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# Cooperative games on antimatroids

E. Algaba[a], J.M. Bilbao[a], R. van den Brink[b], A. Jiménez-Losada[a]

[a] *Escuela Superior de Ingenieros, Camino de los Descubrimientos s/n, Sevilla 41092, Spain*
[b] *Department of Econometrics, Free University, De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands*

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**Abstract**

The aim of this paper is to introduce cooperative games with a feasible coalition system which is called *antimatroid*. These combinatorial structures generalize the *permission structures*, which have nice economical applications. With this goal, we first characterize the approaches from a permission structure with special classes of antimatroids. Next, we use the concept of *interior operator* in an antimatroid and we define the restricted game taking into account the limited possibilities of cooperation determined by the antimatroid. These games extend the restricted games obtained by permission structures. Finally, we provide a computational method to obtain the *Shapley* and *Banzhaf values* of the players in the restricted game, by using the worths of the original game.


*MSC:* 91A12

*Keywords:* Permission structures; Antimatroids; Shapley and Banzhaf values

## 1. Introduction

A *TU-game* or *transferable utility game* describes a situation in which a finite set of players $N$ can generate certain payoffs by cooperation. In a TU-game the players are assumed to be socially identical in the sense that every player can cooperate with every other player. However, in practice there exist social asymmetries among the players. For this reason, the game theoretic analysis of decision processes in which one imposes asymmetric constraints on the behavior of the players has been and continue being an important subject to make a study. Some models, in which have been analyzed social asymmetries among players in a TU-game, are described in e.g. [18,20,3]. In these models the possibilities of coalition formation are determined by the positions of the players in a *communication graph*.

Another type of asymmetry among the players in a TU-game is introduced in Gilles, Owen and van den Brink [13,14,5] In these models, the possibilities of coalition formation are determined by the positions of the players in the so-called *permission structure*. Other related models can be found in Faigle, Kern, Derks, Gilles and Peters [6,7,12].

In the present paper, we use the restricted cooperation model derived from an *antimatroid*. Section 2 introduces permission structures and antimatroids and we show that given a permission structure, the approaches from it are an antimatroid but not every antimatroid is an approach from a permission structure. Moreover, we identify the approaches from an acyclic permission structure with antimatroids satisfying specific properties. This study gives rise to a new class of antimatroids obtained through permission structures. Two new concepts in these structures are essential, on one hand the *path property* and on the other hand the *feasible hull* which are both based on the path characterization of antimatroids. Section 3 introduces the restricted games on antimatroids which generalize the ones studied on permission structures. Using the structural properties from the antimatroid we will be able to express the dividends in terms of the original game. This result will be essential in the last section to provide some formulas to compute the Shapley and Banzhaf

*E-mail addresses:* mbilbao@us.es (J.M. Bilbao), jrbrink@feweb.vu.nl (R. van den Brink).

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values for restricted games on antimatroids. In these formulas these values are computed by means of the original game without having to calculate the restricted game and taking into account only the coalitions in the antimatroid.

## 2. Permission structures and antimatroids

Permission structures were defined by Gilles et al. [14]. They assume that players who participate in a TU-game are part of a hierarchical organization in which there are players that need permission from certain other players before they are allowed to cooperate.

For a finite set of players $N=\{1,2,\dots,n\}$ such a hierarchical organization is given by a mapping $S:N\to 2^N$ which is called a *permission structure* on $N$. The players in $S(i)$ are called the *successors* of player $i$ in the permission structure $S$ and the players in $S^{-1}(i)=\{j\in N: i\in S(j)\}$ are named the *predecessors* of $i$ in $S$. A *chain of players* is an ordered list $(h_1,\dots,h_t)$ where $h_{k+1}\in S(h_k)$, for all $k=1,\dots,t-1$. The *transitive closure* of a player $i$ in $S$ is the set

$$\hat{S}(i)=\{j: \text{ there exists a chain } (h_1,\dots,h_t) \text{ with } h_1=i,\ h_t=j\},$$

whose players are called the *subordinates* of player $i$ in $S$. The players in the set $\hat{S}^{-1}(i)=\{j\in N: i\in\hat{S}(j)\}$ are named the *superiors* of $i$ in $S$. A permission structure $S$ on $N$ is *acyclic* if $i\notin\hat{S}(i)$, for all $i\in N$.

In [14] the conjunctive approach to games with a permission structure is defined. In this approach it is assumed that each player needs permission from all his predecessors before it is allowed to cooperate with other players. Alternatively, in the disjunctive approach as discussed in [4,13] it is assumed that each player needs permission from at least one of his predecessors before he is allowed to cooperate with others players. Thus, the feasible coalitions in the conjunctive and disjunctive approaches, respectively, are given by the sets

$$\Phi_S^{c}=\{E\subseteq N: \text{ for every } i\in E \text{ it holds that } S^{-1}(i)\subseteq E\},$$

$$\Phi_S^{d}=\{E\subseteq N: \text{ for every } i\in E, S^{-1}(i)\neq\emptyset \text{ it holds that } S^{-1}(i)\cap E\neq\emptyset\}.$$

We will show that the feasible coalitions in the conjunctive and disjunctive approaches of permission structures are identified with special classes of set systems called antimatroids. *Antimatroids* were introduced by Dilworth [8] as particular examples of semimodular lattices. Several authors have obtained the same concept by abstracting various combinatorial situations. A systematic study of these structures was started by Edelman and Jamison [11] emphasizing the combinatorial abstraction of convexity. The latter was then shown by Edelman [10] to be a crucial property of closures induced by what he called *convex geometries*, a dual concept of antimatroids (see [2]). Jiménez-Losada [16] introduced antimatroids in games, defining games on the coalitions of the set system given by the antimatroid.

**Definition 1.** An antimatroid $\mathscr{A}$ on $N$ is a family of subsets of $2^N$, satisfying

A1. $\emptyset\in\mathscr{A}$.
A2. (Accessibility) If $E\in\mathscr{A}$, $E\neq\emptyset$, then there exists $i\in E$ such that $E\setminus\{i\}\in\mathscr{A}$.
A3. (Closed under union) If $E,F\in\mathscr{A}$ then $E\cup F\in\mathscr{A}$.

Now, we need to introduce some well-known concepts about antimatroids, which can be found in [17].

The definition of antimatroid implies the following *augmentation property*, i.e., if $E,F\in\mathscr{A}$ with $|E|>|F|$ then there exists $i\in E\setminus F$ such that $F\cup\{i\}\in\mathscr{A}$. From now on we assume that the antimatroid $\mathscr{A}$ is *normal*, i.e.,

A4. For every $i\in N$ there exists an $E\in\mathscr{A}$ such that $i\in E$.

In particular, this last property implies that $N\in\mathscr{A}$ and therefore this means that the whole group of players decides to cooperate.

Let $\mathscr{A}$ be an antimatroid on $N$. This set family allows to define the interior operator $\mathrm{int}:2^N\to\mathscr{A}$, given by

$$\mathrm{int}(E)=\bigcup_{\{F\subseteq E:\,F\in\mathscr{A}\}} F\in\mathscr{A}$$

for all $E\subseteq N$. This operator is the dual one of the closure operator in a convex geometry and satisfies the following properties which characterize it:

I1. $\mathrm{int}(\emptyset)=\emptyset$,
I2. $\mathrm{int}(E)\subseteq E$,

I3. if $E \subseteq F$ then $\text{int}(E) \subseteq \text{int}(F)$,
I4. $\text{int}(\text{int}(E)) = \text{int}(E)$,
I5. if $i, j \in \text{int}(E)$ and $j \notin \text{int}(E \backslash \{i\})$ then $i \in \text{int}(E \backslash \{j\})$.

