Mathematical Social Sciences 36 (1998) 157–173

mathematical social sciences

The Banzhaf power index on convex geometries

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Received 21 February 1997; received in revised form 30 September 1997; accepted 30 March 1998

Abstract

In this paper, we introduce the Banzhaf power indices for simple games on convex geometries. We define the concept of swing for these structures, obtaining *convex swings*. The number of convex swings and the number of coalitions such that a player is an extreme point are the basic tools to define the *convex Banzhaf indices*, one *normalized* and other *probabilistic*. We obtain a family of axioms that give rise to the Banzhaf indices. In the last section, we present a method to calculate the convex Banzhaf indices with the computer program Mathematica, and we apply this to compute power indices in the Spanish and Catalan parliaments and in the Council of Ministers of the European Union.

Keywords: Voting; Political power; Banzhaf index; Convex geometry

1. Introduction

The analysis of power is central in political science. In general, it is difficult to define the idea of power, but for the special case of voting power there are mathematical power indices that have been used. The first such power index was proposed by Shapley and Shubik (1954). Another concept for measuring voting power was introduced by Banzhaf (1965), a lawyer, whose work has appeared mainly in law journals, and whose index has been used in arguments in various legal proceedings. In this paper, we introduce the Banzhaf power index for cooperative games in which only certain coalitions are allowed to form. We will study the structure of such allowable coalitions using the theory of *convex geometries*, a notion developed to combinatorially abstract geometric convexity. The Shapley–Shubik index and the Shapley value on these structures are studied by Bilbao and Edelman (1998), and Edelman (1998). We will define the Banzhaf index and generalize it to the Banzhaf value on convex geometries.

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0165-4896/98/$19.00
PII: S0165-4896(98)00021-3

Let N be a finite set of n elements and $\mathcal{L} \subseteq 2^N$. Edelman and Jamison (1985) introduced the convex geometry as the pair $(N, \mathcal{L})$ where the following axioms hold:

1. $\emptyset \in \mathcal{L}$, and $\mathcal{L}$ is closed under intersection.
2. If $S \in \mathcal{L}$ and $S \neq N$, then there exists $j \in N \backslash S$ such that $S \cup j \in \mathcal{L}$.

Axiom 1 implies that intersections of feasible coalitions should also be feasible, since the players agree on a profile of cooperation. In the model of *conference structures* by Myerson (1980), two players are connected if they can be coordinated by meeting in separate conferences which have some members in common to serve as intermediaries. In our model, the coalitions of intermediaries are in the cooperation structure.

A maximal chain of $\mathcal{L} \subseteq 2^N$ is an ordered collection of convex sets that is not contained in any larger chain. From Axiom 2 and by induction, Edelman and Jamison (1985) showed that every maximal chain contains $n+1$ convex sets

$$\emptyset = S_0 \subset S_1 \subset \cdots \subset S_{n-1} \subset S_n = N,$$

and the cardinal $|S_k| = k$, for all $k = 0, 1, \ldots, n$. Thus, the *hierarchical situations* (see Moulin and Shenker, 1996), when users pay their incremental costs according to a ordering of N, can be modeled by convex geometries.

The elements of $\mathcal{L}$ are called *convex coalitions*, and the elements of N are called *players*. An element i of $S \in \mathcal{L}$ is an *extreme point* of S if $S \backslash i \in \mathcal{L}$. The set of extreme points of the convex set S is denoted by $ex(S)$. If $(N, \mathcal{L})$ is a convex geometry then there exists a closure operator on 2^N, defined by

$$\bar{A} = \bigcap_{\{S \in \mathcal{L} : S \supseteq A\}} S.$$

Definition 1. A simple game on a convex geometry $\mathcal{L} \subseteq 2^N$ is a set function $v : \mathcal{L} \to \{0, 1\}$, such that $v(\emptyset) = 0$, and v is monotone ($v(S) \leq v(T)$, whenever $S \subseteq T$).

The collection of the simple games on $\mathcal{L}$ is denoted by $\Omega(\mathcal{L})$. If v satisfies

$$v(\overline{S \cup T}) \geq v(S) + v(T), \text{ whenever } S \cap T = \emptyset,$$

then we say that v is *superadditive*. This collection of games is denoted by $\Omega_{sa}(\mathcal{L})$. On the set $\Omega(\mathcal{L})$ we define the internal operations meet $\wedge$ and join $\vee$ by

$$(v \wedge w)(S) = \min\{v(S), w(S)\}, (v \vee w)(S) = \max\{v(S), w(S)\}.$$

In $\Omega_{sa}(\mathcal{L})$ the operation join is not internal. The games $\hat{1}$ and $\hat{0}$ are simple games such that $\hat{1}(S) = 1$, and $\hat{0}(S) = 0$, for every nonempty $S \in \mathcal{L}$.

