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An axiomatization of the Banzhaf value for cooperative games on antimatroids*

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Abstract. Cooperative games on *antimatroids* are cooperative games in which coalition formation is restricted by a combinatorial structure which generalizes *permission structures*. These games group several well-known families of games which have important applications in economics and politics. The current paper establishes axioms that determine the restricted *Banzhaf value* for cooperative games on antimatroids. The set of given axioms generalizes the axiomatizations given for the *Banzhaf permission values*. We also give an axomatization of the restricted Banzhaf value for the smaller class of *poset antimatroids*. Finally, we apply the above results to *auction situations*.

Key words: Antimatroid, Cooperative game, Permission structure, Banzhaf value

Mathematics Subject Classification 2000: 91A12

1 Introduction

A cooperative game describes a situation in which a finite set of n players can generate certain payoffs by cooperation. A one-point solution concept for cooperative games is a function which assigns to every cooperative game a n-dimensional real vector which represents a payoff distribution over the players. The study of solution concepts is central in cooperative game theory. Two well-known solution concepts are the *Shapley value* as proposed by Shapley (1953), and the *Banzhaf value*, initially introduced in the context of voting games by Banzhaf (1965), and later on extended to arbitrary games by, e.g., Owen (1975) and Dubey and Shapley (1979). These values satisfy a set of

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intuitively reasonable axioms that characterizes each one of them. The assessment that both values assign to a player is the average of the marginal contribution to any coalition which the player belongs to, although they associate different weights to each coalition. The *Banzhaf value* considers that every player is equally likely to enter to any coalition whereas the Shapley value assumes that every player is equally likely to join to any coalition of the same size and all coalitions with the same size are equally likely.

In a cooperative game the players are assumed to be socially identical in the sense that every player can cooperate with every other player. However, in practice there exist social asymmetries among the players. For this reason, the game theoretic analysis of decision processes in which one imposes asymmetric constraints on the behavior of the players has been and continues to be an important subject to study. Important consequences have been obtained of adopting this type of restrictions on economic behavior. Some models which analyze social asymmetries among players in a cooperative game are described in, e.g., Myerson (1977), Owen (1986) and Borm, Owen and Tijs (1992). In these models the possibilities of coalition formation are determined by the positions of the players in a *communication graph*.

Another type of asymmetry among the players in a cooperative game is introduced in Gilles, Owen and van den Brink (1992), Gilles and Owen (1999), van den Brink and Gilles (1996) and van den Brink (1997). In these models, the possibilities of coalition formation are determined by the positions of the players in a hierarchical *permission structure*. Two different approaches were introduced for these games: *conjunctive* and *disjunctive*. Games on antimatroids were introduced in Jiménez-Losada (1998). Later, Algaba, Bilbao, van den Brink and Jiménez-Losada (2000) showed that the feasible coalition systems derived from both the conjunctive and disjunctive approach to games with a permission structure were identified to certain families of antimatroids: *poset antimatroids* and *antimatroids* with the *path property*, respectively. On the onther hand, Branzei, Fragnelli and Tijs (2002) introduced *peer group games* that were described by a rooted tree. This type of games allows to study particular cases of auction situations, communication situations, sequencing situations and flow games. These games are restricted games on poset antimatroids wiht the path property. This class of antimatroids are the *permission forest* and *permission tree structures* which are often encountred in the economic literature. So, the study of games on antimatroids allows to unify several research lines. Another model in which cooperation possibilities in a game are limited by some hierarchical structure on the set of players can be found in Faigle and Kern (1992) who consider feasible rankings of the players.

A relevant aspect in permission structures has been the study and characterization of solution concepts, the so-called *permission values*, defined with the aid of the Shapley value of two different *permission games* (see van den Brink and Gilles (1996) and van den Brink (1997)) and the *permission Banzhaf values* based on the Banzhaf value for these games (see van den Brink (2000)). However, as we have already pointed out antimatroids extend this model allowing us to study new situations and at the same time unify different approaches. One of the important aspects of paying attention to cooperative games on antimatroids is revealed in the study of a characterization of the Shapley value for cooperative games on antimatroids in Algaba, Bilbao, van den Brink and Jiménez-Losada (2003) where in particular we unify the fairness axioms used in both the conjunctive and disjunctive approaches. This allows us at the same time to simplify the existing literature as well as provide new insights. The current paper complements this study by focusing on the Banzhaf value obtaining an axiomatization for Banzhaf value by means of intuitive reasonable axioms and giving answers to the questions that were studied for the Shapley value. In the last section, we emphasize by means of an example both the importance of the axiomatization as well as the differences of interpretation between the two values studied.