Let $\mathscr{A}$ be an antimatroid on $N$. A *feasible continuation* or *augmentation point* of $E \in \mathscr{A}$ is a player $i \in N \backslash E$ such that $E \cup \{i\} \in \mathscr{A}$, i.e., those players that can be joined to a feasible coalition keeping feasibility. This elements are denoted as $\text{au}(E)$. In a dual way, a player that can leave a feasible coalition $E$ keeping feasibility is called *endpoint* or *extreme point* [11]. The endpoints of $E$ are denoted by $\text{ex}(E)$. By condition A2 the sets $\text{ex}(E) \neq \emptyset$, for all $E \in \mathscr{A}$, $E \neq \emptyset$. In the same way, using antimatroid definition and A4, the sets $\text{au}(E) \neq \emptyset$, for all $E \in \mathscr{A}$, $E \neq N$.

A set $E$ in $\mathscr{A}$ is a *path* if it has a single endpoint. So, the set $E$ in $\mathscr{A}$ is called an *i-path* if it has $i$ as unique endpoint. An $i$-path can be considered as a rooted set denoted by $(E, i)$, i.e., $E$ is a minimal set in $\mathscr{A}$ containing $i$. In particular, a set is in $\mathscr{A}$ or is feasible in $\mathscr{A}$ if and only if it is a union of paths. This leads to a characterization of antimatroids in terms of paths (see [17]).

As a generalization of a rooted path in a player, we can give the following concept about coalitions.

**Definition 2.** Let $\mathscr{A}$ be an antimatroid on $N$ and $E \subseteq N$. A set $F \subseteq N$ is a feasible hull of $E$ if $F \in \mathscr{A}$, $E \subseteq F$ and there exists no $H \in \mathscr{A}$ with $E \subseteq H \subset F$, i.e., it is a feasible minimal set in $\mathscr{A}$ and contains the set $E$.

We denote by $A(E)$ the family of feasible hulls of $E \subseteq N$. Notice that $A(E) \neq \emptyset$. We show in the following result that this definition generalizes the path definition.

**Proposition 1.** *Let $\mathscr{A}$ be an antimatroid on N. The feasible hulls of $\{i\}$, with $i \in N$ are the family of the i-paths of $\mathscr{A}$.*

**Proof.** If $E$ is a feasible hull of $\{i\}$, then $i \in E$ and there exists no $F \in \mathscr{A}$ with $i \in F \subset E$. We have to prove that $E$ is an $i$-path, by condition A2 there exists some endpoint of $E$, suppose that the possible endpoints $j$ of $E$ are different from $i$. In this case, it holds that $i \in E \backslash \{j\} \subset E$ and as $j$ is an endpoint $E \backslash \{j\} \in \mathscr{A}$, what led us to contradiction. So, the unique endpoint of $E$ is $i$.

If $E$ is an $i$-path and it is not a feasible hull of $\{i\}$ then there exists $F \in \mathscr{A}$ with $i \in F$ and $F \subset E$. Applying the augmentation property, there exists $j \in E \backslash \{i\}$ such that $E \backslash \{j\} \in \mathscr{A}$. This contradicts the fact that $E$ is an $i$-path. □

Now we consider the relations between acyclic permission structures and antimatroids.

**Proposition 2.** *If the feasible coalition system $\mathscr{A}$ is derived from an disjunctive or conjunctive approach of an acyclic permission structure then $\mathscr{A}$ is an antimatroid.*

**Proof.** Gilles and Owen [13] and Gilles et al. [14] showed that the feasible coalitions system $\mathscr{A}$ derived from the conjunctive or disjunctive approach contains the empty set and that it is closed under union. Therefore, it suffices to prove that the system $\mathscr{A}$ is accessible. Let $S$ be an acyclic permission structure on $N$. As $N$ is a finite set, given a feasible coalition $E$, $E \neq \emptyset$, there exists a player $i \in E$ such that $S(i) \cap E = \emptyset$ by acyclicity of $S$. Then, the coalition $E \backslash \{i\}$ is also feasible, and hence we obtain the condition A2. □

The reverse of Proposition 2 is not true (see Example 1). This result will later be useful to generalize the restricted games on permission structures and introduce in the next section the restricted games on antimatroids. Next, some specific antimatroids are defined to characterize the feasible coalition systems derived from the conjunctive approach of an acyclic permission structure.

The so-called *poset antimatroids* are a particular case of antimatroid, which are formed by the ideals of a *poset* $\mathscr{P}$ or, in an equivalent way, by the filters. The poset antimatroids can be characterized as the unique antimatroids which are closed under intersection, i.e., if $E, F \in \mathscr{A}$ then $E \cap F \in \mathscr{A}$.

**Lemma 1.** *Let $\mathscr{A}$ be an antimatroid. $\mathscr{A}$ is a poset antimatroid if and only if every $i \in N$ has a unique i-path in $\mathscr{A}$.*

**Proof.** Suppose that $\mathscr{A}$ is a poset antimatroid. Let $E, F$, $E \neq F$, be two distinct $i$-paths for $i \in N$. Then $E \cap F \in \mathscr{A}$ with $i \in E \cap F$. Assume without loss of generality that $E \backslash F \neq \emptyset$. By the augmentation property there exists a $j \in E \backslash (E \cap F) = E \backslash F \subseteq E \backslash \{i\}$ such that $E \backslash \{j\} \in \mathscr{A}$. This is in contradiction with $E$ being an $i$-path.

Suppose that every $i \in N$ has a unique $i$-path in $\mathscr{A}$. Take $E, F \in \mathscr{A}$. If $E \cap F = \emptyset$ then $E \cap F \in \mathscr{A}$ by A1. If $E \cap F \neq \emptyset$ then by Proposition 1, for every $i \in E \cap F$ there exists an $i$-path $H_1^i \subseteq E$ and there exists an $i$-path $H_2^i \subseteq F$. By assumption

$H_1^i = H_2^i = H^i$ is the unique $i$-path in $\mathscr{A}$. So, $H^i \in \mathscr{A}$ and $H^i \subseteq E \cap F$, for all $i \in E \cap F$. Therefore, $E \cap F = \bigcup_{i \in E \cap F} H^i$ and by A3 $E \cap F \in \mathscr{A}$. Thus, $\mathscr{A}$ is a poset antimatroid. □

In the following result we identify the feasible coalition system derived from the conjunctive approach on an acyclic permission structure with a poset antimatroid.

**Theorem 1.** *$\mathscr{A}$ is a poset antimatroid if and only if there is an acyclic permission structure $S$ such that $\mathscr{A} = \Phi_S^{\mathrm{c}}$.*

**Proof.** Let $\mathscr{A}$ be a poset antimatroid. For $i \in N$, denote the unique $i$-path in $\mathscr{A}$ by $P_i^{\mathscr{A}}$. Now, define permission structure $S : N \to 2^N$ by $S^{-1}(i) = P_i^{\mathscr{A}} \setminus \{i\}$, for all $i \in N$. We have to prove that $S$ is acyclic. Suppose $i \in \hat{S}(i)$ then there exists $(h_1, \ldots, h_t)$ with $h_1 = i$, $h_t = i$ and such that $h_{k+1} \in S(h_k)$, for all $k = 1, \ldots t-1$, i.e., $h_k \in P_{k+1}^{\mathscr{A}}$. Moreover, $P_k^{\mathscr{A}} \subseteq P_{k+1}^{\mathscr{A}}$, for all $k$. Since $P_{k+1}^{\mathscr{A}} \in \mathscr{A}$ by Proposition 1 we have that the unique $h_k$-path is contained in $P_{k+1}^{\mathscr{A}}$. But $P_k^{\mathscr{A}} = P_{k+1}^{\mathscr{A}}$ is impossible and hence, we get $P_i^{\mathscr{A}} \subsetneq P_i^{\mathscr{A}}$. So, $S$ is acyclic. We are left to show that $\mathscr{A} = \Phi_S^{\mathrm{c}}$.

Let $E \in \Phi_S^{\mathrm{c}}$ then $S^{-1}(i) \subseteq E$, for all $i \in E$, i.e., $P_i^{\mathscr{A}} \subseteq E$, for all $i \in E$, therefore $\bigcup_{i \in E} P_i^{\mathscr{A}} \subseteq E$. Since $i \in P_i^{\mathscr{A}}$, for all $i \in N$ we also have $E \subseteq \bigcup_{i \in E} P_i^{\mathscr{A}}$. Thus, $E = \bigcup_{i \in E} P_i^{\mathscr{A}} \in \mathscr{A}$ since a union of paths is feasible by A3. Clearly, if $E \in \mathscr{A}$ with $i \in E$ then $S^{-1}(i) = P_i^{\mathscr{A}} \setminus \{i\} \subseteq E$, and thus $E \in \Phi_S^{\mathrm{c}}$.