Example. We consider the set of players $N = \{1, 2, \ldots, 2k+1\}$ and let $\mathcal{L}_{2k+1}$ be the convex geometry whose convex sets are the empty set and the intervals $[i, j] = \{i, i+1, \ldots, j-1, j\}$. The game v, defined by

$$v(S) = 1, \text{ if } |S| \geq k+1.$$

$$v(S) = 0, \text{ if } |S| \leq k,$$

for all $S \in \mathscr{L}_{2k+1}$, is the *majority simple game* on the convex coalitions for the policy order (see Edelman, 1998).

Example. A graph $G=(N, E)$ is connected if any two vertices can be joined by a path. A maximal connected subgraph of G is a *component* of G. A *cutvertex* is one whose removal increases the number of components, and a *bridge* is an edge with the same property. A graph is *2-connected* if it is connected, has at least 3 vertices and contains no cutvertex. A subgraph B of a graph G is a *block* of G if either B is a bridge or else it is a maximal 2-connected subgraph of G. A graph G is a *block graph* if every block is a complete graph. The block graphs are denoted by *cycle-complete* graphs in van den Nouweland and Borm (1991). Let $G=(N, E)$ be a connected block graph and let us consider the collection

$$\mathscr{L} = \{S \subseteq N\colon (S, E(S)) \text{ is a connected subgraph of } G\}.$$

Edelman and Jamison (1985) showed that $\mathscr{L}$ is a convex geometry.

2. Fundamental concepts

Convex coalitions $S \in \mathscr{L}$ with $v(S)=1$ are called *winning* and convex coalitions with $v(S)=0$ *losing*. In the classical theory (see Dubey and Shapley, 1979) a *swing* for player i is a pair of coalitions $(S, S\backslash i)$ such that S is winning and $S\backslash i$ is not. This concept allows us to define convex swings for games on convex geometries.

Definition 2. For a player i and a game v, we say that the pair $(S, S\backslash i)$ is a convex swing if $S \in \mathscr{L}$, $i \in ex(S)$, $v(S)=1$, $v(S\backslash i)=0$. The number of convex swings of player i is denoted by $cs_i(v)$, and the total number of convex swings for game v is

$$cs(v) = \sum_{i \in N} cs_i(v).$$

The editor's referee proposes the following variation: ''*giving player i a swing whenever S is winning and allowable, and S|i is losing or not allowable*''.

This idea is closely related with the additive extension of a game on a partition convex geometry $v\colon\mathscr{L} \to \{0, 1\}$ to the simple game $v^{\mathscr{L}}\colon 2^N \to \{0, 1\}$, defined by

$$v^{\mathscr{L}}(S) := \max\{v(T)\colon T \subseteq S, T \in \mathscr{L}\}.$$

In this model, player $i \in S$ is a $v^{\mathscr{L}}$-swing whenever S contains a winning and feasible coalition and $v(T)=0$, for all $T \subseteq S\backslash i$, $T \in \mathscr{L}$. Thus, we can define the Banzhaf value by $\beta(\mathscr{L}, v) := \beta(v^{\mathscr{L}})$, where β is the classical Banzhaf index (see Bilbao, 1998).

The issue under consideration can be summed up thus: *There are two different models for situations in which not all coalitions are allowable*. In the present paper, we study an *extremal model* in which the power of a player depends on his extremal position in the feasible and winning coalitions. Another model, the *restricted model*, considers that the power of a player is related with the number of feasible winning coalitions such that the player is a cutvertex in a communication graph.

In the extremal model, it is obvious that the convex coalitions where a player is extreme are the only coalitions to take part in the game. The following numbers will usually appear,

$$\mathscr{E}_i = |\{S \in \mathscr{L}: i \in ex(S)\}|, \text{ for all } i \in N,$$

$$\mathscr{E}_i(T) = |\{S \in \mathscr{L}: i \in ex(S), S \supseteq T\}|, \text{ for all } i \in N.$$

The number $\mathscr{E}_i$ will be called *extremal power* of the player i and $\mathscr{E}_i(T)$ is the *extremal power of i on T*. The quotient $\mathscr{E}_i(T)/\mathscr{E}_i$ is the *relative extremal power of i on T*.

We consider the relationship between the convex swings of a player and the set of all convex coalitions such that the player i is an extreme point.

Definition 3. For every player $i \in N$, the number

$$cs_i'(v) = \frac{cs_i(v)}{\mathscr{E}_i},$$

is called convex swing probability of i for the game v. The sum of all these numbers is denoted $cs'(v)$.

The pair $(S, S \setminus i)$ is a convex swing if and only if $[v(S) - v(S \setminus i)] = 1$. Therefore, the following proposition holds.

Proposition 1. *Let* $v \in \Omega(\mathscr{L})$ *and* $i \in N$. *Then,*

$$cs_i(v) = \sum_{\{S \in \mathscr{L}: i \in ex(S)\}} [v(S) - v(S \setminus i)].$$

Definition 4. A player i with $cs_i(v) = 0$ is called a dummy in v.

Notice that a dummy player can never help a convex coalition to win.

For any $S \in \mathscr{L}$, $S \neq \emptyset$ the *unanimity game* on S is the superadditive simple game defined by

$$\zeta_S(T) = 1, \text{ if } S \subseteq T$$

$$\zeta_S(T) = 0, \text{ otherwise.}$$

Proposition 2. *Let S be a convex coalition and* $i \notin ex(S)$, *then i is a dummy in* ζ_S.