The paper is organized as follows. To make the paper self-contained Section 2 recalls some preliminaries on cooperative games, antimatroids and permission structures. Section 3 contains the main result which is focussed on an axiomatization of the restricted Banzhaf value for games on antimatroids. The set of given axioms generalizes the axiomatizations of both the conjunctive and disjunctive Banzhaf permission values for games with a permission structure. In particular, with respect to these we unify the *fairness* axioms as well as the *predecessor fairness* used in both conjunctive and disjunctive approaches to a stronger fairness axiom. In Section 4, we restrict our attention to a special class of antimatroids the so-called poset antimatroids, showing that deleting the *strong fairness* axiom characterizes the restricted Banzhaf value for the class of cooperative games on poset antimatroids. Moreover, it turns out that the class of games on poset antimatroids is characterized as that class of games on which the restricted Banzhaf value is the unique solution satisfying these axioms. Finally, we apply the obtained results to *auction situations*. In particular, we obtain an axiomatization of the Banzhaf value in this context as well as a nice formula to compute it.

2 Cooperative games on antimatroids

A cooperative game is a pair (N, v), where $N \subseteq \mathbb{N}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ is a *characteristic function* satisfying $v(\emptyset) = 0$. A cooperative game (N, v) is *monotone* if $v(E) \leq v(F)$ whenever $E \subseteq F \subseteq N$. Antimatroids were introduced by Dilworth (1940) as particular examples of semimodular lattices. A symmetric study of these structures was started by Edelman and Jamison (1985) emphasizing the combinatorial abstraction of convexity. The convex geometries are a dual concept of antimatroids (see Bilbao, 2000).

An *antimatroid* $\mathcal{A}$ on N is a family of subsets of 2^N, satisfying

A1. $\emptyset \in \mathcal{A}$.
A2. (Accessibility) If $E \in \mathcal{A}$, $E \neq \emptyset$, then there exists $i \in E$ such that $E \backslash \{i\} \in \mathcal{A}$.
A3. (Closed under union) If $E, F \in \mathcal{A}$ then $E \cup F \in \mathcal{A}$.

The definition of antimatroid implies the following *augmentation property*: if $E, F \in \mathcal{A}$ with $|E| > |F|$ then there exists $i \in E \backslash F$ such that $F \cup \{i\} \in \mathcal{A}$.

From now on, we only consider antimatroids satisfying

A4. (*Normality*) *For every* $i \in N$ *there exists an* $E \in \mathcal{A}$ *such that* $i \in E$.

In particular, this implies that $N \in \mathcal{A}$. Now we introduce some well-known concepts about antimatroids which can be found in Korte, Lovász and

Schrader (1991, Chapter III). Let $\mathcal{A}$ be an antimatroid on N. This set family allows to define the *interior operator* $int_{\mathcal{A}} : 2^N \to \mathcal{A}$, given by $int_{\mathcal{A}}(E) = \bigcup_{F \subseteq E, F \in \mathcal{A}} F \in \mathcal{A}$, for all $E \subseteq N$.

Let $\mathcal{A}$ be an antimatroid on N. An *augmentation point* (Jiménez-Losada, 1998) of $E \in \mathcal{A}$ is a player $i \in N \backslash E$ such that $E \cup \{i\} \in \mathcal{A}$, i.e., those players that can be joined to a feasible coalition keeping feasibility. In a dual way, an *extreme point* (Edelman and Jamison, 1985) of $E \in \mathcal{A}$ is a player $i \in E$ such that $E \backslash \{i\} \in \mathcal{A}$, i.e., those players that can leave a feasible coalition E keeping feasibility. By condition A2 (Accessibility) every non-empty coalition in $\mathcal{A}$ has at least one extreme point. A set $E \in \mathcal{A}$ is a *path* in $\mathcal{A}$ if it has a single extreme point. The path $E \in \mathcal{A}$ is called a i-path in $\mathcal{A}$ if it has $i \in N$ as unique extreme point. A coalition $E \in \mathcal{A}$ if and only if E is a union of paths. Moreover, for every $E \in \mathcal{A}$ with $i \in E$ there exists an i-path F such that $F \subseteq E$. The set of i-paths for a given player $i \in N$ will be denoted by $A(i)$.

A special class of antimatroids are the *poset antimatroids* being antimatroids that are closed under intersection. An antimatroid $\mathcal{A}$ is a poset antimatroid if $E \cap F \in \mathcal{A}$ for every $E, F \in \mathcal{A}$. Alternatively, the poset antimatroids are characterized as those antimatroids such that for every $i \in N$ there is exactly one i-path.