By Proposition 2, the feasible coalition system from a conjunctive approach of an acyclic permission structure is an antimatroid. Gilles et al. [14] showed that the feasible coalition system from the conjunctive approach is closed under intersection, therefore this approach is a poset antimatroid. □

Now we are interested in characterizing the antimatroids that can be the set of disjunctive feasible coalitions of some acyclic permission structure. With this goal we introduce the following definition.

**Definition 3.** An antimatroid $\mathscr{A}$ on $N$ is said to have the path property if

P1. Every path $E$ has a unique feasible ordering, i.e. $E := (i_1 > \cdots > i_t)$ such that $\{i_1, \ldots, i_k\} \in \mathscr{A}$ for all $1 \leqslant k \leqslant t$. Furthermore, the union of these orderings for all paths is a partial ordering of $N$.

P2. If $E, F$ and $E \setminus \{i\}$ are paths such that the endpoint of $F$ equals the endpoint of $E \setminus \{i\}$, then $F \cup \{i\} \in \mathscr{A}$.

Observe that every path has a unique feasible ordering if and only if for any $i$-path $E$ with $|E| > 1$ we have that $E \setminus \{i\}$ is a path. In the next theorem, we identify those antimatroids that satisfy the path property in relation to the feasible coalition systems derived from acyclic permission structures.

**Theorem 2.** *An antimatroid $\mathscr{A}$ has the path property if and only if there is an acyclic permission structure $S$ such that $\mathscr{A} = \Phi_S^{\mathrm{d}}$.*

**Proof.** Let $\mathscr{A}$ be an antimatroid with the path property. Define $S : N \to 2^N$ by

$$S^{-1}(i) = \{ j \in N\colon \text{ there exists an } i\text{-path } E \text{ such that } E \setminus \{i\} \text{ is a } j\text{-path}\}.$$

First, we show that the permission structure $S$ is acyclic. Suppose not, then let $i \in N$ such that $i \in \hat{S}(i)$ and so there exists a chain $(h_1, \ldots, h_t)$ such that $h_1 = h_t = \{i\}$ and $h_{k+1} \in S(h_k)$, for all $k = 1, \ldots, t-1$. Hence

$$i \in S^{-1}(h_2), \ldots, h_k \in S^{-1}(h_{k+1}), \ldots, h_{t-1} \in S^{-1}(i).$$

Since $\mathscr{A}$ satisfies property P1, we can obtain a collection of paths with the following orderings:

$$(\cdots > i > h_2), \ldots, (\cdots > h_k > h_{k+1}), \ldots, (\cdots > h_{t-1} > i).$$

Thus, the cycle $i > h_2 > \cdots > h_{t-1} > i$ contradicts property P1.

Next, we prove that the antimatroid $\mathscr{A} = \Phi_S^{\mathrm{d}}$. If $E \in \mathscr{A}$ we have that for every $i \in E$ there exists an $i$-path $F \subseteq E$. Thus, for every $i \in E$, $\{i\} \in \mathscr{A}$ or there exists an $i$-path $F \subseteq E$ and a $j \in F \setminus \{i\}$ such that $F \setminus \{i\}$ is a $j$-path in $\mathscr{A}$. Therefore, for every $i \in E$, as $\{i\}$ is the unique $i$-path, $S^{-1}(i) = \emptyset$ or $S^{-1}(i) \cap E \neq \emptyset$ and hence $E \in \Phi_S^{\mathrm{d}}$.

Suppose that $E \in \Phi_S^{\mathrm{d}}$. We prove that $E \in \mathscr{A}$ by induction on $|E|$. If $|E| = 1$, i.e., $E = \{i\}$, then $S^{-1}(i) = \emptyset$. Thus, the unique $i$-path is $E = \{i\} \in \mathscr{A}$. Proceeding by induction, assume that $E' \in \mathscr{A}$ if $|E'| = |E| - 1$. Proposition 2 implies that $\Phi_S^{\mathrm{d}}$ is accessible and hence there exists a $i \in E$ such that $E \setminus \{i\} \in \Phi_S^{\mathrm{d}}$. The induction hypothesis implies that $E \setminus \{i\} \in \mathscr{A}$.

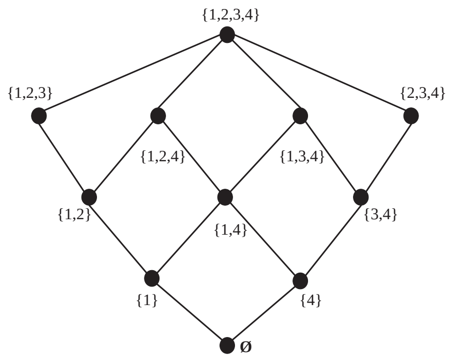


Fig. 1. Antimatroid.

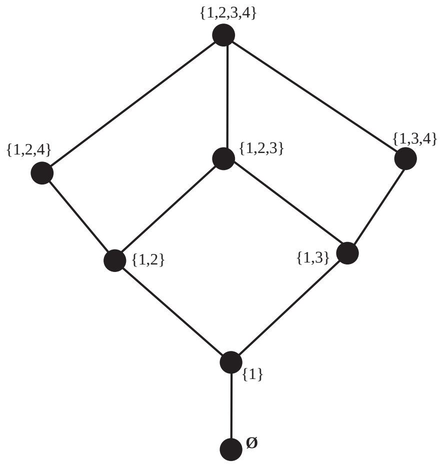


Fig. 2. Antimatroid with the path property.

If $S^{-1}(i)=\emptyset$ then $\{i\}\in\mathscr{A}$ and so $E=(E\backslash\{i\})\cup\{i\}\in\mathscr{A}$. Otherwise, there exists an element $j\in S^{-1}(i)\cap E\neq\emptyset$. Since $E\backslash\{i\}\in\mathscr{A}$ there exists a $j$-path $F\subseteq E\backslash\{i\}$. Observe that $j\in S^{-1}(i)$ implies that there exists an $i$-path $E'$ such that $E'\backslash\{i\}$ is a $j$-path. Property 2 then yields $F\cup\{i\}\in\mathscr{A}$. In view of A3 we obtain that $(F\cup\{i\})\cup(E\backslash\{i\})=E\in\mathscr{A}$.

Conversely, let $S$ be an acyclic permission structure. By Proposition 2, the family $\mathscr{A}:=\Phi_S^{\mathrm{d}}$ is an antimatroid. We have to show that $\mathscr{A}$ satisfies the path property.

We first prove that if $E$ is an $i$-path with $|E|>1$ then $S(i)\cap E=\emptyset$ and $S^{-1}(j)=\emptyset$ or $|S^{-1}(j)\cap E|=1$ for all $j\in E$. Suppose that $S(i)\cap E\neq\emptyset$. By the acyclicity of $S$ there exists a $j\in\hat{S}(i)\cap E$ such that $S(j)\cap E=\emptyset$, and hence $E\backslash\{j\}\in\Phi_S^{\mathrm{d}}$. Since $j\neq i$ we arrive at a contradiction, since $E$ is an $i$-path.