Proof. If $i \notin ex(S)$, then there are two possibilities for i. If $i \notin S$ then $\zeta_S(T) = \zeta_S(T \setminus i)$, thus $cs_i(\zeta_S) = 0$. If $i \in S \setminus ex(S)$ and $S \subseteq T$, then we have that $i \notin ex(T)$. Otherwise $T \setminus i$ and S are convex coalitions and $(T \setminus i) \cap S = S \setminus i$ is a convex coalition, hence $i \in ex(S)$, a contradiction. □

Proposition 3. *Let $S \in \mathscr{L}$, then the number of convex swings for ζ_S is*

$$cs_i(\zeta_S) = 0, \text{ if } i \notin ex(S)$$

$$cs_i(\zeta_S) = \mathscr{E}_i(S), \text{ if } i \in ex(S).$$

Proof. By Proposition 1 we have

$$cs_i(\zeta_S) = \sum_{\{T \in \mathscr{L} : i \in ex(T)\}} [\zeta_S(T) - \zeta_S(T \setminus i)].$$

If $i \notin ex(S)$, then Proposition 2 implies the result. If $i \in ex(S)$, note that $\zeta_S(T) - \zeta_S(T \setminus i) = 1$ if and only if $T \in \{S \in \mathscr{L}: i \in ex(S)\}$ and $T \supseteq S$. □

Let us define two different types of indices: the normalized and probabilistic convex Banzhaf indices.

Definition 5. The vector whose components are the numbers of convex swings $cs_i(v)$ is called the convex Banzhaf index. When this number is normalized then we obtain the normalized convex Banzhaf index $c\beta$: $\Omega(\mathscr{L}) \to \mathbb{R}^n$,

$$c\beta(v) = (c\beta_1(v), \ldots, c\beta_n(v)), \text{ where } c\beta_i(v) = \frac{cs_i(v)}{cs(v)}.$$

The interest of these numbers lies in their ratios and it has been useful to normalize them with the total number of swings (see Dubey and Shapley, 1979). Another possibility is to consider the measure given by the expected number of convex swings $cs'_i(v) = cs_i(v)/\mathscr{E}_i$, for all $i \in N$.

Definition 6. The probabilistic convex Banzhaf index is $c\beta' : \Omega(\mathscr{L}) \to \mathbb{R}^n$,

$$c\beta'(v) = (c\beta'_1(v), \ldots, c\beta'_n(v)), \text{ where } c\beta'_i(v) = cs'_i(v).$$

If we apply Proposition 1 then these indices satisfy

$$c\beta_i(v) = \sum_{\{S \in \mathscr{L} : i \in ex(S)\}} \frac{1}{cs(v)} [v(S) - v(S \setminus i)],$$

$$c\beta'_i(v) = \sum_{\{S \in \mathscr{L} : i \in ex(S)\}} \frac{1}{\mathscr{E}_i} [v(S) - v(S \setminus i)],$$

and we can show that $c\beta'_i(v)$ is a probabilistic index because

$$\sum_{\{S \in \mathscr{L} : i \in ex(S)\}} \frac{1}{\mathscr{E}_i} = 1.$$

The relation of these indices with the internal operations of $\Omega(\mathscr{L})$ is the transfer property.

Proposition 4. *The convex Banzhaf index satisfies the transfer property, i.e.*

$$cs_i(v \vee w) + cs_i(v \wedge w) = cs_i(v) + cs_i(w), \text{ for all } v, w \in \Omega(L).$$

Proof. In the following table we observe that $(v \vee w) + (v \wedge w) = v + w$.

v	w	$v \vee w$	$v \wedge w$	$v + w$	$(v \vee w) + (v \wedge w)$
1	1	1	1	2	2
1	0	1	0	1	1
0	1	1	0	1	1
0	0	0	0	0	0

Since Proposition 1 implies that cs_i is an additive operator, the proof is complete. □

Remark 1. *The normalized index does not satisfy this property because its denominator depends on the game. However the probabilistic index offers no problems. This proposition is true in $\Omega_{sa}(\mathcal{L})$ when the operation join is an internal operation.*

3. Axioms for the convex Banzhaf indices

Let φ: $\Omega(\mathcal{L}) \to \mathbb{R}^n$ be a value with $\varphi(v) = (\varphi_i(v))_{i=1,...,n}$. We use four axioms to obtain a characterization of the convex Banzhaf indices. If a player $i \in N$ is a dummy for v then $c\beta_i(v) = c\beta'_i(v) = cs_i(v) = 0$. Thus, the first axiom is:

Axiom 1. *Dummy*: If $i \in N$ is a dummy in v then $\varphi_i(v) = 0$.