For a cooperative game (N, v) and an antimatroid $\mathcal{A}$ on N we define the restricted characteristic function $v_{\mathcal{A}}$ which assigns to every coalition E the worth generated by the interior of E, i.e., $v_{\mathcal{A}}(E) = v(int_{\mathcal{A}}(E))$, for all $E \subseteq N$. A solution for games on antimatroids is a function f that assigns a payoff distribution $f(N, v, \mathcal{A}) \in \mathbb{R}^n$ to every cooperative game (N, v) and antimatroid $\mathcal{A}$ on N. Algaba et al. (2003) characterized the restricted Shapley value for cooperative games on antimatroids. In this paper we characterize the *restricted Banzhaf value* $\bar{B}(N, v, \mathcal{A})$ for a cooperative game (N, v) and an antimatroid $\mathcal{A}$ on N which is obtained by applying the *Banzhaf value* to game $(N, v_{\mathcal{A}})$, i.e.,

$$\bar{B}_i(N, v, \mathcal{A}) = B_i(N, v_{\mathcal{A}}) = \frac{1}{2^{|N|-1}} \sum_{\{E \subseteq N : i \in E\}} (v_{\mathcal{A}}(E) - v_{\mathcal{A}}(E \backslash \{i\})).$$

The Banzhaf value was introduced as a power index for voting games by Banzhaf (1965), and later was generalized to arbitrary games by, e.g., Owen (1975) and Dubey and Shapley (1979).

As we have already indicated games on antimatroids generalize cooperative games with an acyclic permission structure. In the *conjunctive approach* as developed in Gilles et al. (1992), it is assumed that each player needs permission from *all* its predecessors before it is allowed to cooperate. Alternatively, in the *disjunctive approach* as discussed in Gilles and Owen (1999) it is assumed that each player that has predecessors only needs permission from *at least one* of its predecessors before it is allowed to cooperate with other players. Algaba et al. (2000) show that for every acyclic permission structure, both the feasible coalition set derived from the conjunctive and disjunctive approaches are antimatroids. Moreover, the class of all sets of feasible coalitions that can be obtained as conjunctive feasible coalitions is exactly the class of poset antimatroids. The class of all sets of feasible coalitions that can be obtained as disjunctive feasible coalitions is exactly the class of antimatroids satisfying the so-called path property.

A solution for games with a permission structure is a function f that assigns a payoff distribution $f(N, v, S) \in \mathbb{R}^n$ to every cooperative game

(N, v) and permission structure S on N. The conjunctive Banzhaf permission value is obtained by applying the Banzhaf value to the conjunctive restricted games, while the disjunctive Banzhaf permission value is obtained by applying the Banzhaf value to the disjunctive restricted games. (See van den Brink (2000)).

The aim in the next sections will be to generalize axiomatizations given for the conjunctive and disjunctive Banzhaf permission values to obtain an axiomatization of the restricted Banzhaf value for cooperative games on antimatroids.

3 An axiomatization of the restricted Banzhaf value

In this section an axiomatization of the restricted Banzhaf value for games on antimatroids is provided. This result generalizes the axiomatizations of the conjunctive and disjunctive Banzhaf permission values given in van den Brink (2000). The first axiom is a weaker version of the usual efficiency axiom.

Axiom 1 (*One-player efficiency*). *For every cooperative game (N, v) and antimatroid $\mathcal{A}$ on N, if $N = \{i\}$ then $f_i(N, v, \mathcal{A}) = v(\{i\})$.*

The next three axioms are generalizations of corresponding axioms for cooperative games with a permission structure and are already stated in Algaba et al. (2003). For two cooperative games (N, v) and (N, w) the game $(N, v + w)$ is given by $(v + w)(E) = v(E) + w(E)$ for all $E \subseteq N$.

Axiom 2 (*Additivity*). *For every pair of cooperative games $(N, v), (N, w)$ and antimatroid $\mathcal{A}$ on N, $f(N, v + w, \mathcal{A}) = f(N, v, \mathcal{A}) + f(N, w, \mathcal{A})$.*

Axiom 3 (*Necessary player property*). *For every monotone cooperative game (N, v) and antimatroid $\mathcal{A}$ on N, if $i \in N$ satisfies $v(E) = 0$ for all $E \subseteq N \backslash \{i\}$ then $f_i(N, v, \mathcal{A}) \geq f_j(N, v, \mathcal{A})$ for all $j \in N$.*

Given the antimatroid $\mathcal{A}$ on N, the *basic path group P_i* of player i is given by those players that are in every i-path, i.e., $P_i = \bigcap_{E \in A(i)} E$. This set is formed by those players that totally control player i in $\mathcal{A}$, i.e., without them player i can not be in any feasible coalition. Obviously, $i \in P_i$ for all $i \in N$.

