal footing with the Council when legislating on a whole series of important issues. In 2009, the Treaty of Lisbon gave to the European Parliament new legislative powers (over forty new items were included by the co-decision procedure). While the Council of the European Union represents the national governments of members states, the European Parliament pretends to be the ideologic representation of the European citizens. But actually the channel of voting is the set of national political parties in each member state. Hence, the relations among these groups are partial because of the national interests. Historically, the first paper on model analysis of the EU institutions, Holler and Kellermann [6], was focused on national distribution of voting power in the European Parliament (even before the first election in 1979). Hosli [7] and Noury [13] analyze voting power with national and ideological dimension of voting in the European Parliament. Later, other studies about the power in the European Parliament are Hix et al. [5] and Turnovec et al. [19]. Particularly, the Banzhaf index was used in Nurmi [15] to analyze the power of the political parties in the European Parliament.

In this paper we quantify the power of the political groups in the European Parliament taking into account their partial communications. The voting system in the Parliament is described by a weighted voting game, a specific simple cooperative game. For each voting in the chamber we can take, following Myerson [12], a graph representing the communication situation among the groups. We use our Banzhaf value for games with fuzzy communication structure to determine the power index of the groups in the European Parliament. The number of players in this game is too large to use the original formulas of the values. We show new algorithms to determine the indices and we study the time complexity of them.

Section 1 presents in short the background which allows the reader to follow the paper: cooperative games, games with communication structure, games with fuzzy communication structure. We define the fuzzy graph Banzhaf value following the Choquet by graph (cg) model in Section 2. We obtain in this section an axiomatization of the value. Section 3 is dedicated to the computation of cg-fuzzy graph Banzhaf value and the time complexity of the algorithm. Finally, in Section 4, we apply the values to study the power of the political groups in the European Parliament.

1. Preliminaries

1.1. Cooperative games

A *cooperative game with transferable utility* is a pair (N, v) where N is a finite set and $v : 2^N \to \mathbb{R}$ is a function with $v(\emptyset) = 0$. The elements of $N = \{1, 2, \dots, n\}$ are called *players*, the subsets $S \subseteq N$ *coalitions* and $v(S)$ is the *worth* of S. Let (N, v) be a game. A *dummy player* $i \in N$ satisfies $v(S) - v(S \setminus \{i\}) = v(\{i\})$ for all $S \subseteq N$ with $i \in S$. If $S \subseteq N$ then (S, v) represents the restriction of the characteristic function v to 2^S. The *amalgamated game* of (N, v) for $i, j \in N$ is other game (N^{ij}, v^{ij}) where $N^{ij} = N \setminus \{i, j\} \cup \{ij\}$ and for all $S \subseteq N^{ij}$,

$$v^{ij}(S) = \begin{cases} v(S \setminus \{ij\} \cup \{i, j\}), & \text{if } ij \in S \\ v(S), & \text{if } ij \notin S. \end{cases} \tag{1}$$

Player ij $(i \neq j, ij = ji)$ represents the coordinated action in the game of players i, j as a unique one. The *unanimity game* for coalition $T \subseteq N$ is (N, u_T) with $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise. The characteristic function of every game (N, v) is a linear combination of unanimity games in N, that is

$$v = \sum_{\{T \subseteq N:\ T \neq \emptyset\}} \Delta_T^v u_T, \quad \text{with } \Delta_T^v = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S). \tag{2}$$

The coefficients of the above combination, Δ_T^v for all non-empty coalition $T \subseteq N$, are named *Harsanyi dividends* [4] of the game.

A *payoff vector* for the game (N, v) is any $x \in \mathbb{R}^N$ where, for each player $i \in N$, the number x_i represents the payment of i owing to his cooperation possibilities. A *value* for cooperative games assigns to each game (N, v) a payoff vector in $\mathbb{R}^N$. The *Banzhaf value*, Owen [16], of a game (N, v) is defined for any player $i \in N$ as

$$\beta_i(N, v) = \frac{1}{2^{n-1}} \sum_{\{S \subseteq N:\ i \in S\}} \bigl[v(S) - v\bigl(S \setminus \{i\}\bigr)\bigr]. \tag{3}$$

This value satisfies the following properties.

(B1) *Linearity*. $\beta(N, a_1 v_1 + a_2 v_2) = a_1 \beta(N, v_1) + a_2 \beta(N, v_2)$ for all $a_1, a_2 \in \mathbb{R}$ and games $(N, v_1), (N, v_2)$.
(B2) *Dummy player*. If $i \in N$ is a dummy player in the game (N, v) then $\beta_i(N, v) = v(\{i\})$.
(B3) 2-*Efficiency*. $\beta_{ij}(N^{ij}, v^{ij}) = \beta_i(N, v) + \beta_j(N, v)$ for all $i, j \in N$.

Nowak [14] characterized the Banzhaf value using, among other axioms, 2-efficiency and dummy player condition. The Banzhaf value can be calculated using the Harsanyi dividends,

$$\beta_i(N, v) = \sum_{\{S \subseteq N:\ i \in S\}} \frac{\Delta_S^v}{2^{|N|-1}}. \tag{4}$$

We consider a *simple game* as a game (N, v) satisfying the following properties: 1) $v(S) \in \{0, 1\}$ for all $S \subseteq N$, 2) $v(S) \leqslant v(T)$ if $S \subseteq T$, and 3) $v(N) = 1$. There are two kinds of coalitions in a simple game, *winning* coalitions (the worth is 1) and *losing* coalitions (the worth is 0). Unanimity games are examples of simple games. A payoff vector of a simple game is interpreted as a *power index*, moreover the Banzhaf value was introduced at the beginning as a power index, called the Banzhaf–Coleman index. A *weighted voting game* (N, v) with $v = [q; w_1, \dots, w_n]$ is a specific simple game given by

$$v(S) = \begin{cases} 1, & \text{if } \sum_{i \in S} w_i \geqslant q \\ 0, & \text{if } \sum_{i \in S} w_i < q, \end{cases} \tag{5}$$

where $q \in \mathbb{R}_+$ is named *quota* and $w_i \in \mathbb{R}_+$ is the *weight* of each player i with $\sum_{i \in N} w_i \geqslant q$.

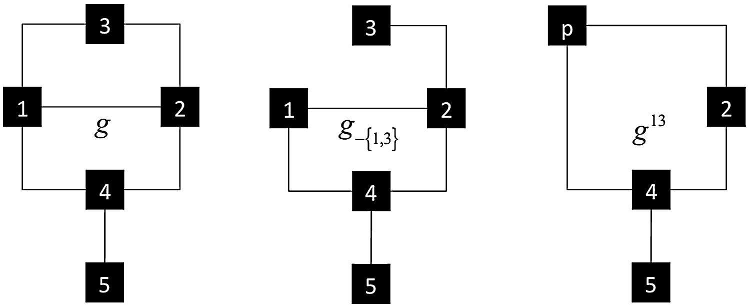

Fig. 1. Deletion and amalgamation ($p = 13$).

1.2. Communication structures

Myerson [12] considered that the bilateral communications among players can modify the solution of a game. He established the players' communication by a graph. An undirected *graph* $g = (S, A)$ is defined by a finite set S and a set A of unordered pairs of different members of S. The elements of S are named *vertices* and the elements of A are called *links*. Let $L = \{\{i, j\}: i \neq j; i, j \in N\}$ denote the set of bilateral relations among the players in our game. Myerson defined a *communication structure* over N as a graph $g = (S, A)$ where $S \subseteq N$ is the subset of players who are genuinely active in the game and $A \subseteq L$ is the set of feasible communications among them. Therefore we will use throughout the paper graph or communication structure alike. The set of all the communication structures over N is denoted by CS^N. Particularly $gN = (N, L)$ is the complete graph representing the total cooperation among all the players. Let $g = (S, A) \in CS^N$ be a graph. A *subgraph* $g' = (S', A')$ of g is another graph which satisfies that $S' \subseteq S$ and $A' \subseteq A$. A path in g is defined by a sequence of vertices $(i_k)_{k=1}^{k=m}$ satisfying that $\{i_k, i_{k+1}\} \in A$ is a different link in g for each $k = 1, \dots, m-1$. The graph g is *connected* if for all pair of vertices there is a path in g containing them. If $T \subseteq N$ is a coalition then we denote g_T as the subgraph of g using only the vertices in $T \cap S$ and the links in A among them. Coalition T is *feasible* iff $T \subseteq S$ and g_T is connected. The family of all the feasible coalitions in g is denoted as $\mathcal{F}^g$. The *connected components* of g are the maximal subgraphs of g which are connected. The coalitions of players in the connected components of g are $N/g = \{H \subseteq S: g_H$ is a connected component of $g\}$. Coalitions in N/g form a partition of S. A vertex i is *isolated* in the graph g iff $\{i\} \in N/g$. For each link $\{i, j\} \in A$ in g we will use the following graphs

$$g_{-\{i,j\}} = \big(S, A \setminus \{i, j\}\big) \quad \text{and} \quad g^{ij} = \big(S^{ij}, A^{ij}\big)$$

where $S^{ij} = S \setminus \{i, j\} \cup ij$ and

$$A^{ij} = \big\{\{i', j'\} \in A: i', j' \neq i, j\big\} \cup \big\{\{i', ij\}: i' \neq i, j \text{ and } \big[\{i', i\} \in A \text{ or } \{i', j\} \in A\big]\big\}.$$

Example 1. Fig. 1 shows a graph g with five vertices and $g_{-\{1,3\}}, g^{13}$.