Axiom 2a. *Total swings*: The sum of all the components of $\varphi(v)$ is equal to the total number of the convex swings,

$$\sum_{i \in N} \varphi_i(v) = cs(v).$$

Axiom 2b. *Total swing probabilities*: The sum of all the components of $\varphi(v)$ is equal to the total number of the convex swing probabilities,

$$\sum_{i \in N} \varphi_i(v) = cs'(v).$$

We will now consider the value of unanimity games in the *extremal* axioms.

Axiom 3a. *Extremal power*: Let ζ_S be the unanimity game on $S \in \mathcal{L}$. Then, for all i, $j \in S$,

$$\mathcal{E}_i(S)\varphi_j(\zeta_S) = \mathcal{E}_j(S)\varphi_i(\zeta_S).$$

Axiom 3b. *Relative extremal power*: Let ζ_S be the unanimity game on $S \in \mathcal{L}$. Then, for all $i,\ j \in S$,

$$\frac{\mathcal{E}_i(S)}{\mathcal{E}_i}\,\varphi_j(\zeta_S) = \frac{\mathcal{E}_j(S)}{\mathcal{E}_j}\,\varphi_i(\zeta_S).$$

The last axiom is based on the transfer property.

Axiom 4. *Transfer*: If $v,\ w \in \Omega(\mathcal{L})$ then

$$\varphi(v) + \varphi(w) = \varphi(v \vee w) + \varphi(v \wedge w).$$

We will now prove that the normalized and probabilistic convex Banzhaf indices can be characterized with these axioms. In the next theorem, the normalized index is not obtained directly by Axiom 4 (see Remark 1).

Theorem 1. *There exists a unique function φ that satisfies the dummy, total swings, extremal power and transfer axioms. Moreover,*

$$c\beta(v) = \frac{1}{cs(v)}\,\varphi(v), \text{ for all } v \in \Omega(\mathcal{L}).$$

Proof. Let S be a convex coalition, if $j \notin ex(S)$ then j is a dummy in ζ_S and Axiom 1 implies

$$\sum_{i \in N} \varphi_i(\zeta_S) = \sum_{j \in ex(S)} \varphi_j(\zeta_S).$$

If $i \in ex(S)$ then $S \in \{T \in \mathcal{L}\colon i \in ex(T)\}$ and $S \supseteq S$, hence $\mathcal{E}_i(S) \neq 0$ and by Axiom 3a we obtain that

$$\varphi_j(\zeta_S) = \frac{\mathcal{E}_j(S)}{\mathcal{E}_i(S)}\,\varphi_i(\zeta_S).$$

Therefore, φ is uniquely determined over the unanimity game since Axiom 2a leads us to

$$cs(v) = \sum_{j \in ex(S)} \varphi_j(\zeta_S) = \sum_{j \in ex(S)} \frac{\mathcal{E}_j(S)}{\mathcal{E}_i(S)}\,\varphi_i(\zeta_S) = \frac{\varphi_i(\zeta_S)}{\mathcal{E}_i(S)} \sum_{j \in ex(S)} \mathcal{E}_j(S).$$

Using Proposition 3 we show that

$$\sum_{j \in ex(S)} \mathcal{E}_j(S) = cs(\zeta_S) \text{ and } \mathcal{E}_i(S) = cs_i(\zeta_S).$$

So, we obtain $\varphi_i(\zeta_S) = cs_i(\zeta_S)$.

Finally, we can extend φ_i for all $\Omega(\mathcal{L})$ applying Axiom 4, because the calculation of $\varphi_i(v)$ can be reduced to $\varphi_i(\zeta_S)$. Let $S_1, \ldots, S_p$ be the minimal winning convex coalitions for $v \in \Omega(\mathscr{L})$, $v \neq \hat{0}$. By using the monotonicity of v and the decomposition showed by Dubey and Shapley (1979), we obtain

$$v = \zeta_{S_1} \vee \zeta_{S_2} \vee \ldots \vee \zeta_{S_p},$$

and this is the unique decomposition of v given by the operation join in $\Omega(\mathscr{L})$. By the transfer axiom,

$$\varphi(v) = \varphi(\zeta_{S_1}) + \varphi(\zeta_{S_2} \vee \cdots \vee \zeta_{S_p}) - \varphi(\zeta_{S_1} \wedge (\zeta_{S_2} \vee \cdots \vee \zeta_{S_p})).$$

Each game that appears in the second member is a game with fewer minimal winning convex coalitions than v. So, we can perform an induction on the number of minimal winning convex coalitions. If $v = \hat{0}$, then all the players are dummy and therefore $\varphi_i(\hat{0}) = 0$. We have proved the uniqueness and the existence is obtained because the vector of components $cs_i(v)$ satisfies the axioms. □

We obtain the probabilistic convex Banzhaf index with a similar proof.