Now assume $j\in E$ with $|S^{-1}(j)\cap E|>1$. We claim that $S(h)\cap E\neq\emptyset$, for all $h\in E\backslash\{i\}$. Otherwise we have $E\backslash\{h\}\in\Phi_S^{\mathrm{d}}$ where $h\neq i$ contradicting that $E$ is an $i$-path. Then there exists a $g\in E$ with $|S^{-1}(g)\cap E|>1$ and $|S^{-1}(h)\cap E|=1$ for all $h\in\hat{S}(g)\cap E$. Since $S(i)\cap E=\emptyset$ we obtain $i\notin S^{-1}(g)\cap E$. By acyclicity of $S$ it holds that $S(k)\cap E=\{g\}$, for all $k\in S^{-1}(g)\cap E$. Thus $E\backslash\{k\}\in\Phi_S^{\mathrm{d}}$, for all $k\in S^{-1}(g)\cap E$, contradicting that $E$ is an $i$-path. Therefore, for any $i$-path $E$ with $|E|>1$ we have that $E\backslash\{i\}$ is a path. Then each $i$-path $E$ has a unique feasible ordering $E:=(i_1>\cdots>i_t)$ where $i_t=i$ and

$$\{i_1\}=S^{-1}(i_2)\cap E,\ldots,\ \{i_{t-1}\}=S^{-1}(i)\cap E.$$

The union of these linear orderings (as binary relations) for all paths is a reflexive and transitive relation on $N$. If the relation is not a partial order then there exists a cycle and this is a contradiction with the acyclicity of $S$. Thus the requirements of property P1 are fulfilled.

To show property P2, let $E$ be an $i$-path and assume that $E\backslash\{i\}$ is a $j$-path with $\{j\}=S^{-1}(i)\cap E$. For every $j$-path $F$ we have that $S^{-1}(h)=\emptyset$ or $|S^{-1}(h)\cap F|=1$ for all $h\in F$. Moreover, $S^{-1}(i)\cap F=\{j\}\neq\emptyset$. Thus, $F\cup\{i\}\in\Phi_S^{\mathrm{d}}=\mathscr{A}$. □

**Example 1** (Fig. 1). Let $N=\{1,2,3,4\}$ and the following family of subsets:

$$\mathscr{A}=\{\emptyset,\{1\},\{4\},\{1,2\},\{1,4\},\{3,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},N\}.$$

Observe that $\mathscr{A}$ is an antimatroid and it does not satisfy the path property. The path $E=\{1,2,3\}$ has the ordering $(1>2>3)$ and the path $F=\{2,3,4\}$ has the ordering $(4>3>2)$. Then the union of the above orderings is not a partial order on $N$.

**Example 2** (Fig. 2). Consider the antimatroid $\mathscr{A}$ on $N=\{1,2,3,4\}$, where

$$\mathscr{A}=\{\emptyset,\{1\},\{1,2\},\{1,3\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{1,2,3,4\}\}.$$

It can be checked that $\mathscr{A}$ satisfies the path property.

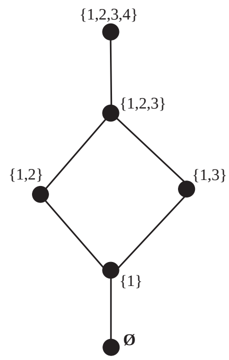


Fig. 3. Poset antimatroid.

**Example 3** (Fig. 3). Consider the poset antimatroid $\mathscr{B}$ on $N=\{1,2,3,4\}$ where

$$\mathscr{B}=\{\emptyset,\{1\},\{1,2\},\{1,3\},\{1,2,3\},\{1,2,3,4\}\}.$$

Note that $\mathscr{B}$ does not satisfy the path property since the 4-path $E=\{1,2,3,4\}$ is such that $E\backslash\{4\}=\{1,2,3\}$ is not a path and so $E$ has two feasible orderings.

As we have seen in Example 3, posets antimatroids do not satisfy the path property in general. Next, we characterize the class of antimatroids which satisfies the path property. To obtain this result, we first provide a lemma on games with permission structures. This lemma states that the permission structures for which the sets of conjunctive and disjunctive feasible coalitions coincide are exactly the *permission forest* structures.

**Definition 4.** An acyclic permission structure $S$ is a permission forest structure if $|S^{-1}(i)| \leqslant 1$, for all $i \in N$.

**Lemma 2.** *Let S be an acyclic permission structure. Then S is a permission forest structure if and only if* $\Phi_S^{\rm c}=\Phi_S^{\rm d}$.

**Proof.** Let $S$ be an acyclic permission structure. If $S$ is a permission forest structure then for every $i\in N$ we have that $S^{-1}(i)=\emptyset$ or $S^{-1}(i)\cap E\neq\emptyset$ if and only if $S^{-1}(i)\subseteq E$. But then $\Phi_S^{\rm c}=\Phi_S^{\rm d}$.

If $S$ is not a permission forest structure then there exists $j\in N$ with $|S^{-1}(j)|\geqslant 2$. If $h\in S^{-1}(j)$ then $\hat{S}^{-1}(j)\backslash\{h\}\in\Phi_S^{\rm d}\backslash\Phi_S^{\rm c}$, implying that $\Phi_S^{\rm c}\neq\Phi_S^{\rm d}$. □

The next theorem states that the poset antimatroids satisfying the path property, are exactly those antimatroids that can be obtained as the set of conjunctive or disjunctive feasible coalitions of some permission forest structure. Note that given $\mathscr{A}$ a poset antimatroid, $\mathscr{A}$ satisfies the path property if and only if it satisfies P1.

**Theorem 3.** *$\mathscr{A}$ is a poset antimatroid satisfying the path property if and only if there exists a permission forest structure S such that* $\mathscr{A}=\Phi_S^{\rm c}=\Phi_S^{\rm d}$.

**Proof.** Suppose that $\mathscr{A}$ is a poset antimatroid satisfying the path property. Define the permission structure $S:N\to 2^N$ by

$$S^{-1}(i)=\{j\in N\colon \text{ there exists an } i\text{-path } E \text{ such that } E\backslash\{i\} \text{ is a } j\text{-path}\}.$$

Since $\mathscr{A}$ is a poset antimatroid, Lemma 1 implies that there is a unique $i$-path for every $i\in N$. But then $S$ is a permission forest structure. We are left to show that $\Phi_S^{\rm c}=\mathscr{A}$. For every $i\in N$, let $H^i$ be the unique $i$-path in $\mathscr{A}$. Then $E\in\Phi_S^{\rm c}$ if and only if $S^{-1}(i)\subseteq E$, for all $i\in E$ if and only if $H^i\subseteq E$, for all $i\in E$ if and only if $\bigcup_{i\in E}H^i=E$ if and only if $E\in\mathscr{A}$. By Lemma 2 we conclude that $\mathscr{A}=\Phi_S^{\rm c}=\Phi_S^{\rm d}$.

Assume that $\mathscr{A}=\Phi_S^{\rm c}=\Phi_S^{\rm d}$ for some acyclic permission structure $S$. By Theorem 1, $\Phi_S^{\rm c}$ is a poset antimatroid. By Theorem 2 it is satisfied the path property. So, $\mathscr{A}$ is a poset antimatroid satisfying the path property. □

### 3. Restricted games on antimatroids

Let us denote by $\Gamma^N$ the real vector space of all TU-games $(N, v)$, i.e. functions $v: 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. The elements of $N = \{1, 2, \ldots, n\}$ are called *players*, the subsets $E \in 2^N$ *coalitions* and $v(E)$ is the *worth* of $E$. Every TU-game $(N, v)$ is uniquely determined by the collection of its worths $\{v(E) : E \subseteq N,\ E \neq \emptyset\}$. Then $\Gamma^N$ will be identified with $\mathbb{R}^{2^n-1}$. For any $E \subseteq N,\ E \neq \emptyset$, we define the *unanimity* game

$$u_E(F) = \begin{cases} 1 & \text{if } E \subseteq F, \\ 0 & \text{otherwise.} \end{cases}$$

Every game is a unique linear combination of unanimity games (cf. [21]),

$$v = \sum_{F \in 2^N \setminus \{\emptyset\}} d_v(F) u_F, \text{ where } d_v(F) = \sum_{E \subseteq F} (-1)^{|F|-|E|} v(E).$$

We shall call $d_v(F)$ the *dividend* of $F$ in the game $v$ and $d_v(\emptyset) = 0$ [15].