A *game with communication structure* is a triple (N, v, g). A *communication value* ψ assigns a payoff vector to each game with communication structure. Myerson [12] introduced for each (N, v, g) the *graph game* (N, v^g) with

$$v^g(S) = \sum_{H \in N/g_S} v(H) \quad \forall S \subseteq N, \tag{6}$$

a new game incorporating the communication structure in the game. Owen [17] defined the *graph Banzhaf value* of a game with communication structure as the Banzhaf value of the graph game,

$$\xi(N, v, g) = \beta\big(N, v^g\big) \tag{7}$$

for every game with communication structure (N, v, g). Particularly, $\xi(N, v, gN) = \beta(N, v)$. Alonso-Meijide and Fiestras-Janeiro [1] proved that the graph Banzhaf value is the only communication value satisfying the following properties.

(GB1) *Isolation*. If $i \in N$ is isolated in g then $\xi_i(N, v, g) = v(\{i\})$.

(GB2) *Fairness*. Let $\{i, j\}$ be a link in g. Then

$$\xi_i(N, v, g) - \xi_i(N, v, g_{-\{i,j\}}) = \xi_j(N, v, g) - \xi_j(N, v, g_{-\{i,j\}}).$$

(GB3) *Pairwise merging*. The amalgamated game of the game (N, v) for players $i, j \in N$ with $\{i, j\}$ a link in $g \in CS^N$ satisfies

$$\xi_i(N, v, g) + \xi_j(N, v, g) = \xi_{ij}\big(N^{ij}, v^{ij}, g^{ij}\big).$$

1.3. Fuzzy communication structures

Let K be a finite set. A *fuzzy set* in K is a function $\tau : K \to [0, 1]$. The family of fuzzy sets in K is denoted as $[0, 1]^K$. Each subset $Q \subseteq K$ is associated to the fuzzy set $e^Q \in [0, 1]^K$ with $e^Q(i) = 1$ if $i \in Q$ and $e^Q(i) = 0$ otherwise. Specifically, we denote $e^\emptyset = 0$. The *support* of τ is $\text{supp}(\tau) = \{i \in K\colon\ \tau(i) \neq 0\}$ and the *image* of τ is the set $\text{im}(\tau) = \{\lambda \in \mathbb{R}\colon\ \exists i \in K \text{ with } \tau(i) = \lambda\}$.

Jiménez-Losada et al. [8] introduced fuzzy communication structures for games as fuzzy graphs. In this paper we use the operators $\wedge, \vee$ as the minimum and the maximum respectively. Let L be the set of bilateral communications among the players in N. A *fuzzy communication structure* for the game v is an undirected fuzzy graph [11] over N, that is a pair $\gamma = (\tau, \rho)$ with $\tau \in [0, 1]^N$ the fuzzy set of vertices and $\rho \in [0, 1]^L$ the fuzzy set of links satisfying $\rho(i, j) \leqslant \tau(i) \wedge \tau(j)$ for all $\{i, j\} \in L$. The set of fuzzy communication structures over N is denoted by FCS^N. Hence we will use fuzzy graph or fuzzy communication structure alike. We denote as $\gamma = 0$ the null fuzzy graph where $\tau = 0$ and $\rho = 0$. Every communication structure $g = (S, A) \in CS^N$ is identified with the fuzzy graph $g = (\tau, \rho)$ where $\tau = e^S$ and $\rho = e^A$. Let $\gamma = (\tau, \rho) \in FCS^N$ be a fuzzy communication structure. The number $\tau(i)$ is interpreted as the real level of involvement of player $i \in N$ in the game v and the number $\rho(i, j)$ represents the maximal level to which the link $\{i, j\}$ can be used. The set of vertices in γ is $\text{vert}(\gamma) = \text{supp}(\tau)$ and the set of links is $\text{link}(\gamma) = \text{supp}(\rho)$. So, the *crisp version* of γ is the graph $g^\gamma = (\text{vert}(\gamma), \text{link}(\gamma))$. We use the notation $N/\gamma = N/g^\gamma$ and $\mathcal{F}^\gamma = \mathcal{F}^{g^\gamma}$. The *minimal level* in γ is

$$\wedge\gamma = \Big(\bigwedge_{i \in \text{vert}(\gamma)} \tau(i)\Big) \wedge \Big(\bigwedge_{\{i,j\} \in \text{link}(\gamma)} \rho(i, j)\Big).$$

A player i is *isolated* in γ if he is isolated in g^γ. Another fuzzy graph $\gamma' = (\tau', \rho')$ over N is a *subgraph* of γ iff $\tau' \leqslant \tau$ and $\rho' \leqslant \rho$. We use in that case $\gamma' \leqslant \gamma$. If $S \subseteq N$ is a coalition then $\gamma_S = (\tau_S, \rho_S) \in FCS^N$ is the subgraph of γ defined as

$$\tau_S(i) = \begin{cases} \tau(i), & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \rho_S(i, j) = \begin{cases} \rho(i, j), & \text{if } i, j \in S \\ 0, & \text{otherwise.} \end{cases}$$

We defined three binary operations for fuzzy graphs in Jiménez-Losada et al. [8]. Let $\gamma = (\tau, \rho), \gamma' = (\tau', \rho') \in FCS^N$ be two fuzzy graphs over N:

(1) If $\tau(i) + \tau'(i) \leqslant 1$ for all $i \in N$ then the sum is $\gamma + \gamma' = (\tau + \tau', \rho + \rho')$ where for each player $i, j \in N$ we have $(\tau + \tau')(i) = \tau(i) + \tau'(i)$ and

$$\big(\rho + \rho'\big)(i, j) = \rho(i, j) + \rho'(i, j).$$

(2) If $\gamma' \leqslant \gamma$ then the subtraction is a new fuzzy graph $\gamma - \gamma' = (\tau - \tau', \rho - \rho')$ where for all $i, j \in N$ we have $(\tau - \tau')(i) = \tau(i) - \tau'(i)$ and

$$\big(\rho - \rho'\big)(i, j) = \big[\rho(i, j) - \rho'(i, j)\big] \wedge \big[\tau(i) - \tau'(i)\big] \wedge \big[\tau(j) - \tau'(j)\big].$$

(3) If $t \in [0, 1]$ then there is a new fuzzy graph defined by $t\gamma = (t\tau, t\rho)$ where for every $i, j \in N$ we have $(t\tau)(i) = t\tau(i)$ and

$$(t\rho)(i, j) = t\rho(i, j).$$

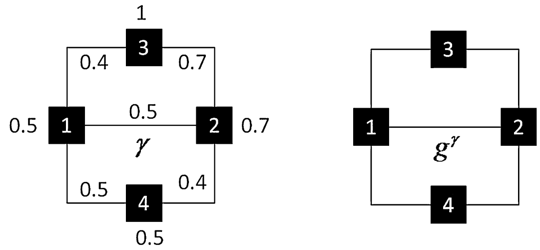

Fig. 2. Fuzzy graph and crisp version.