Theorem 2. *There exists a unique function φ that satisfies the dummy, total swing probabilities, relative extremal power and transfer axioms. Moreover, $c\beta'(v) = \varphi(v)$, for all v in $\Omega(\mathscr{L})$.*

Remark 2. *We can restrict $\Omega_{sa}(\mathscr{L})$ if the transfer property is applicable to those games of $\Omega_{sa}(\mathscr{L})$ such that the operation join applied to them belongs to $\Omega_{sa}(\mathscr{L})$.*

Proposition 5. *Let v be the majority simple game on the convex geometry $\mathscr{L}_{2k+1}$. Then,*

1. *The normalized convex Banzhaf indices are*

$$c\beta_i(v) = \frac{1}{2k+2}, \text{ if } i \neq k+1$$
$$c\beta_i(v) = \frac{1}{k+1}, \text{ if } i = k+1.$$

2. *The probabilistic convex Banzhaf indices are*

$$c\beta'_i(v) = \frac{1}{2k+1}, \text{ if } i \neq k+1$$
$$c\beta'_i(v) = \frac{2}{2k+1}, \text{ if } i = k+1.$$

Proof. (1) This formula is showed by Edelman (1998). (2) First, we obtain the extremal power $\mathscr{E}_i$, of each player $i \in N$. For all $j \geq i$ the interval $[i, j]$ is a coalition where i is an extreme point. So, there are $2k+1-(i-1) = 2k+2-i$ convex coalitions. For all $j < i$, the interval $[j, i]$ is a coalition with i as an extreme point, and therefore there are $i-1$ more convex coalitions. Then, the number $\mathscr{E}_i$, for every player i, is $\mathscr{E}_i = 2k+1$. The number of swings is

$$cs_i(v) = 1, \text{ if } i \neq k+1$$

$cs_i(v) = 2$, if $i = k + 1$,

and the proof is complete. □

We can try to generalize the probabilistic convex Banzhaf power index to the convex Banzhaf value. A game on the convex geometry $\mathcal{L}$ is defined by a real characteristic function $v:\mathcal{L}\to\mathbb{R}$, with $v(\emptyset)=0$.

Definition 7. The convex Banzhaf value of the game $v:\mathcal{L}\to\mathbb{R}$ is the vector whose components are

$$c\beta'_i(v) = \sum_{\{S\in\mathcal{L}:i\in ex(S)\}} \frac{1}{\mathscr{E}_i}[v(S) - v(S\setminus i)].$$

The dummy, total swing probabilities, extremal power axioms and, in this case, also the linearity axiom, hold for this value.

Theorem 3. *There exists a unique value on the set of games on $\mathcal{L}$ that satisfies the linearity, dummy, relative extremal power and total swing probabilities axioms. This value is the convex Banzhaf value.*

Proposition 6. *Let v be a game on $\mathcal{L}_n$. The components of the convex Banzhaf value are*

$$c\beta'_i(v) = \frac{1}{n}\left\{\sum_{j=1}^{i-1}[v([j,i]) - v([j,i-1])] + \sum_{j=i+1}^{n}[v([i,j]) - v([i+1,j])]\right\}.$$

4. The Banzhaf indices computed with Mathematica

The package *DiscreteMath'Combinatorica'* extends the program Mathematica to combinatorics and graph theory. The best guide of this package is the book by Skiena (1990). The package *Cooperative.m* included in Carter (1993) presents tools for solving cooperative games. We define the following functions for calculating the normalized Banzhaf indices for games on 2^N.

```
ba[S_List]:=1;

Banzhaf[game_, i_]:=Plus @@ (ba[#] & /@

Select [Coalitions, (v[#]==1 && v[DeleteCases [#, i] ]==0) & ])

Attributes [Banzhaf]={Listable}; Banzhaf[game_]:=Banzhaf[game, T]

BanzhafIndice[game_]:=(Banzhaf[game])/(Plus @@ Banzhaf[game])
```

We present the functions ConvexBanzhafIndex[] and ConvexBanzhafProb[] for games on a convex geometry F. The family of convex coalitions $\mathscr{L}$ is denoted by F in the Input [3].

In[1]: =

≪ DiscreteMath′Combinatorica′

In [2]: =

≪ Cooperat′Cooperat′

In[3]: =

F = {{ }, {1}, {2}, {3}, {4}, {5}, {1, 2}, {2, 3}, {3, 4}, {4, 5}, {1, 2, 3}, {2, 3, 4}, {3, 4, 5}, {1, 2, 3, 4}, {2, 3, 4, 5}, {1, 2, 3, 4, 5}};

In[4]: =

lattice [F_List]: = MakeGraph[F,

((Intersection [#2, #1] = = #1)&&(#1! = #2))&];

In[5]: =

ShowLabeledGraph [HasseDiagram [lattice [F]], F];

The convex geometry $\mathscr{L}_5$ is shown in Fig. 1.