According to the previous section, the interior of a coalition $E$ in an antimatroid is the largest part of $E$ that is active or feasible. In other words and taking into account that the antimatroid structure limits the possibilities of coalitional formation in a TU-game, the interior of $E$ are those players of $E$ that are allowed to cooperate. Therefore, the interior is what Gilles and Owen [13] and Gilles et al. [14] called (conjunctive or disjunctive) sovereign part in a permission structure. So, we define a restricted game on an antimatroid for a coalition $E$ as the value obtained on the interior of $E$.

**Definition 5.** Let $\mathscr{A}$ be an antimatroid on $N$. If $v$ is a TU-game then the restricted game on $\mathscr{A}$ is defined by $v_{\mathscr{A}}(E) = v(\text{int}(E)),\ E \subseteq N$.

Our next aim is to obtain the expression of the dividends of the restricted game in terms of the dividends of the original game. For that, we first need some results.

**Proposition 3.** *Let $\mathscr{A}$ be an antimatroid on $N$ and $E \subseteq N$. The restricted game of the unanimity game $u_E$ on $\mathscr{A}$ is $(u_E)_{\mathscr{A}} = \bigvee_{F \in A(E)} u_F$. Moreover, its dividends are given, for all $F \subseteq N$, by*

$$d_{(u_E)_{\mathscr{A}}}(F) = \sum_{\{H \subseteq F : \exists T \in A(E), H \supseteq T\}} (-1)^{|F|-|H|}.$$

*In particular, $d_{(u_E)_{\mathscr{A}}}(F) = 0$, if $F \not\supseteq E$ and $d_{(u_E)_{\mathscr{A}}}(F) = 1$, if $F \in A(E)$.*

**Proof.** Let $H \subseteq N$, then

$$(u_E)_{\mathscr{A}}(H) = u_E(\text{int}(H)) = \begin{cases} 1 & \text{if } E \subseteq \text{int}(H), \\ 0 & \text{otherwise.} \end{cases}$$

If $E \subseteq \text{int}(H) \in \mathscr{A}$ then there exists $F \in A(E)$, such that $F \subseteq \text{int}(H)$, therefore $u_F(H) = 1$. In this case $\bigvee_{F \in A(E)} u_F(H) = 1$. Otherwise, $H$ does not contain any feasible hull of $E$ and therefore $u_F(H) = 0$, for all $F \in A(E)$, which implies that $\bigvee_{F \in A(E)} u_F(H) = 0$.

Let us consider $F \subseteq N$,

$$d_{(u_E)_{\mathscr{A}}}(F) = \sum_{H \subseteq F} (-1)^{|F|-|H|} (u_E)_{\mathscr{A}}(H) = \sum_{H \subseteq F} (-1)^{|F|-|H|} \bigvee_{T \in A(E)} u_T(H) = \sum_{\{H \subseteq F\,:\, \exists T \in A(E), H \supseteq T\}} (-1)^{|F|-|H|}.$$

If $F \not\supseteq E$ then there is not any $H \subseteq F$ which contains some feasible hull of $E$ and hence $d_{(u_E)_{\mathscr{A}}}(F) = 0$. Finally, if $F \in A(E)$, by definition, the unique term in the above formula is obtained for $H = F$ and therefore $d_{(u_E)_{\mathscr{A}}}(F) = 1$. □

Next, we consider the particular case from a poset antimatroid. We will take into account that for these antimatroids each coalition has a unique feasible hull (the set obtained as the intersection of all feasible coalitions that contain the coalition) since the intersection is a closed operation.

**Corollary 1.** *Let $\mathscr{A}$ be a poset antimatroid on $N$ and $E \subseteq N$. The restricted game $(u_E)_{\mathscr{A}} = u_T$, where $T$ is the unique feasible hull of $E$. Moreover, all dividends are null except $d_{(u_E)_{\mathscr{A}}}(T) = 1$.*

**Proof.** By Proposition 3, if $F=T$ then $d_{(u_E)_{\mathscr{A}}}(T)=1$. Moreover, if $F\not\supseteq E$ then $d_{(u_E)_{\mathscr{A}}}(F)=0$. Finally, if $F\supseteq E$ and $F\neq T$, since $F\supsetneq T$, we have that

$$d_{(u_E)_{\mathscr{A}}}(F)=\sum_{T\subseteq H\subseteq F}(-1)^{|F|-|H|}=\sum_{h=|T|}^{|F|}\binom{|F|-|T|}{h-|T|}(-1)^{|F|-h}=(1-1)^{|F|-|T|}=0.$$

This completes the proof. □

Using the dividends of the restricted games of unanimity games we can obtain the dividends of any restricted game on the antimatroid.

**Theorem 4.** *Let $\mathscr{A}$ be an antimatroid on N and let v be a TU-game. The dividends of the game $v_{\mathscr{A}}$ are, for all $F\subseteq N$,*

$$d_{v_{\mathscr{A}}}(F)=\begin{cases}\sum_{E\subseteq F}d_{(u_E)_{\mathscr{A}}}(F)\,d_v(E) & \text{if } F\in\mathscr{A},\\ 0 & \text{otherwise.}\end{cases}$$

**Proof.** We first suppose that $F\notin\mathscr{A}$, then

$$d_{v_{\mathscr{A}}}(F)=\sum_{E\subseteq F}(-1)^{|F|-|E|}v(\mathrm{int}(E))=\sum_{\{E\subseteq F\,:\,E\in\mathscr{A}\}}\left(\sum_{\{H\subseteq F:\mathrm{int}(H)=E\}}(-1)^{|F|-|H|}\right)v(E).$$

Observe that given $E\subseteq F$ such that $E\in\mathscr{A}$ the possible elements in the set $\{H\subseteq F\,:\,\mathrm{int}(H)=E\}$ are those coalitions $H$ contained in $F$ that contain $E$ but that do not contain any augmentation player of $E$ (by condition A2). Let $k_E=|F-(E\cup\mathrm{au}(E))|$ then

$$d_{v_{\mathscr{A}}}(F)=\sum_{\{E\subseteq F\,:\,E\in\mathscr{A}\}}\left(\sum_{p=0}^{k_E}\binom{k_E}{p}(-1)^{|F|-p}\right)v(E)=(-1)^{|F|}\sum_{\{E\subseteq F\,:\,E\in\mathscr{A}\}}(1-1)^{k_E}v(E)=0.$$

It is satisfied that $k_E>1$ since $F\notin\mathscr{A}$ and moreover the set $E\cup[\mathrm{au}(E)\cap F]$ is feasible because $E\cup[\mathrm{au}(E)\cap F]=\bigcup_{j\in\mathrm{au}(E)\cap F}(E\cup\{j\})\in\mathscr{A}$, by condition A3 and the augmentation player definition.

Suppose that $F\in\mathscr{A}$, now we use induction on the cardinal of $F$. If $|F|=1$, then $F=\{i\}$ and we have that

$$d_{v_{\mathscr{A}}}(\{i\})=\sum_{E\subseteq\{i\}}(-1)^{1-|E|}v_{\mathscr{A}}(E)=-v(\mathrm{int}(\emptyset))+v(\mathrm{int}(\{i\}))=v(\{i\}),$$

since by I1 $\mathrm{int}(\emptyset)=\emptyset$ and as $F\in\mathscr{A}$ then $\mathrm{int}(\{i\})=\{i\}$. On the other hand,

$$\sum_{E\subseteq\{i\}}d_{(u_E)_{\mathscr{A}}}(\{i\})d_v(E)=d_{(u_\emptyset)_{\mathscr{A}}}(\{i\})d_v(\emptyset)+d_{(u_{\{i\}})_{\mathscr{A}}}(\{i\})d_v(\{i\})=v(\{i\}),$$

because $d_v(\emptyset)=0$, $d_v(\{i\})=v(\{i\})$ and by Proposition 3, $d_{(u_{\{i\}})_{\mathscr{A}}}(\{i\})=1$ since $\{i\}$ is the feasible hull of $\{i\}$. Suppose that the result is true if $|F|\leqslant k$ and we will prove it if $|F|=k+1$. As $F\in\mathscr{A}$, $\mathrm{int}(F)=F$,

$$\sum_{E\subseteq F}d_v(E)=v(F)=v_{\mathscr{A}}(F)=\sum_{E\subseteq F}d_{v_{\mathscr{A}}}(E)=\sum_{E\subsetneq F}d_{v_{\mathscr{A}}}(E)+d_{v_{\mathscr{A}}}(F).$$