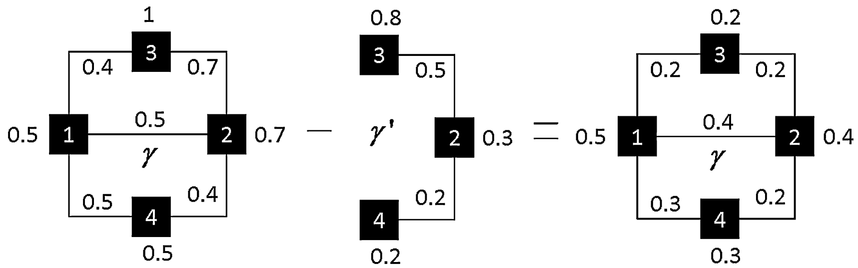

Fig. 3. The difference between fuzzy graphs.

Remark. The reader can see that the sum and the subtraction of fuzzy graphs are not opposite operations. However, the following equality holds, see Jiménez-Losada et al. [8]: If $\gamma, \gamma', \gamma'' \in FCS^N$ are three fuzzy graphs over N such that $\gamma'' \leqslant \gamma - \gamma'$ and $\gamma' \leqslant \gamma$ then $(\gamma - \gamma') - \gamma'' = \gamma - (\gamma' + \gamma'')$. Particularly $(\gamma - \gamma') - \gamma'' = (\gamma - \gamma'') - \gamma'$.

Example 2. We consider the fuzzy graph $\gamma = (\tau, \rho)$ over the set $N = \{1, 2, 3, 4\}$ with $\rho(1, 3) = \rho(2, 4) = 0.4$, $\tau(1) = \tau(4) = \rho(1, 2) = \rho(1, 4) = 0.5$, $\tau(2) = \rho(2, 3) = 0.7$, $\tau(3) = 1$ and $\rho(3, 4) = 0$. Fig. 2 represents this fuzzy graph and its crisp version.

The minimal level is $\wedge\gamma = 0.4$. Fig. 3 calculates the difference with a subgraph of γ.

1.4. The cg-fuzzy model

A *game with fuzzy communication structure* is a triple (N, v, γ) where (N, v) is a game and $\gamma \in FCS^N$. A *fuzzy communication value* assigns to each game with fuzzy communication structure a payoff vector. Aubin [2] introduced partitions by levels to determine the worth of a fuzzy coalition in a game. Jiménez-Losada et al. [9], following Aubin and the Myerson model, defined a way to get fuzzy communication values. They introduced the concept of partition by levels of a fuzzy graph. A *partition by levels* of a fuzzy communication structure $\gamma \in FCS^N$ is the finite set $(g_k, s_k)_{k=1}^{k=m}$ with $s_k \in (0, 1]$ and $g_k \in CS^N$ such that $s_k g_k \leqslant \gamma - \sum_{l=1}^{k-1} s_l g_l$ for all k and $\gamma - \sum_{k=1}^{m} s_k g_k = 0$. A *fuzzy partition selector* f over N chooses a specific partition by levels for each fuzzy communication structure. Let f be a fuzzy partition selector. For each fuzzy graph γ the *f-fuzzy graph game*, following (6), is (N, v_f^γ) defined for any coalition $S \subseteq N$ as

$$v_f^\gamma(S) = \sum_{k=1}^{m} s_k v^{g_k}(S) \tag{8}$$

where $f(\gamma_S) = (g_k, s_k)_{k=1}^{k=m}$. Finally, we can apply any known value for cooperative games. In Jiménez-Losada et al. [9] several selectors are studied using the Shapley value.

Tsurumi et al. [18] used the Choquet integral for calculating the worth of fuzzy coalitions. The *Choquet by graphs* (*cg*) *selector*, defined in Jiménez-Losada et al. [9], is a specific fuzzy partition selector inspirated in the Choquet integral. Players, following this selector, try to get first the biggest graph and second the top level to connect it. The cg-partition for each fuzzy graph can be obtained applying the following algorithm. Let $\gamma = (\tau, \rho) \in FCS^N$ be a fuzzy communication structure.

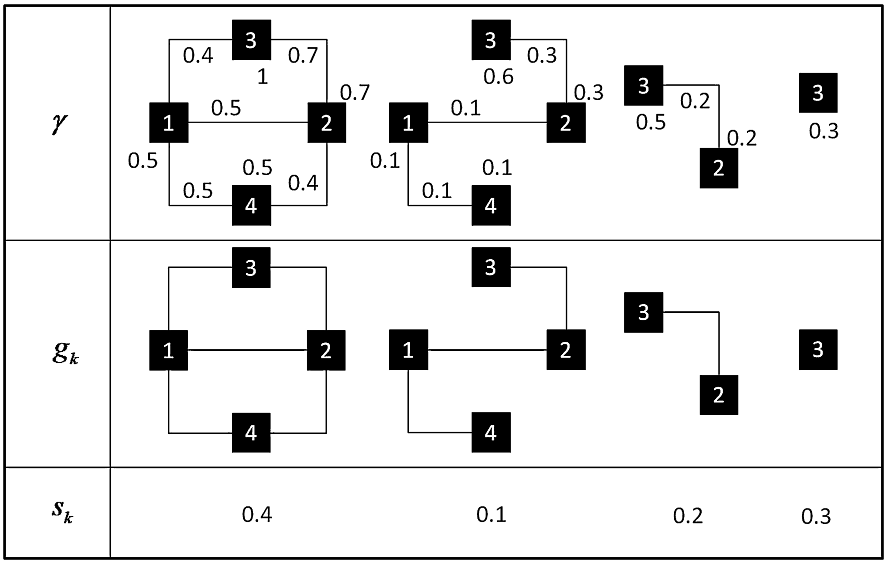

Fig. 4. cg-partition.

Algorithm 1. *cg-partition* (γ)

$k \leftarrow 0$, $cg \leftarrow \emptyset$
while $\gamma \neq 0$ **do**
  $k \leftarrow k+1$
  $s_k \leftarrow \wedge\gamma$
  $g_k \leftarrow g^{\gamma}$
  $cg \leftarrow cg \cup \{(g_k, s_k)\}$
  $\gamma \leftarrow \gamma - s_k g_k$
end

The partition by levels is $cg(\gamma) = cg$.

For each player $i \in N$ the cg-partition $(g_k, s_k)_{k=1}^{k=m}$ of γ satisfies

$$\sum_{\{k:\, i \in \mathrm{vert}(g_k)\}} s_k = \tau(i). \tag{9}$$

Example 3. We can see in Fig. 4 the cg-partition by levels of a fuzzy graph γ.

2. The cg-fuzzy graph Banzhaf value

Following Owen [17], in this section we introduce our version of Banzhaf value as the Banzhaf value of the cg-fuzzy graph game. For each fuzzy graph γ the *cg-fuzzy graph game* (N, v_{cg}^{γ}) is defined, following (8), for any coalition $S \subseteq N$ as

$$v_{cg}^{\gamma}(S) = \sum_{k=1}^{m} s_k v^{g_k}(S) \tag{10}$$

where $cg(\gamma_S) = (g_k, s_k)_{k=1}^{k=m}$.

Definition 1. The cg-fuzzy graph Banzhaf value is the fuzzy communication value defined as $B(N, v, \gamma) = \beta(N, v_{cg}^{\gamma})$ for all game with fuzzy communication structure (N, v, γ).

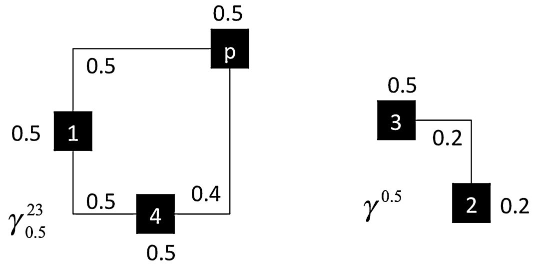

Fig. 5. Fuzzy amalgamation ($p = 23$).

Next, we are going to extend the graph Banzhaf properties (GB1, GB2 and GB3) to fuzzy situations. Consider the following axioms for a given fuzzy communication value F. Let (N, v, γ) be a game with fuzzy communication structure and $\gamma = (\tau, \rho)$. The payment of an isolated player in γ, who cannot cooperate with others, should be the worth of his individual coalition taking into account his maximum capacity of involvement.