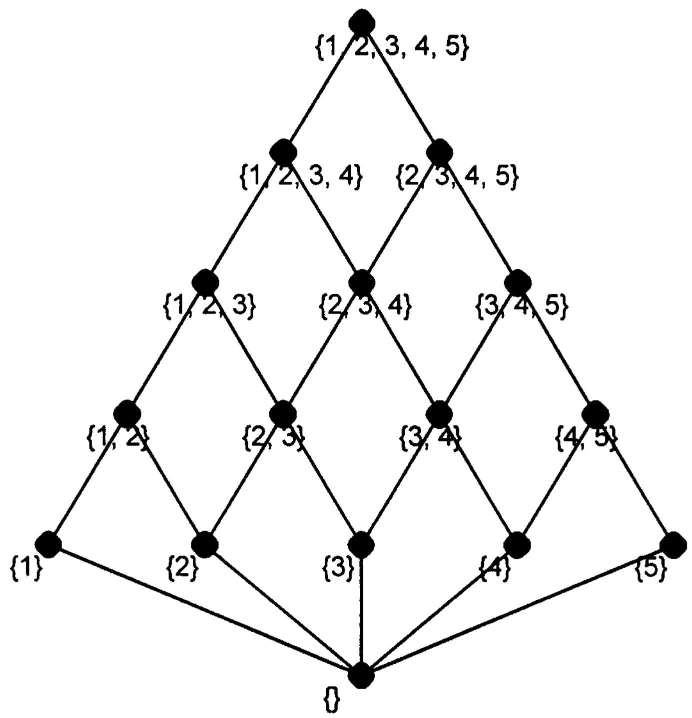

Fig. 1. The convex geometry $\mathscr{L}_5$.

```
In[6]:=
    Extreme [S_List, F_List]: = Sort [Select [S,
    (MemberQ [F, Complement [S, {#}]])&]]
In[7]:=
    Extreme [{1, 2, 3}, F]
Out[7]:=
    {1, 3}
In[8]:=
    Convex [i_, F_List]: = Sort [Select [F,
    (Intersection [{i}, Extreme [#, F]] = = ={i})&]]
In[9]:=
    Convex [1, F]
Out[9]:=
    {{1}, {1, 2}, {1, 2, 3}, {1, 2, 3, 4}, {1, 2, 3, 4, 5}}
In[10]:=
    ConvexBanzhaf [game_, i_]: = Length [
    Select [Convex [i, F], (v[#] = = 1&&
    v[DeleteCases [#, i] = = 0)&]]
In[11]:=
    Attributes [ConvexBanzhaf] ={Listable};
In[12]:=
    ConvexBanzhaf [game_]: = ConvexBanzhaf [game, T]
In[13]:=
    ConvexBanzhafIndex [game_]: =
    ConvexBanzhaf [game_]/(Plus@@ConvexBanzhaf [game])
In[14]:=
    ConvexBanzhafProb [game_, i_]: = Length [
    Select [Convex [i, F], (v[#] = = 1&&
```

v[DeleteCases [#, i]] = = 0)&]]/Length [Convex [i, F]

In[15]:=

Attributes [ConvexBanzhafProb] ={Listable};

ConvexBanzhafProb [game_]: = ConvexBanzhafProb [game, T]

We will analyze two voting games, the first is the simple majority rule in the Spanish parliament (176 out of 350 votes), the second is the simple majority rule in the Catalan parliament (68 out of 135 votes). In both cases, the convex geometry is $\mathscr{L}_n$, that is, the convex coalitions comprise parties that are adjacent to one another in the left–right ordering. First, we obtain the convex Banzhaf indices for the 11 political parties with seats in the present Spanish parliament (1996–). The voting game is defined by

[176; 156(PP), 141(PSOE), 21(IU), 16(CiU), 5(PNV), 4(CC), 2(BNG), 2(HB), 1(-ERC), 1(EA), 1(UV)],

and the ideological positions in the left–right ordering are

HB—BNG—ERC—EA − IU—PSOE—PNV—CiU—PP—CC—UV

In[16]:=

Game1: = (T = Range[11]; Clear [x, y, v]; x[1]: = 156;

x[2]: = 141; x[3]: = 21; x[4]: = 16; x[5]: = 5; x[6]: = 4;

x[7]: = 2; x[8]: = 2; x[9]: = 1; x[10]: = 1; x[11]: = 1;

y[S_List]: = Plus@@x/@S;

v[{ }]: = 0; v[S_/; y[S] ≥ 17]: = 1;

v[S_/; y[S] < 176]: = 0);

In[17]:=

Banzhaf [Game1]

Out[17] =

{743, 281, 281, 231, 25, 25, 15, 15, 7, 7, 7}

In[18]:=

BanzhafIndice [Game1]

Out[18]:=

$$\left\{\frac{743}{1637}, \frac{281}{1637}, \frac{281}{1637}, \frac{231}{1637}, \frac{25}{1637}, \frac{25}{1637}, \frac{15}{1637}, \frac{15}{1637}, \frac{7}{1637}, \frac{7}{1637}, \frac{7}{1637}\right\}$$

In[19]: =

ConvexBanzhaf [Game1]

Out[19] =

{2, 0, 1, 7, 1, 1, 0, 0, 0, 0, 0}

In[20]: =

ConvexBanzhafIndex [Game1]

Out[20] =

$$\left\{\frac{1}{6}, 0, \frac{1}{12}, \frac{7}{12}, \frac{1}{12}, \frac{1}{12}, 0, 0, 0, 0, 0\right\}$$

In[21]: =

ConvexBanzhafProb [Game1]