By using the induction hypothesis and the case $E\notin\mathscr{A}$ considered above we have that

$$\begin{aligned}d_{v_{\mathscr{A}}}(F)&=\sum_{E\subseteq F}d_v(E)-\sum_{\{E\subsetneq F\,:\,E\in\mathscr{A}\}}d_{v_{\mathscr{A}}}(E)\\&=\sum_{E\subseteq F}d_v(E)-\sum_{\{E\subsetneq F\,:\,E\in\mathscr{A}\}}\left(\sum_{H\subseteq E}d_{(u_H)_{\mathscr{A}}}(E)\,d_v(H)\right)\end{aligned}$$

$$= \sum_{E\subseteq F} d_v(E) - \sum_{E\subsetneq F}\left(\sum_{E\subseteq H\subsetneq F} d_{(u_E)_{\mathscr{A}}}(H)\right) d_v(E)$$

$$= d_v(F) + \sum_{E\subsetneq F}\left(1 - \sum_{E\subseteq H\subsetneq F} d_{(u_E)_{\mathscr{A}}}(H)\right) d_v(E).$$

As $d_{(u_F)_{\mathscr{A}}}(F) = 1$ by Proposition 3 and $F\in\mathscr{A}$, it remains to prove

$$1 - \sum_{E\subseteq H\subsetneq F} d_{(u_E)_{\mathscr{A}}}(H) = d_{(u_E)_{\mathscr{A}}}(F) \quad \forall E \subsetneq F,$$

since $F$ is the unique feasible hull of $F$. Applying Proposition 3, we get

$$1 - \sum_{E\subseteq H\subsetneq F} d_{(u_E)_{\mathscr{A}}}(H) = 1 - \sum_{E\subseteq H\subsetneq F}\ \sum_{\{T\subseteq H\,:\,\exists R\in A(E), T\supseteq R\}} (-1)^{|H|-|T|}$$

$$= (-1)^{|F|-|F|} - \sum_{\{T\subsetneq F\,:\,\exists R\in A(E), T\supseteq R\}}\ \sum_{T\subseteq H\subsetneq F} (-1)^{|H|-|T|}.$$

Taking into account that

$$\sum_{T\subseteq H\subsetneq F} (-1)^{|H|-|T|} = \sum_{T\subseteq H\subseteq F} (-1)^{|H|-|T|} - (-1)^{|F|-|T|} = -(-1)^{|F|-|T|}$$

since

$$\sum_{T\subseteq H\subseteq F} (-1)^{|H|-|T|} = (1-1)^{|F|-|T|} = 0, \tag{1}$$

we conclude that

$$1 - \sum_{E\subseteq H\subsetneq F} d_{(u_E)_{\mathscr{A}}}(H) = (-1)^{|F|-|F|} + \sum_{\{T\subsetneq F\,:\,\exists R\in A(E), T\supseteq R\}} (-1)^{|F|-|T|} = \sum_{\{T\subseteq F:\exists R\in A(E), T\supseteq R\}} (-1)^{|F|-|T|} = d_{(u_E)_{\mathscr{A}}}(F).$$

Note that in formula (1) we may take $|F| - |T| \neq 0$, because otherwise $F$ would be a feasible hull of $E$ and $\sum_{E\subseteq H\subsetneq F} d_{(u_E)_{\mathscr{A}}}(H) = 0$. □

In particular, in Proposition 3 the dividends of a non-feasible coalition for the restricted game of a unanimity game are also zero.

**Corollary 2.** *Let $\mathscr{A}$ be a poset antimatroid on $N$ and let $v$ be a TU-game. The dividends of the game $v_{\mathscr{A}}$ are given, for all $F \subseteq N$, by*

$$d_{v_{\mathscr{A}}}(F) = \sum_{\{E\subseteq F:\ F=A(E)\}} d_v(E),$$

*where $A(E)$ denote the unique feasible hull of $E$.*

**Proof.** It is straightforward obtained using Corollary 1 and Theorem 4. Notice that if $F\notin\mathscr{A}$ then $d_{v_{\mathscr{A}}}(F) = 0$, since in this case $F$ is not a feasible hull. □

In the next result we will show that the dividends can be expressed in terms of only the characteristic function evaluated on the feasible coalitions in the antimatroid. For this, we will use the Möbius inversion formula for partially ordered sets (see [22]). Notice that an antimatroid $\mathscr{A}$ with the inclusion relation is, in particular, a lattice taking into account the following operations, $E \vee F = E \cup F$, $E \wedge F = \text{int}(E \cap F)$, for all $E, F \in \mathscr{A}$. We denote by $[E,F]_{\mathscr{A}}$ the interval of coalitions in $\mathscr{A}$ that contain $E$ and are contained in $F$. Given $E\in\mathscr{A}$ the set $E^+$ is the coalition formed by $E$ and the augmentation players, i.e., $E^+ = E \cup \text{au}(E)$.

**Lemma 3.** *Let $\mathscr{A}$ be an antimatroid on $N$, it holds that*

1. *If $F \subseteq E$, $E, F \in \mathscr{A}$, the interval $[F,E]_{\mathscr{A}}$ is a Boolean algebra if and only if $E \backslash F = \mathrm{au}(F) \cap E$.*
2. *If $E \in \mathscr{A}$ then $[E,E^+]_{\mathscr{A}}$ is a Boolean algebra.*

**Proof.** 1. In the lattice $\mathscr{A}$, a coalition $T$ covers $F$ if $T = F \cup \{i\}$, where $i \in \mathrm{au}(F)$ (by the augmentation property such $i$ exists). We prove that $[F,E]_{\mathscr{A}}$ is a Boolean algebra if and only if $E$ can be written as $E = \bigcup_{i\in \mathrm{au}(F)\cap E}(F \cup \{i\})$. If $[F,E]_{\mathscr{A}}$ is a Boolean algebra then $F \cup \{i\} \in \mathscr{A}$ for all $i \in E \backslash F$ and therefore $i \in \mathrm{au}(F) \cap E$. The other inclusion is straightforward. So, we can conclude that $[F,E]_{\mathscr{A}}$ is a Boolean algebra if and only if $E \backslash F = \mathrm{au}(F) \cap E$.

2. For all $i \in E^+ \backslash E$ it holds that $i \in \mathrm{au}(E)$ and therefore $E \cup \{i\} \in \mathscr{A}$. By property (A3) it is concluded that each coalition of $[E,E^+]$ is feasible. □

**Lemma 4.** *Let $\mathscr{A}$ be an antimatroid on $N$ and $E, F \in \mathscr{A}$, with $F \subseteq E$. Then its Möbius function is given by*

$$\mu(F,E) = \begin{cases} (-1)^{|E|-|F|} & \text{if } E \backslash F = \mathrm{au}(F) \cap E, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** It is well-known [22] that the Möbius function of an interval $[F,E]_{\mathscr{A}}$ is zero unless $E$ is the union of those coalitions that cover $F$ in the interval. By the proof of Lemma 3, this is equivalent to $E \backslash F = \mathrm{au}(F) \cap E$. By Lemma 3 the equality is true if and only if $[F,E]$ is a Boolean algebra. The Möbius function in the Boolean algebra $[F,E]$ is $\mu(F,E) = (-1)^{|E|-|F|}$. So, we conclude the result. □