Fuzzy isolation. If $i \in N$ is an isolated player in γ then

$$F(N, v, \gamma) = \tau(i) v(\{i\}).$$

The subgraph $\gamma^t_{-\{i,j\}}$ represents the fuzzy graph γ modified by reducing $t \in [0, \rho(i, j)]$ the capacity of $\{i, j\} \in \text{link}(\gamma)$, that is, $\tau^t_{-\{i,j\}} = \tau$ and $\rho^t_{-\{i,j\}} = \rho$ except for

$$\rho^t_{-\{i,j\}}(i, j) = \rho(i, j) - t. \tag{11}$$

Fuzzy fairness. Let $\{i, j\} \in \text{link}(\gamma)$ and $t \in [0, \rho(i, j)]$. Then

$$F_i(N, v, \gamma) - F_i\big(N, v, \gamma^t_{-\{i,j\}}\big) = F_j(N, v, \gamma) - F_j\big(N, v, \gamma^t_{-\{i,j\}}\big).$$

We define from γ the fuzzy communication structure $\gamma^{ij}_t = (\tau^{ij}_t, \rho^{ij}_t) \in FCS^{N^{ij}}$ for all $i, j \in N$ and $t \in [0, \rho(i, j)]$. If $i', j' \in N^{ij}$ then

$$\tau^{ij}_t(i') = \begin{cases} t, & \text{if } i' = ij \\ \tau(k) \wedge t, & \text{if } i' \neq ij \end{cases} \qquad \rho^{ij}_t(i', j') = \begin{cases} \rho(i', j') \wedge t, & \text{if } i', j' \neq ij \\ (\rho(i', i) \vee \rho(i', j)) \wedge t, & \text{if } j' = ij. \end{cases} \tag{12}$$

This new fuzzy graph represents the amalgamation of players i, j in the structure below level t. If we suppose that the merging situation of these players is only possible down to level t then the fuzzy communication structure $\gamma^t = (\tau^t, \rho^t)$ represents the situation above level t. If $i, j \in N$ then

$$\tau^t(i) = \big(\tau(i) - t\big) \vee 0 \qquad \rho^t(i, j) = \big(\rho(i, j) - t\big) \vee 0. \tag{13}$$

Example 4. Fig. 5 shows the fuzzy graphs $\gamma^{23}_{0.5}$ and $\gamma^{0.5}$ for the fuzzy communication structure γ in Fig. 2.

The following axiom guarantees the merging neutrality by levels of the outcome.

Pairwise fuzzy merging. The amalgamated game of (N, v) for players $i, j \in N$ satisfies for all $t \in [0, \rho(i, j)]$

$$F_i(N, v, \gamma) + F_j(N, v, \gamma) = F_{ij}\big(N^{ij}, v^{ij}, \gamma^{ij}_t\big) + F_i\big(N, v, \gamma^t\big) + F_j\big(N, v, \gamma^t\big).$$

If we take a usual graph, $g \in CS^N$ then $g^{ij}_1 = g^{ij}$ and $g^1 = 0$. Hence pairwise fuzzy merging coincides with pairwise merging on games with communication structure when level 1 is reached.

The next theorem says that there are not two fuzzy communication values satisfying these three axioms. The proof is similar to the crisp version, see Theorem 6 in Alonso-Meijide and Fiestras-Janeiro [1].

Theorem 1. *There is at most one fuzzy communication value satisfying fuzzy isolation, fuzzy fairness and pairwise fuzzy merging.*

Given $T \subset N$ and $x \in \mathbb{R}^T$ then $x^0 \in \mathbb{R}^N$ is the vector with components $x_i^0 = x_i$ if $i \in T$ and $x_i^0 = 0$ otherwise.

Lemma 2. *Let (N, v) be a game. If there is a coalition $T \subseteq N$ such that $v(S) = v(S \cap T)$ for all $S \subseteq N$ then $\beta(N, v) = \beta^0(T, v)$.*

Proof. If $i \notin T$ then i is a dummy player in (N, v) by the particular definition of v. Property (B2) of the Banzhaf value implies $\beta_i(N, v) = 0$.

We denote $t = |T|$. For each $R \subseteq T$ there are 2^{n-t} different coalitions $S \subseteq N$ with $S \cap T = R$. Therefore, if $i \in T$ then

$$\beta_i(N, v) = \frac{1}{2^{n-1}} \sum_{\{S \subseteq N:\, i \in S\}} \bigl[v(S) - v\bigl(S \setminus \{i\}\bigr)\bigr]$$
$$= \frac{1}{2^{n-1}} \sum_{\{R \subseteq T:\, i \in R\}} 2^{n-t} \bigl[v(R) - v\bigl(R \setminus \{i\}\bigr)\bigr] = \beta_i(T, v). \qquad \square$$

We relate the cg-fuzzy graph Banzhaf value with the graph Banzhaf value (7). Theorem 3 provides a formula in Choquet form to calculate our version of fuzzy Banzhaf value by a linear combination of graph Banzhaf values.

Theorem 3. *Let (N, v, γ) be a game with fuzzy communication structure. If the cg-partition of γ is $cg(\gamma) = (g_k, s_k)_{k=1}^m$ then*

$$B(N, v, \gamma) = \sum_{k=1}^{m} s_k \xi^0\bigl(\mathrm{vert}(g_k), v, g_k\bigr).$$

Proof. The cg-algorithm determines $cg(\gamma) = (g_k, s_k)_{k=1}^m$ for a given fuzzy graph $\gamma \in FCS^N$.

Let $S \subseteq N$ be a coalition. It is possible to get the cg-partition of γ_S from $cg(\gamma)$. If $cg(\gamma_S) = (g'_p, s'_p)_{p=1}^q$ then there are indices $(k_p)_{p=1}^q$ with $(g_{k_p+1})_S \neq (g_{k_p})_S$ ($k_q = m$ if $(g_m)_S \neq 0$) such that

$$g'_p = (g_k)_S \quad \forall k_p \leqslant k < k_{p+1} \quad \text{and} \quad s'_p = \sum_{k=k_p}^{k_{p+1}-1} s_k.$$

So, we obtain from (10)

$$v_{cg}^{\gamma}(S) = \sum_{p=1}^{q} s'_p v^{g'_p}(S) = \sum_{p=1}^{q} \sum_{k=k_p}^{k_{p+1}-1} s_k v^{g_k}(S) = \sum_{k=1}^{k_q} s_k v^{g_k}(S).$$

Observe that $S \cap \mathrm{vert}((g_k)_S) = \emptyset$ for all $k > k_q$, thus $v_{cg}^{\gamma} = \sum_{k=1}^{m} s_k v^{g_k}$. By property (B1) and the previous lemma we have

$$B(N, v, \gamma) = \beta\bigl(N, v_{cg}^{\gamma}\bigr) = \sum_{k=1}^{m} s_k \beta\bigl(N, v^{g_k}\bigr) = \sum_{k=1}^{m} s_k \beta^0\bigl(\mathrm{vert}(g_k), v^{g_k}\bigr)$$
$$= \sum_{k=1}^{m} s_k \xi^0\bigl(\mathrm{vert}(g_k), v, g_k\bigr),$$

because $v^{g_k}(S \cap \mathrm{vert}(g_k)) = v^{g_k}(S)$ for all k and S. $\square$

Now we prove that the cg-fuzzy graph Banzhaf value is the only fuzzy communication value satisfying our axioms, taking into account Theorem 1.

Theorem 4. *The cg-fuzzy graph Banzhaf value satisfies fuzzy isolation, fuzzy fairness and pairwise fuzzy merging.*

Proof. Suppose (N, v, γ) with $\gamma = (\tau, \rho)$ and $cg(\gamma) = (s_k, g_k)_{k=1}^m$ the cg-partition by levels of γ.