Out[21] =

$$\left\{\frac{2}{11}, 0, \frac{1}{11}, \frac{7}{11}, \frac{1}{11}, \frac{1}{11}, 0, 0, 0, 0, 0\right\}$$

The voting game in the Catalan parliament is defined by

[68; 60(CiU), 34(PSC), 17(PP), 13(ERC), 11(IC − LV)],

and the ideological positions in the left–right ordering are

ERC—IC/LV—PSC—CiU—PP

In[22]: =

CatalanGame: = (T = Range[5];

Clear [x, y, v]; x[1]: = 60;

x[2]: = 34; x[3]: = 17; x[4]: = 13; x[5]: = 11;

y[S_List]: = Plus@@x/@S;

v[{ }]: = 0; v[S_/; y[S] ≥ 68]: = 1;

v[S_/; y[S] < 68]: = 0);

In[23]: =

BanzhafIndice [CatalanGame]

Out[23] =

$$\left\{\frac{7}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}\right\}$$

In[24]: =

ConvexBanzhafIndex [CatalanGame]

Out[24] =

$$\left\{\frac{2}{3}, \frac{1}{6}, \frac{1}{6}, 0, 0\right\}$$

In[25]: =

ConvexBanzhafProb [CatalanGame]

Out[25] =

$$\left\{\frac{4}{5}, \frac{1}{5}, \frac{1}{5}, 0, 0\right\}.$$

5. The Banzhaf power in the European Union

We will now study the voting power in the Council of Ministers of the European Union. Herne and Nurmi (1993); Widgrén (1994) and Lane and Mæland (1995) have studied the power indices (Shapley–Shubik and Banzhaf) in the EU Council. In our model, we also consider the convex geometry of the connected subgraphs of a *communication situation* defined by the graph in Fig. 2. There are two blocks containing the following countries:

{GE, BE, NE, LU, AU, SW, FI, DE} and {FR, IT, SP, GR, PO, IR},

and a bridge between them with {IR, UK, DE}.

Table 1 and Figs. 3 and 4 contain the population, votes, normalized and convex

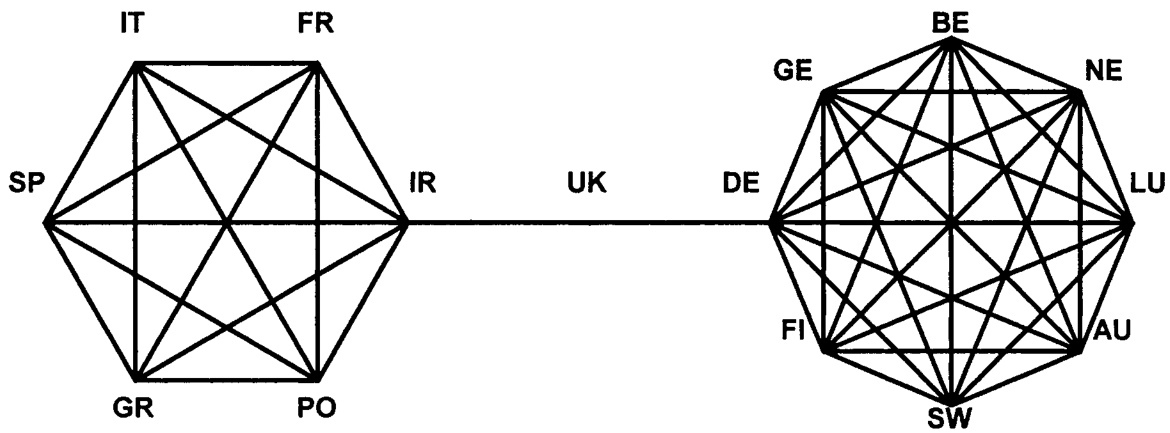

Fig. 2. Graph of the relations in the EU.

Table 1
Population, votes and Banzhaf power in the EU

Country	Population	Votes	Ba 62	Ba 65	CoBa 62	CoBa 65
Germany	0.219	0.115	0.112	0.110	0.140	0.140
UK	0.157	0.115	0.112	0.110	0	0
France	0.156	0.115	0.112	0.110	0.140	0.140
Italy	0.155	0.115	0.112	0.110	0.140	0.140
Spain	0.106	0.092	0.092	0.093	0.113	0.112
Netherlands	0.041	0.058	0.059	0.060	0.069	0.071
Greece	0.028	0.058	0.059	0.060	0.069	0.071
Belgium	0.027	0.058	0.059	0.060	0.069	0.071
Portugal	0.027	0.058	0.059	0.060	0.069	0.071
Sweden	0.024	0.046	0.048	0.045	0.060	0.057
Austria	0.022	0.046	0.048	0.045	0.060	0.057
Denmark	0.014	0.035	0.036	0.037	0	0
Finland	0.014	0.035	0.036	0.037	0.043	0.043
Ireland	0.010	0.035	0.036	0.037	0	0
Luxembourg	0.001	0.023	0.023	0.025	0.025	0.026