**Theorem 5.** *Let $\mathscr{A}$ be an antimatroid on $N$ and let $v$ be a TU-game. The dividends of $v_{\mathscr{A}}$ are, for all $E \subseteq N$,*

$$d_{v_{\mathscr{A}}}(E) = \begin{cases} \displaystyle\sum_{\{F\in\mathscr{A}: E\in[F,F^+]\}} (-1)^{|E|-|F|} v(F) & \text{if } E \in \mathscr{A}, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** For every $E \subseteq N$ it holds that

$$v_{\mathscr{A}} = \sum_{F\subseteq N} d_{v_{\mathscr{A}}}(F) u_F.$$

By Theorem 4, the dividends of any non-feasible coalition are zero. Given a feasible coalition $E \in \mathscr{A}$ then $v_{\mathscr{A}}(E) = v(E) = \sum_{\{F\in\mathscr{A}: F\subseteq E\}} d_{v_{\mathscr{A}}}(F)$. Using the Möbius inversion formula [22] applied to $v$ and $d_{v_{\mathscr{A}}}$ on $\mathscr{A}$ we get

$$d_{v_{\mathscr{A}}}(E) = \sum_{\{F\in\mathscr{A}: F\subseteq E\}} \mu(F,E) v(F).$$

Then, replacing the Möbius function obtained in Lemma 4 we have that

$$d_{v_{\mathscr{A}}}(E) = \sum_{\{F\in\mathscr{A}: F\subseteq E, E\backslash F = \mathrm{au}(F)\cap E\}} (-1)^{|E|-|F|} v(F).$$

The following equality is immediate:

$$\{F \in \mathscr{A} : F \subseteq E,\ E\backslash F = \mathrm{au}(F) \cap E\} = \{F \in \mathscr{A} : E \in [F,F^+]_{\mathscr{A}}\},$$

and by Lemma 3, the interval $[F,F^+]_{\mathscr{A}}$ is a Boolean algebra. □

## 4. The restricted Shapley and Banzhaf values on antimatroids

In this section, we consider the Shapley and Banzhaf values of the restricted game on an antimatroid to obtain new values for a game. Taking into account the cooperation possibilities, the values that we define generalize the Shapley and Banzhaf values defined on permission structures and studied in [4,5,13,14]. Given the game $v$ the Shapley [21] and Banzhaf [1] values for the player $i$ in the game $v$ are denoted by $\mathrm{Sh}_i(v)$ and $\mathrm{Ba}_i(v)$, respectively, and in terms of dividends they are given by

$$\mathrm{Sh}_i(v) = \sum_{\{E \subseteq N : i \in E\}} \frac{d_v(E)}{|E|},$$

$$\mathrm{Ba}_i(v) = \sum_{\{E \subseteq N : i \in E\}} \frac{d_v(E)}{2^{|E|-1}}.$$

The Banzhaf power index is introduced for voting situations. A generalization for arbitrary TU-games as in, e.g., [9,19] is considered.

**Definition 6.** Let $\mathscr{A}$ be an antimatroid on $N$ and let $v$ be a TU-game. The restricted Shapley value is defined by $\overline{\mathrm{Sh}}(v, \mathscr{A}) = \mathrm{Sh}(v_{\mathscr{A}})$. The restricted Banzhaf value is given by $\overline{\mathrm{Ba}}(v, \mathscr{A}) = \mathrm{Ba}(v_{\mathscr{A}})$.

The linearity of these values and Theorem 4 imply that, for every $i \in N$,

$$\overline{\mathrm{Sh}}_i(v, \mathscr{A}) = \sum_{\{E \in \mathscr{A} : i \in E\}} \frac{d_{v_{\mathscr{A}}}(E)}{|E|}, \tag{2}$$

$$\overline{\mathrm{Ba}}_i(v, \mathscr{A}) = \sum_{\{E \in \mathscr{A} : i \in E\}} \frac{d_{v_{\mathscr{A}}}(E)}{2^{|E|-1}}. \tag{3}$$

From Theorem 5 we can get a formula to compute these values using directly the characteristic function of the original game.

**Theorem 6.** *Let $\mathscr{A}$ be an antimatroid on $N$ and let $v$ be a TU-game. We consider the following collections, for $i \in N$,*

$$\mathscr{A}_i = \{E \in \mathscr{A} : i \in E\},$$

$$\mathscr{A}_i^+ = \{E \in \mathscr{A} : i \in \mathrm{ex}(E), (E \backslash \{i\})^+ = E^+\},$$

$$\mathscr{A}_i^* = \{E \in \mathscr{A} : i \in \mathrm{au}(E), (E \cup \{i\})^+ \neq E^+\}.$$

*Then*

1. *The restricted Shapley value for player i is given by*

$$\overline{\mathrm{Sh}}_i(v, \mathscr{A}) = \sum_{E \in \mathscr{A}_i^+} \frac{(e-1)!(e^+ - e)!}{e^+!} [v(E) - v(E \backslash \{i\})]$$
$$+ \sum_{E \in \mathscr{A}_i \backslash \mathscr{A}_i^+} \frac{(e-1)!(e^+ - e)!}{e^+!} v(E) - \sum_{E \in \mathscr{A}_i^*} \frac{e!(e^+ - e - 1)!}{e^+!} v(E).$$

2. *The restricted Banzhaf value for player i is given by*

$$\overline{\mathrm{Ba}}_i(v, \mathscr{A}) = \sum_{E \in \mathscr{A}_i^+} \frac{1}{2^{e^+ - 1}} [v(E) - v(E \backslash \{i\})] + \sum_{E \in \mathscr{A}_i \backslash \mathscr{A}_i^+} \frac{1}{2^{e^+ - 1}} v(E) - \sum_{E \in \mathscr{A}_i^*} \frac{1}{2^{e^+ - 1}} v(E),$$

*where $e = |E|$ and $e^+ = |E^+|$.*

**Proof.** 1. By Theorem 5 and formula (2)

$$\overline{\mathrm{Sh}}_i(v, \mathscr{A}) = \sum_{\{F \in \mathscr{A} : i \in F\}} \frac{1}{|F|} \left[ \sum_{\{E \in \mathscr{A} : F \in [E, E^+]_{\mathscr{A}}\}} (-1)^{|F|-|E|} v(E) \right].$$

Hence,

$$\overline{\mathrm{Sh}}_i(v,\mathscr{A}) = \sum_{E\in\mathscr{A}} \left[ \sum_{\{F\in[E,E^+]_{\mathscr{A}}:i\in F\}} \frac{1}{|F|}(-1)^{|F|-|E|} \right] v(E) = \sum_{E\in\mathscr{A}} c_i(E)v(E),$$

where we denote by $c_i(E)$ the coefficient of $v(E)$ in the above sum. We distinguish two cases: $i\in E$ and $i\notin E$.

If $i\in E$, as $[E,E^+]_{\mathscr{A}}$ is a Boolean algebra then

$$\begin{aligned} c_i(E) &= \sum_{k=e}^{e^+} \binom{e^+-e}{k-e} \frac{(-1)^{k-e}}{k} = \sum_{k=0}^{e^+-e} \binom{e^+-e}{k} \frac{(-1)^k}{k+e} \\ &= \sum_{k=0}^{e^+-e} \binom{e^+-e}{k} (-1)^k \int_0^1 x^{k+e-1}\,\mathrm{d}x \\ &= \int_0^1 x^{e-1}(1-x)^{e^+-e}\,\mathrm{d}x = \frac{(e-1)!(e^+-e)!}{e^+!}. \end{aligned}$$

If $i\notin E$ then $i\in \mathrm{au}(E)$ otherwise $c_i(E)=0$. In that case, $[E\cup\{i\},E^+]$ is a Boolean algebra and hence

$$\begin{aligned} c_i(E) &= \sum_{k=e+1}^{e^+} \binom{e^+-e-1}{k-e-1} \frac{(-1)^{k-e}}{k} = \sum_{k=0}^{e^+-e-1} \binom{e^+-e-1}{k} \frac{(-1)^{k+1}}{k+e+1} \\ &= -\sum_{k=0}^{e^+-e-1} \binom{e^+-e-1}{k} (-1)^k \int_0^1 x^{k+e}\,\mathrm{d}x \\ &= -\int_0^1 x^{e}(1-x)^{e^+-e-1}\,\mathrm{d}x = -\frac{e!(e^+-e-1)!}{e^+!}. \end{aligned}$$

Replacing the coefficients in the formula it boils down to

$$\overline{\mathrm{Sh}}_i(v,\mathscr{A}) = \sum_{E\in\mathscr{A}_i} \frac{(e-1)!(e^+-e)!}{e^+!} v(E) - \sum_{\{E\in\mathscr{A}:i\in\mathrm{au}(E)\}} \frac{e!(e^+-e-1)!}{e^+!} v(E). \tag{4}$$

For every $E\in\mathscr{A}_i$, if $i\in\mathrm{ex}(E)$ and $(E\backslash\{i\})^+=E^+$, then $E\in\mathscr{A}_i^+$. In that case, we have that $E\backslash\{i\}\in\mathscr{A}$ and $c_i(E\backslash\{i\}) = -c_i(E)$, and therefore we can group both coefficients. If $E\in\mathscr{A}_i\backslash\mathscr{A}_i^+$ its coefficient appears in the first sum but it can not be grouped. Finally, if none of these possibilities happen then $E\in\mathscr{A}$, $i\in\mathrm{au}(E)$ and $(E\backslash\{i\})^+\neq E^+$, i.e., $E\in\mathscr{A}_i^*$. So, these appear in the second sum.