Fuzzy isolation. Let $i \in N$ be an isolated player in γ. As $g_k \leqslant g^\gamma$ for each k then i is an isolated player in g_k when $i \in \mathrm{vert}(g_k)$. We get $\xi_i(\mathrm{vert}(g_k), v, g_k) = v(\{i\})$ using property (GB1). Theorem 3 and (9) imply

$$B_i(N, v, \gamma) = \sum_{k=1}^m s_k \xi_i^0\big(\mathrm{vert}(g_k), v, g_k\big) = v\big(\{i\}\big) \sum_{\{k:\, i \in \mathrm{vert}(g_k)\}} s_k = \tau(i) v\big(\{i\}\big).$$

Fuzzy fairness. Now we take $i, j \in N$ with $\rho(i, j) > 0$. Let $t \in (0, \rho(i, j)]$. We define the game (N, w) as $w = v_{cg}^\gamma - v_{cg}^{\gamma_{-\{i,j\}}^t}$. If i or j is not in S then $w(S) = 0$, using that $\gamma_S = (\gamma_{-\{i,j\}}^t)_S$. Therefore,

$$w(S) - w\big(S \setminus \{i\}\big) = \left\{ \begin{array}{l} w(S), \text{ if } i, j \in S \\ 0, \text{ otherwise} \end{array} \right\} = w(S) - w\big(S \setminus \{j\}\big).$$

The Banzhaf formula (3) implies $\beta_i(N, w) = \beta_j(N, w)$. As the Banzhaf value is a linear function (B1) we have

$$\begin{aligned} B_i(N, v, \gamma) - B_i\big(N, v, \gamma_{-\{i,j\}}^t\big) &= \beta_i\big(N, v_{cg}^\gamma\big) - \beta_i\big(N, v_{cg}^{\gamma_{-\{i,j\}}^t}\big) \\ &= \beta_i(N, w) = \beta_j(N, w) \\ &= B_j(N, v, \gamma) - B_j\big(N, v, \gamma_{-\{i,j\}}^t\big). \end{aligned}$$

Pairwise fuzzy merging. Let us consider $i, j \in N$ and $t \in [0, \rho(i, j)]$. There exists $q \in \{1, \dots, m\}$ such that $s_q \leqslant t < s_{q+1}$. It is possible to determine the cg-partition of γ_t^{ij} from $cg(\gamma)$,

$$cg\big(\gamma_t^{ij}\big) = \big(s_k, (g_k)^{ij}\big)_{k=1}^q \cup \big(t - s_q, (g_{q+1})^{ij}\big).$$

The cg-partition of γ^t from $cg(\gamma)$ is

$$cg\big(\gamma^t\big) = \big(s_k, (g_k)^{ij}\big)_{k=q+2}^m \cup (s_{q+1} - t + s_q, g_{q+1}).$$

We will use $Q_k = \mathrm{vert}(g_k)$ in the following calculations. Theorem 3 allows us to describe the cg-fuzzy graph Banzhaf value by graph Banzhaf values. Then we can use property (GB3) to get

$$\begin{aligned} B_i(N, v, \gamma) + B_j(N, v, \gamma) &= \sum_{k=1}^m s_k\big[\xi_i^0(Q_k, v, g_k) + \xi_j^0(Q_k, v, g_k)\big] \\ &= \sum_{k=1}^q s_k\big[\xi_i^0(Q_k, v, g_k) + \xi_j^0(Q_k, v, g_k)\big] \\ &\quad + (t - s_q)\big[\xi_i^0(Q_{q+1}, v, g_{q+1}) + \xi_j^0(Q_{q+1}, v, g_{q+1})\big] \\ &\quad + (s_{q+1} - t + s_q)\big[\xi_i^0(Q_{q+1}, v, g_{q+1}) + \xi_j^0(Q_{q+1}, v, g_{q+1})\big] \\ &\quad + \sum_{k=q+2}^m s_k\big[\xi_i^0(Q_k, v, g_k) + \xi_j^0(Q_k, v, g_k)\big] \\ &= \sum_{k=1}^q s_k \xi_{ij}^0\big(Q_k^{ij}, v^{ij}, (g_k)^{ij}\big) + (t - s_q)\xi_{ij}^0\big(Q_{q+1}^{ij}, v^{ij}, (g_{q+1})^{ij}\big) \\ &\quad + B_i\big(N, v, \gamma^t\big) + B_j\big(N, v, \gamma^t\big) \\ &= B_{ij}\big(N^{ij}, v^{ij}, \gamma_t^{ij}\big) + B_i\big(N, v, \gamma^t\big) + B_j\big(N, v, \gamma^t\big). \qquad \square \end{aligned}$$

3. Computing the cg-fuzzy graph Banzhaf value

The goal of this section is to formulate an algorithm to determine the cg-fuzzy graph Banzhaf using the dividends of the game, and to study the complexity of it. Let A be an algorithm. The *time complexity* of A is measured by a function $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ where $f(n)$ is the maximal number of iterations in a universal Turing machine (before halting) in relation to the size of the input n. Let $f, g : \mathbb{Z}_+ \to \mathbb{Z}_+$. Following $\mathcal{O}\Omega\Theta$-notation, proposed by Knuth [10], we use $f = \mathcal{O}(g)$ if there are $c, n_0 \in \mathbb{Z}_+$ such that $f(n) \leqslant cg(n)$ for all $n \geqslant n_0$. In that case we say f *is of the order of* g.

We consider a game with fuzzy communication structure (N, v, γ). The cg-fuzzy graph Banzhaf value can be obtained by formula (4) if the dividends of the cg-fuzzy graph game (10) are calculated previously. It is possible to get first the new game and second its dividends by (2), but the complexity order is too large. We will look for an algorithm to compute the dividends of the cg-fuzzy graph game without calculating the characteristic function.

Owen [17] proved for a communication structure $g \in CS^N$ that the dividends of a graph game (N, v^g) (6) can be obtained from the dividends of (N, v) and the dividends of the unanimity games for feasible coalitions,

$$\Delta_S^{v^g} = \begin{cases} \sum_{\{T \subseteq S:\, T \in \mathcal{F}^g\}} \Delta_S^{u_T} \Delta_T^{v}, & \text{if } S \in \mathcal{F}^g \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

This formula implies to determine before the dividends of (N, v) and then it involves with a very large computation effort. Nevertheless, it is possible to calculate the dividends of (N, v^g) directly and then we can determine the dividends of the cg-fuzzy graph game.

Lemma 5. *Let (N, v, g) be a game with communication structure. The Harsanyi dividends of (N, v^g) are*

$$\Delta_S^{v^g} = v(S) - \sum_{\{T \in \mathcal{F}^g:\, T \subset S\}} \Delta_T^{v^g},$$

for each $S \in \mathcal{F}^g$ and $\Delta_S^{v^g} = 0$ if $S \notin \mathcal{F}^g$.

Proof. Following (14), $\Delta_S^{v^g} = 0$ if $S \notin \mathcal{F}^g$. If $S \in \mathcal{F}^g$ then using (2)

$$v(S) = v^g(S) = \sum_{\{T \in \mathcal{F}^g:\, T \subseteq S\}} \Delta_T^{v^g} = \Delta_S^{v^g} + \sum_{\{T \in \mathcal{F}^g:\, T \subset S\}} \Delta_T^{v^g}. \qquad \square$$

We calculate the dividends of a graph game in upward order, and we store them in a table. Let $g \in CS^N$ be a communication structure and $d = \bigvee_{T \in N/g} |T|$. We denote as t_p the number of feasible coalitions in g with cardinal p for all $p = 1, \ldots, d$ and then the set of these coalitions is $\mathcal{F}_p^g = \{S_1^p, \ldots, S_{t_p}^p\}$. The algorithm described in Lemma 5 is as follows:

Algorithm 2. *dividends* $(v, g, \mathcal{F}^g)$

$\Delta_\emptyset^{v^g} \leftarrow 0$
for p **from** 1 **to** d
  for q **from** 1 **to** t_p
  $\Delta_{S_q^p}^{v^g} \leftarrow v(S_q^p) - \sum_{\{T \in \mathcal{F}^g : T \subset S_q^p\}} \Delta_T^{v^g}$
  end
end

Hence we can calculate the dividends of a graph game directly using only the feasible coalitions. We use the following procedure, that we name *CGDL procedure* (Choquet by graphs and dividends by levels), to compute the cg-fuzzy graph Banzhaf value for a given game with fuzzy communication structure: 1) first we determine the cg-partition of the fuzzy graph by Algorithm 1, 2) second we calculate the dividends of the feasible coalitions for each graph in the partition using Algorithm 2 and, 3) finally we get the cg-fuzzy graph Banzhaf value using formula (4) and Theorem 3. The following theorem determines the time complexity of this procedure.