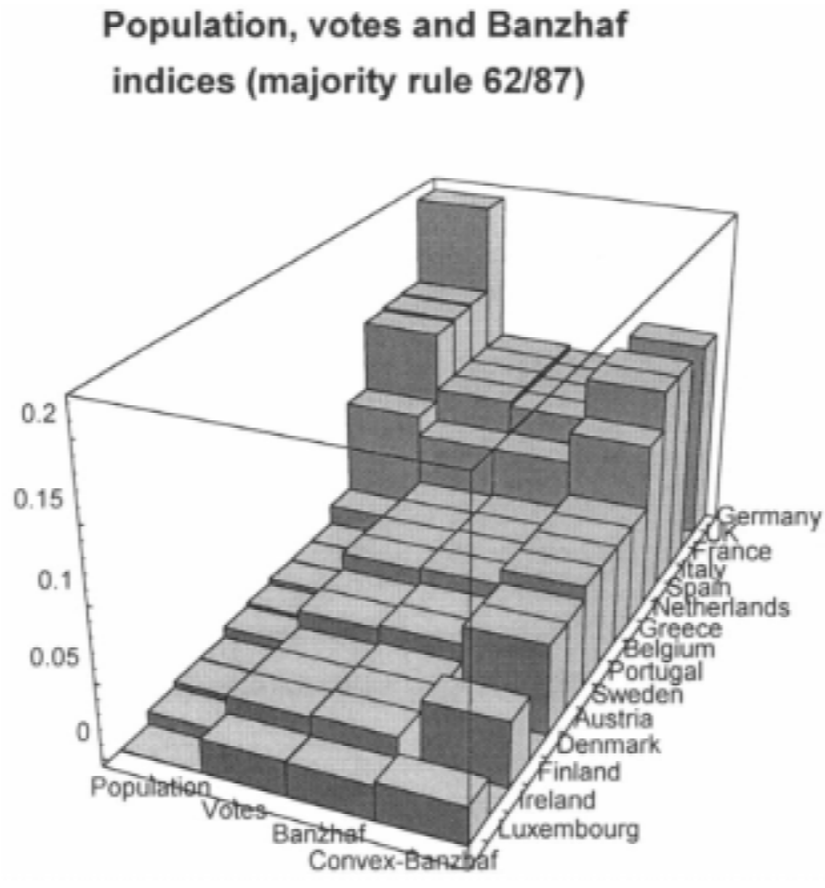

Fig. 3. Power with rule 62/87.

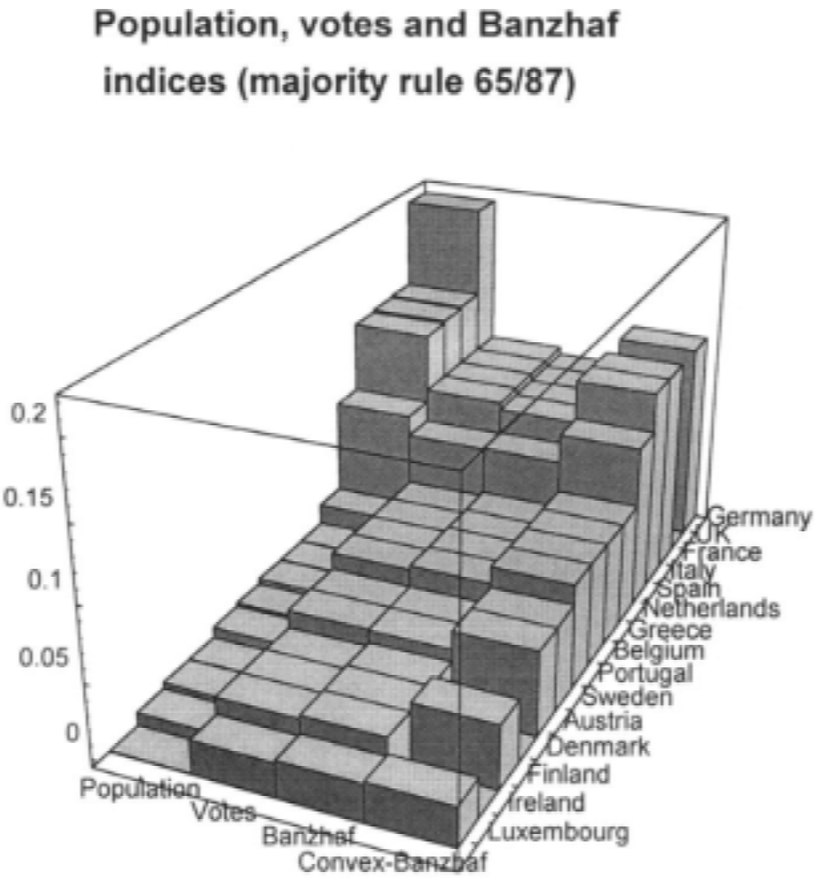

Fig. 4. Power with rule 65/87.

Banzhaf power indices. The majority voting rule is assumed to be 62 and 65 out of 87 votes in the EU Council.

Acknowledgements

The authors are very grateful to an anonymous referee for his/her helpful suggestions and valuable comments on the original version of the present paper.

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