2. By the above part and formula (3), we know that

$$\overline{\mathrm{Ba}}_i(v,\mathscr{A}) = \sum_{E\in\mathscr{A}} c_i(E)v(E),$$

where in this case

$$c_i(E) = \sum_{\{F\in[E,E^+]_{\mathscr{A}}:i\in F\}} \frac{1}{2^{|F|-1}}(-1)^{|F|-|E|}$$

if $i\in E$ then

$$c_i(E) = \sum_{k=e}^{e^+} \binom{e^+-e}{k-e} \frac{(-1)^{k-e}}{2^{k-1}} = \sum_{k=0}^{e^+-e} \binom{e^+-e}{k} \frac{(-1)^k}{2^{k+e-1}} = \frac{1}{2^{e^+-1}}$$

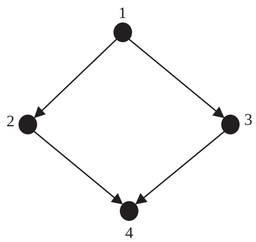


Fig. 4.

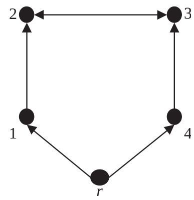


Fig. 5.

and if $i \in \mathrm{au}(E)$ then

$$c_i(E) = \sum_{k=e+1}^{e^+} \binom{e^+ - e - 1}{k - e - 1} \frac{(-1)^{k-e}}{2^{k-1}} = -\sum_{k=0}^{e^+-e-1} \binom{e^+ - e - 1}{k} \frac{(-1)^k}{2^{k+e}} = \frac{-1}{2^{e^+-1}}.$$

Doing the same process as in the case of the Shapley value we get the formula for the Banzhaf value. □

**Remark 1.** Notice that if $\mathscr{A} = 2^N$ then the formulas obtained in Theorem 6 are equal to the Shapley and Banzhaf values. Moreover, Eq. (4) is efficient from the computational point of view and coincides with Eq. (11) of Shapley [21].

To apply the computational method developed in this section we analyze two examples.

**Example 4.** Let $N = \{1,2,3,4\}$, given $v = u_{\{4\}}$ the unanimity game on the coalition $\{4\}$ and $S$ the acyclic permission structure on $N$ given by (see Fig. 4)

$$S(1) = \{2,3\},\ S(2) = S(3) = \{4\},\ S(4) = \emptyset.$$

The disjunctive and the conjunctive approaches in the permission structure are the antimatroids considered in Examples 2 and 3 respectively (see Figs. 2 and 3).

To compute the restricted Shapley value on $\mathscr{A}$ and $\mathscr{B}$ we use formula (4) since so we only have to calculate $\mathscr{A}_i$ and $\{E \in \mathscr{A} : i \in \mathrm{au}(E)\}$. Notice that with this formula, if the game changes but the structure continue being the same the calculations on these sets only are done once. So, the corresponding sets $\mathscr{A}_i$ belong to the disjunctive approach are given by

$$\mathscr{A}_1 = \{\{1\},\{1,2\},\{1,3\},\{1,2,3\},\{1,2,4\},\{1,3,4\},N\},$$

$$\mathscr{A}_2 = \{\{1,2\},\{1,2,3\},\{1,2,4\},N\},$$

$$\mathscr{A}_3 = \{\{1,3\},\{1,2,3\},\{1,3,4\},N\},$$

$$\mathscr{A}_4 = \{\{1,2,4\},\{1,3,4\},N\}$$

and

$$\{E \in \mathscr{A} : 1 \in \mathrm{au}(E)\} = \{\emptyset\},$$

$$\{E \in \mathscr{A} : 2 \in \mathrm{au}(E)\} = \{\{1\},\{1,3\},\{1,3,4\}\},$$

$$\{E \in \mathscr{A} : 3 \in \mathrm{au}(E)\} = \{\{1\},\{1,2\},\{1,2,4\}\},$$

$$\{E \in \mathscr{A} : 4 \in \mathrm{au}(E)\} = \{\{1,2\},\{1,3\},\{1,2,3\}\}$$

thus, applying the above formula we obtain $\overline{\mathrm{Sh}}(v,\mathscr{A}) = (5/12, 1/12, 1/12, 5/12)$ and in a similar way $\overline{\mathrm{Sh}}(v,\mathscr{B}) = (1/4, 1/4, 1/4, 1/4)$.

Now, the following example shows that the generalization of the model of permission structures to antimatroids has applications in other scopes.

**Example 5.** Let $G = (V, A)$ be a directed graph with root $r \in V$ and $c : V \to \mathbb{R}$ a map on the nodes called capacity function. A $c$-compatible directed path with root $r$ is a sequence of vertices $(r, x_1, \ldots, x_k)$, such that $rx_1 \ldots x_k$ is a directed path and such that $c(r) \geqslant k, c(x_1) \geqslant k-1, \ldots, c(x_k) \geqslant 0$. We interpret $N = V \setminus \{r\}$ as the clients of a certain source $r$ and the edges as the directed network that can be established among the clients from the source. The capacity of a vertex is the quantity that can be retransmitted from it. The feasible coalitions that the clients can form are those in which, without influence of others, can connect to the source through a directed $c$-compatible path. The structure that is defined is an antimatroid called capacitated point search,

$$\mathscr{A} = \left\{ E \subseteq N : \forall x \in E, \exists c\text{-compatible path } (r, x_1, \ldots, x_k) \text{such that } \{x_1, \ldots, x_k\} \subseteq E \text{ and } x_k = x \right\}.$$

These antimatroids do not satisfy the path property that characterize to the antimatroids which derived from a disjunctive approach on an acyclic permission structure, further they are not poset antimatroids, as it is seen in the following example. Consider the directed graph corresponding to Fig. 5, whose antimatroid of feasible coalitions is the one described in Example 1. If we suppose that the source offers a discount in the prices of its service to groups that ask for it, depending on their needs, the clients will obtain a profit cooperating among them. We define a game that prizes this profit for each cooperation assuming that it is proportional to the clients number. Without loss of generality we consider $v(E) = |E| - 1$, for every non-empty coalition $E \subseteq N$ and $v(\emptyset) = 0$. In a cooperation among several players logically will be taken into account the maximal feasible coalition that the players can form. We used the Shapley value on the game $v_{\mathscr{A}}$ to determine how to divide the profits of a hypothetical cooperation among the four clients. In order to calculate the payoff of the player 1 through formula (4) obtained in Theorem 6 we have to calculate before the sets $\mathscr{A}_i$ and $\{E \in \mathscr{A} : i \in \mathrm{au}(E)\}$. So, for instance

$$\mathscr{A}_1 = \{\{1\}, \{1,2\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{1,2,3,4\}\},$$

$$\{E \in \mathscr{A} : 1 \in \mathrm{au}(E)\} = \{\emptyset, \{4\}, \{3,4\}, \{2,3,4\}\}.$$

In this way, the division of the profits of the total cooperation, $v(N) = 3$ is given by $\overline{\mathrm{Sh}}(v, \mathscr{A}) = (5/6, 2/3, 2/3, 5/6)$.

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