Theorem 6. *Let (N, v, γ) be a game with fuzzy communication structure and $\gamma = (\tau, \rho)$. The time complexity of computing its cg-fuzzy graph Banzhaf value by the CGDL procedure is $\mathcal{O}(ce(a+b)3^d)$ where*

$$a = \left|\operatorname{supp}(\tau)\right|, \qquad b = \left|\operatorname{supp}(\rho)\right|, \qquad c = \left|\operatorname{im}(\tau) \cup \operatorname{im}(\rho)\right|, \qquad d = \bigvee_{H \in N/\gamma} |H| \quad \textit{and} \quad e = |N/\gamma|.$$

Proof. We follow the three steps of the CGDL procedure.

First step. Algorithm 1 contains, besides assignments, one *while*. The maximum number of iterations for *while* in this algorithm is c. The computation in each iteration needs at most $\mathcal{O}(a+b)$ to calculate s_k, g_k and $\gamma - s_k g_k$. Hence we conclude that

$$\mathit{time}(\mathit{algorithm}1) = \mathcal{O}\big(c(a+b)\big).$$

Second step. We calculate the dividends of v^{g_k} for each graph g_k (at most c) in the cg-partition of γ. In order to use Algorithm 2 we have to calculate the feasible coalitions in g_k, these are the subsets of vertices which generate a connected subgraph. It is known that we can determine if a graph is connected or not by a Depth First Search (DFS) algorithm with order at most $\mathcal{O}(a+b)$ and then we need at most $\mathcal{O}(e2^d(a+b))$ to obtain $\mathcal{F}^{g_k}$. Algorithm 2 involves two loops ($loop2$ into $loop1$) besides one assignment. A feasible coalition in g_k has at most cardinality d. Therefore the time complexity of each iteration of $loop2$ depends on the cardinality p with $p = 1, \dots, d$. So,

$$\mathit{time}(\mathit{algorithm}2) = 1 + \mathit{time}(\mathit{loop}1) \leqslant 1 + \sum_{p=1}^{d} \mathit{time}(\mathit{loop}2).$$

For each p we have at most $\binom{d}{p}$ feasible coalitions with cardinality p and, for each of them there are at most $2^p - 1$ non-empty feasible coalitions contained in it. As we do in $loop2$ one assignment and a sum with $2^p - 1$ elements at most we have

$$\mathit{time}(\mathit{loop}2) \leqslant \binom{d}{p} 2^p.$$

Hence, we get

$$\mathit{time}(\mathit{algorithm}2) \leqslant 1 + \sum_{p=1}^{d} \binom{d}{p} 2^p = \sum_{p=0}^{d} \binom{d}{p} 2^p = 3^d.$$

The time for computing all the dividends of the graph games in the partition is $\mathcal{O}(c3^d)$ and then we obtain the dividends of all the games v^{g_k} following Algorithm 2 in $\mathcal{O}(ce(a+b)3^d)$.

Third step. Finally we use formula (4) and Theorem 3 to obtain the desirable cg-fuzzy value for each active player $i \in \operatorname{supp}(\tau)$,

$$B_i(N, v, \gamma) = \sum_{\{k \in \{1,\dots,m\}:\, i \in \operatorname{vert}(g_k)\}} \frac{1}{2^{|\operatorname{vert}(g_k)|-1}} \sum_{\{S \in \mathcal{F}^{g_k}:\, i \in S\}} \Delta_S^{v^{g_k}}.$$

The time complexity of this stage is $\mathcal{O}(ca2^d)$.

We consider then $\max\{c(a+b), ce(a+b)3^d, ca2^d\}$ and so the CGDL procedure needs time at most $\mathcal{O}(ce(a+b)3^d)$. □

Evidently, we deduce from the above theorem that the order of the time complexity of algorithm *dividend* can decrease depending on the number of vertices and links which are active, the number of components and the number of coalitions in $\mathcal{F}^{g^\gamma}$. The CGDL procedure allows us to compare the payoffs obtained with the fuzzy graph to the payoffs obtained with the graphs in the cg-partition.

4. The power of the political groups in the European Parliament

The European Parliament is organized in political groups depending on the ideologic feeling. The different political parties of the member countries present a list of candidates in their own countries and later they assume the membership to a specific group in the chamber. Therefore, the behavior of a group is not homogeneous because it is determined by the relationships among the countries. A group needs to verify two conditions: it must contain at least twenty five seats and it must represent at least one quarter of the member countries. The members of a group cannot belong to more than one group. Those members of the chamber who do not belong to any political group are known as non-attached members. There are currently (seventh legislature data) seven political groups in the European Parliament in addition to the non-attached seats. We consider in our example the following groups:

1. European People's Party (Christian Democrats), 265 members.
2. Progressive Alliance of Socialists and Democrats, 183 members.
3. Alliance of Liberals and Democrats for Europe, 84 members.
4. European Conservatives and Reformists, 55 members.
5. Greens/European Free Alliance, 55 members.
6. European United Left – Nordic Green Left, 35 members.
7. Europe of Freedom and Democracy, 29 members.
8. Non-attached Members, 29 members.

There are many voting games to be considered (game of political representations, game of national representations, games of national chapters of European political parties in each particular party decision making etc.). In the following example we consider the game of political representation groups of the European Parliament (seventh legislature data) with 365 seats (there is one vacancy seat) and a quota of 368. The corresponding weighted voting game is represented by, see (5), $v = [368; 265, 183, 84, 55, 55, 35, 29, 29]$, called the EP-game.

After the Lisbon Treaty the European Parliament has more power and responsibility. However, this "power" is blurred by the structure of the groups, consisting of heterogeneous groups of deputies of various nationalities. The factor "national", along with the "ideological", shows us the opportunity to use fuzzy communication structures. Therefore the relationships among the groups which are represented usually by a graph should be shaded. If we use a crisp graph we assume that ideological close groups guarantee the total cooperation among all the members, but this fact is not sure in the European Parliament. We can summarize the bilateral relations among the different groups, and the degree of cohesion of every group, by a fuzzy graph $\gamma = (\tau, \rho)$ over $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Later we can use the cg-fuzzy values to get more realistic power indices for the political groups. In order to illustrate the calculation of the cg-indices we take the following fuzzy graph "ad hoc". Hence we describe the calculation procedure. We provide

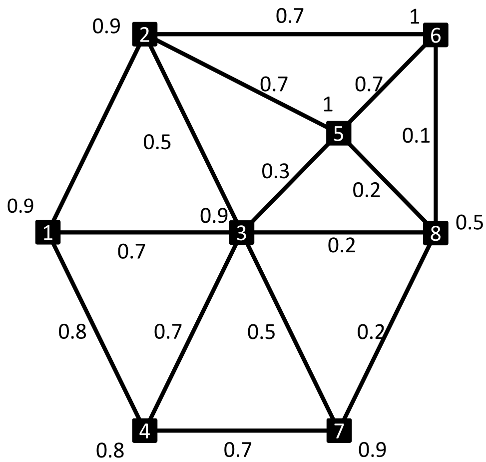

Fig. 6. EP fuzzy graph.

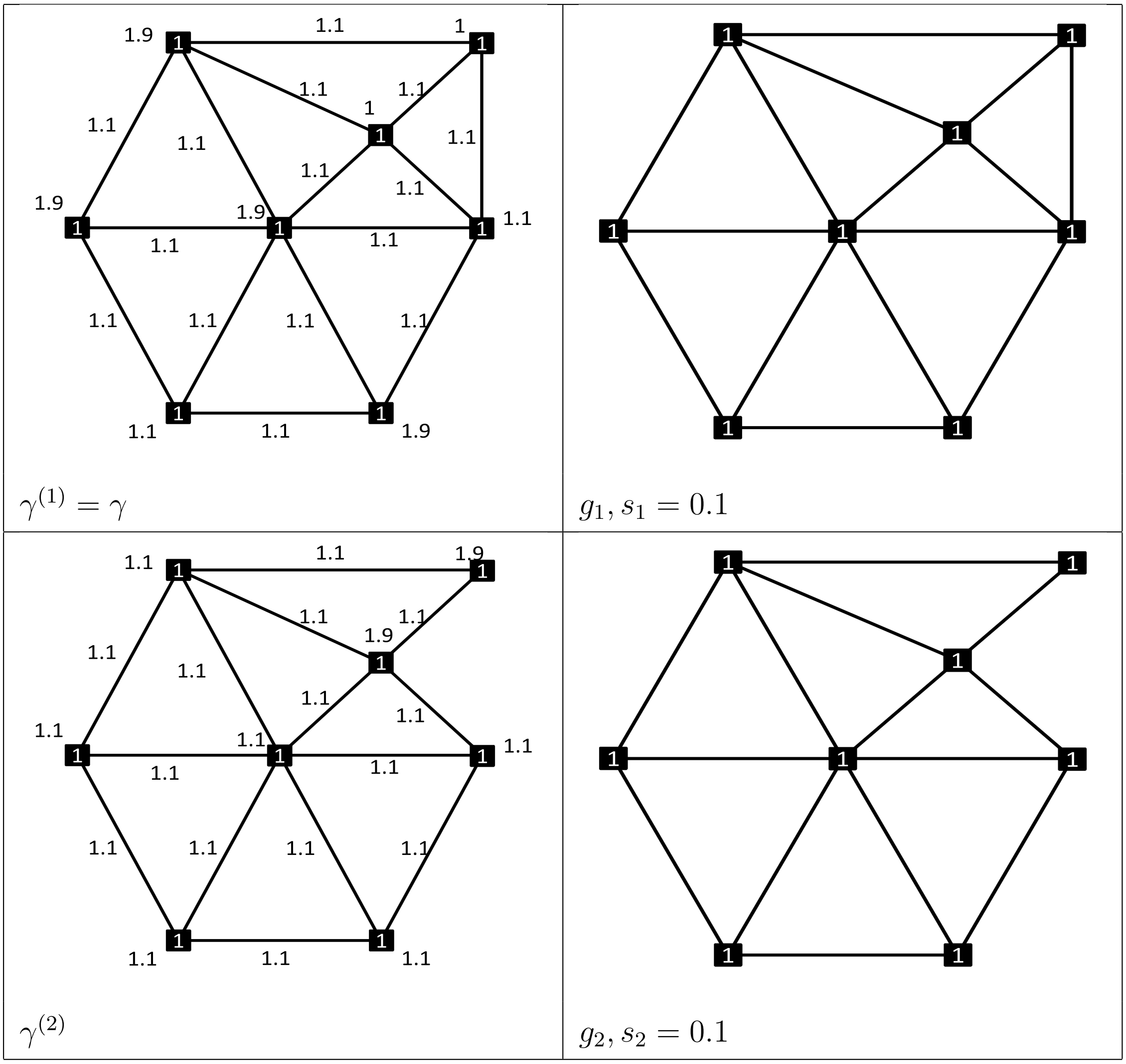

Fig. 7. The cg-partition by levels of the EP fuzzy graph γ.

the fuzzy set of vertices and the fuzzy set of edges by a symmetric matrix where the elements in the diagonal are the levels of the vertices and the rest the levels of the links. So,

$$\gamma = \begin{bmatrix} 0.9 & 0.5 & 0.7 & 0.8 & 0 & 0 & 0 & 0 \\ 0.5 & 0.9 & 0.5 & 0 & 0.7 & 0.7 & 0 & 0 \\ 0.7 & 0.5 & 0.9 & 0.7 & 0.3 & 0 & 0.5 & 0.2 \\ 0.8 & 0 & 0.7 & 0.8 & 0 & 0 & 0.7 & 0 \\ 0 & 0.7 & 0.3 & 0 & 1 & 0.7 & 0 & 0.2 \\ 0 & 0.7 & 0 & 0 & 0.7 & 1 & 0 & 0.1 \\ 0 & 0 & 0.5 & 0.7 & 0 & 0 & 0.9 & 0.2 \\ 0 & 0 & 0.2 & 0 & 0.2 & 0.1 & 0.2 & 0.5 \end{bmatrix}.$$

Fig. 6 shows the corresponding fuzzy communication structure that we name *EP fuzzy graph*. In this case $\tau(i)$ is interpreted as the membership capacity of the groups on the voting day. Number $\rho(i, j)$ means the closeness index among the groups i and j. We are going to apply the CGDL procedure to get the cg-fuzzy graph Banzhaf value for

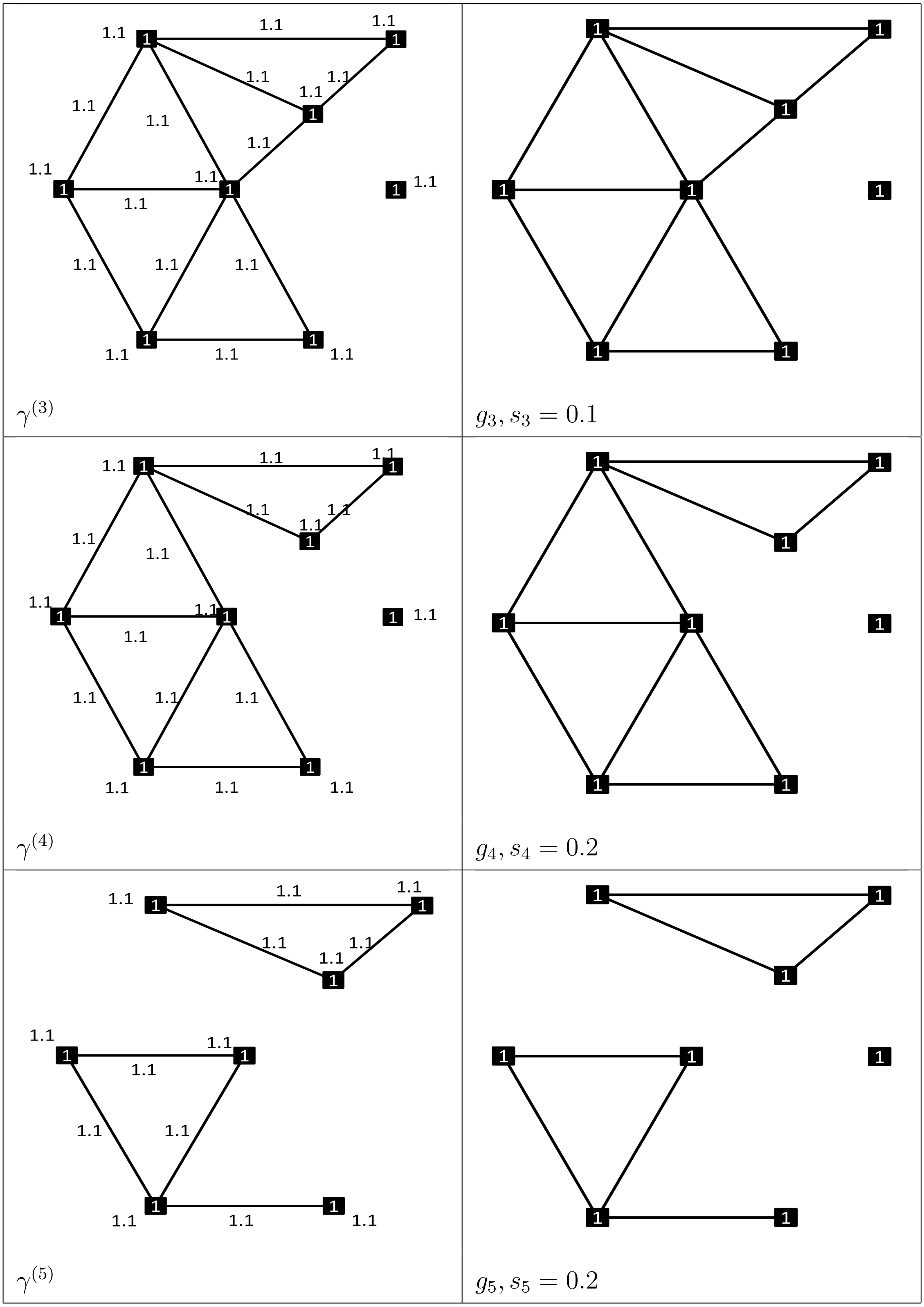

Fig. 7. (*continued*)

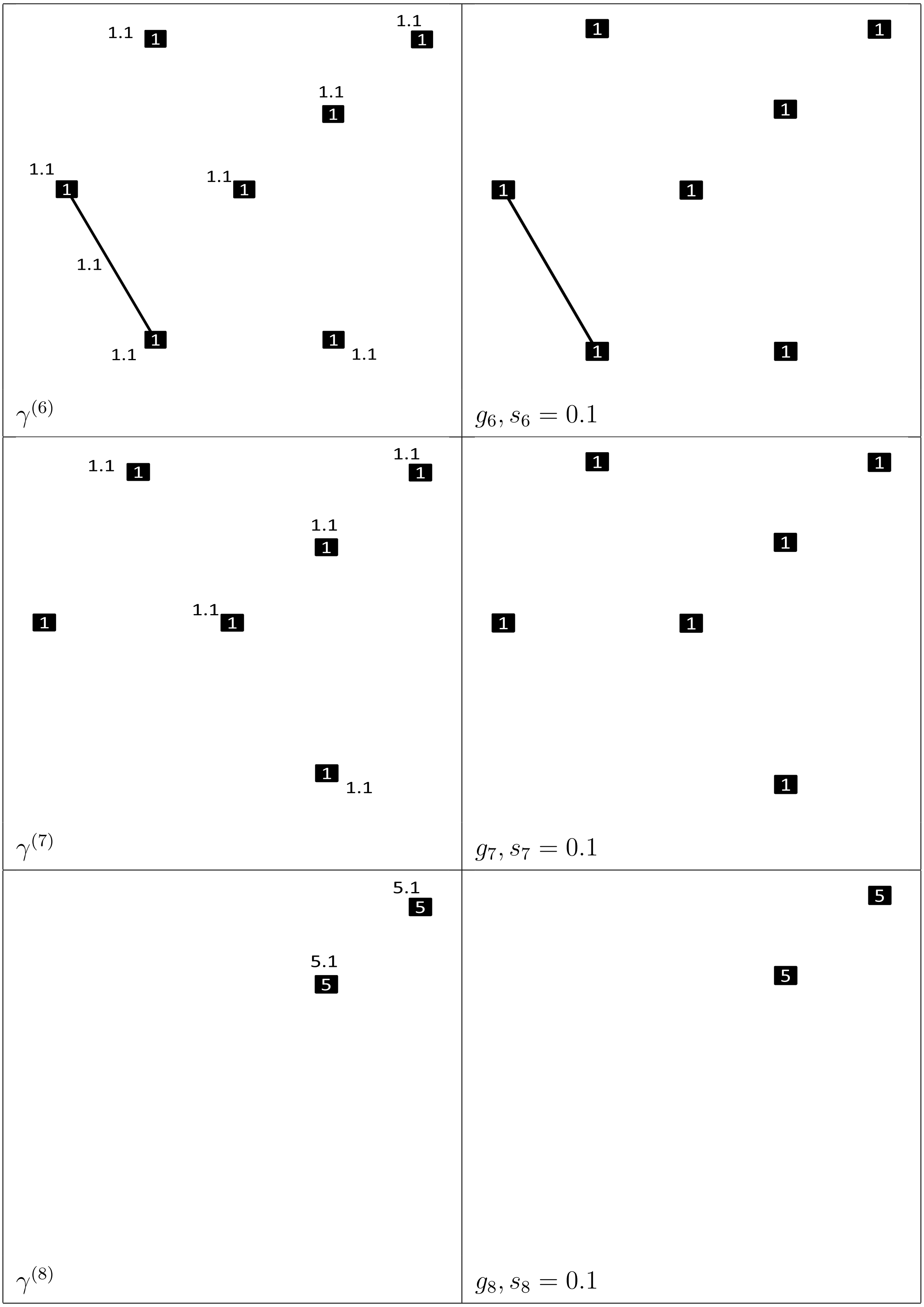

Fig. 7. (*continued*)

Table 1
Graph Banzhaf values of the graphs in the cg-partition of the EP fuzzy graph.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
1	0.6328130	0.632813	0.62500	0.59375	0.25	0	0	0
2	0.3671880	0.367188	0.37500	0.40625	0	0	0	0
3	0.3203130	0.320313	0.31250	0.28125	0.25	0	0	0
4	0.1171880	0.117188	0.09375	0.12500	0.25	0	0	0
5	0.0859375	0.0859375	0.09375	0.06250	0	0	0	0
6	0.0390625	0.0390625	0.03125	0.03125	0	0	0	0
7	0.0859375	0.0859375	0.06250	0.09375	0	0	0	0
8	0.0859375	0.0859375	0	0	0	0	0	0

Table 2
Banzhaf power indices of the groups.

	Groups	Members	Banzhaf	Banzhaf graph	cg-fuzzyBanzhaf
1	PPE	265	0.734375	0.632813	0.357813
2	S&D	183	0.265625	0.367188	0.192188
3	ADLE	84	0.234375	0.320313	0.201563
4	CRE	55	0.140625	0.117188	0.107812
5	Greens-ALE	55	0.140625	0.085938	0.039063
6	GUE/NGL	35	0.078125	0.039063	0.017188
7	EDF	29	0.078125	0.085938	0.042188
8	NI	29	0.078125	0.085938	0.017188

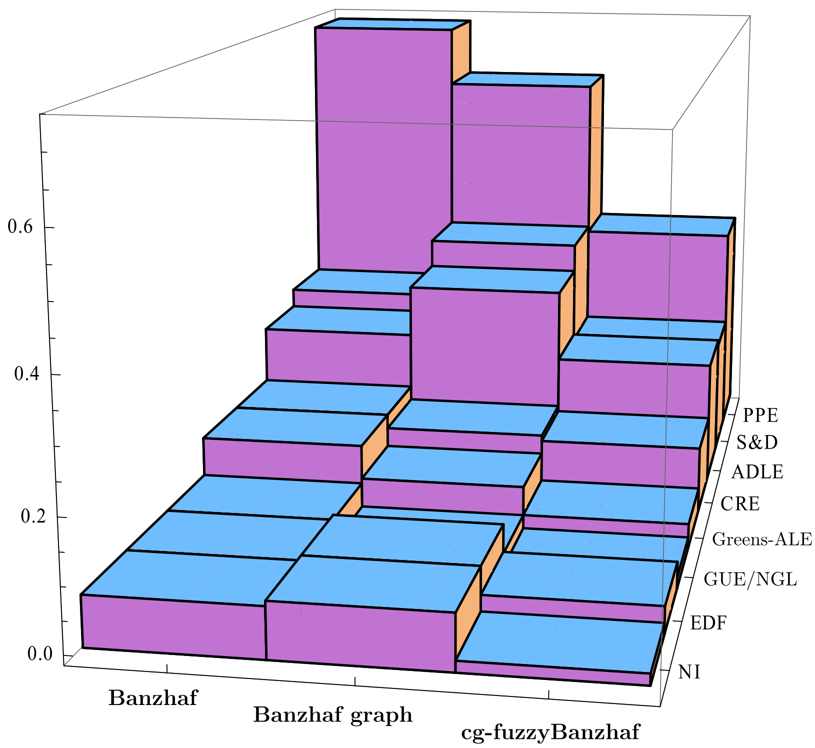

Fig. 8. Comparative graphic of power indices of the groups.

the EP fuzzy graph in the EP game. Fig. 7 develops the algorithm *cg-partition* for this fuzzy communication structure, $(g_k, s_k)_{k=1}^{k=8}$ with $s_k \in (0, 1]$ and $g_k \in CS^N$, of the EP fuzzy graph. In this case the number of steps coincides with the number of vertices.

We calculate now the cg-fuzzy Banzhaf value using the graph Banzhaf values of the communication structures in the cg-partition. So, Table 1 includes the graph Banzhaf value $\beta(N, v, g_k)$ for every $g_k \in CS^N$ of the cg-partition by levels $(g_k, s_k)_{k=1}^{k=8}$ of the EP fuzzy graph γ. Table 2 includes the Banzhaf value $\beta(N, v)$, the graph Banzhaf value $\xi(N, v, g)$ and the cg-fuzzy graph Banzhaf value $B(N, v, \gamma)$. Fig. 8 is a comparative graphic of these indices.

Conclusions

We have defined a version of the known Banzhaf value for games with fuzzy communication situations. Fuzzy graphs allow us to study situations in cooperative games where the communication among the agents should be shared. The cg-fuzzy graph Banzhaf value can be obtained using the graph Banzhaf values (Owen [17]) of a partition of the fuzzy graph. We have provided with an axiomatization of the new value and an algorithm to calculate it. We have shown an application of the value as a power index to determine the power of the groups in the European Parliament. We can see that the fuzziness of the cooperation among the groups implies a drop of the powers of all the groups.

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