Axiom 4 (*Structural monotonicity*). *For every monotone cooperative game (N, v) and antimatroid $\mathcal{A}$ on N, if $j \in N$ then for all $i \in P_j$ we have $f_i(N, v, \mathcal{A}) \geq f_j(N, v, \mathcal{A})$.*

Before introducing three new axioms for cooperative games on antimatroids, we introduce some concepts. For cooperative game (N, v) and $j \in N$ define cooperative game $(N \backslash \{j\}, v_{-j})$ by $v_{-j}(E) = v(E)$ for all $E \subseteq N \backslash \{j\}$. For antimatroid $\mathcal{A}$ on N define the set system $\mathcal{A}_{-j}$ on $N \backslash \{j\}$ by $\mathcal{A}_{-j} = \{E \backslash \{j\}: E \in \mathcal{A}\}$. So, $\mathcal{A}_{-j}$ is obtained by deleting j from all coalitions in $\mathcal{A}$. Next we establish that $\mathcal{A}_{-j}$ is an antimatroid.

Lemma 1. *If $\mathcal{A}$ is an antimatroid on N and $j \in N$ then $\mathcal{A}_{-j}$ is an antimatroid on $N \backslash \{j\}$.*

Proof. (i) As $\emptyset \in \mathcal{A}$ we have that $\emptyset \in \mathcal{A}_{-j}$.

(ii) Suppose that $E, F \in \mathcal{A}_{-j}$. Then we can establish the following: (a) if $E, F \in \mathcal{A}$ then $E \cup F \in \mathcal{A}$, and (since $j \notin E \cup F$), $E \cup F \in \mathcal{A}_{-j}$; (b) if $\{E, F\} \not\subseteq \mathcal{A}$ then $E \cup F \cup \{j\} \in \mathcal{A}$, and thus $E \cup F \in \mathcal{A}_{-j}$. So, $\mathcal{A}_{-j}$ is closed under union.

(iii) Consider $E \in \mathcal{A}_{-j}$. We establish the following: (a) if $E \in \mathcal{A}$ then there exists $i \in E$ such that $E \backslash \{i\} \in \mathcal{A}$. Since $j \notin E$, it holds that $E \backslash \{i\} \in \mathcal{A}_{-j}$; (b) if $E \notin \mathcal{A}$ then $E \cup \{j\} \in \mathcal{A}$, implying that there exists $i \in E$ (clearly $i \neq j$) such that $(E \cup \{j\}) \backslash \{i\} \in \mathcal{A}$. But then $E \backslash \{i\} \in \mathcal{A}_{-j}$. So, $\mathcal{A}_{-j}$ satisfies accessibility.

(iv) Finally, it follows straightforward that $\mathcal{A}_{-j}$ satisfies normality, i.e., $N \backslash \{j\} \in \mathcal{A}_{-j}$. □

Given the antimatroid $\mathcal{A}$ on N, the *path group* P^i of player i is defined as the set of players that are in some i-path, i.e., $P^i = \bigcup_{E \in A(i)} E$. So, the path group of player i are all those players of which i has some dependence, i.e., these players in some sense partially control player i. Obviously, $P_i \subseteq P^i$ for all $i \in N$. Now, given an antimatroid $\mathcal{A}$ on N, we call $i \in N$ an *inessential player* for $\mathcal{A}$ in (N, v) if player i and every player $j \in N$ such that $i \in P^j$ are null players in (N, v).

The first of the three new axioms is related to the *null player out property* of the Banzhaf value (see Derks and Haller, 1994) and states in addition to the *inessential player property* as introduced in Algaba et al. (2003), that deleting an inessential player from a game on an antimatroid does not change the payoffs of the other players.

Axiom 5 (*Strong inessential player property*). *For every cooperative game (N, v) and antimatroid $\mathcal{A}$ on N, if i is an inessential player for $\mathcal{A}$ in (N, v) then*

$$f_j(N, v, \mathcal{A}) = \begin{cases} 0 & \text{if } j = i, \\ f_j(N \backslash \{i\}, v_{-i}, \mathcal{A}_{-i}) & \text{otherwise.} \end{cases}$$

As we have already pointed out the fairness axiom introduced in Algaba et al. (2003) generalizes both conjunctive and disjunctive *fairness* for games with a permission structure (see van den Brink 1997, 1999). In order to provide a characterization of the restricted Banzhaf value we need to strengthen this axiom in a way such that it also generalizes *predecessor fairness* for games with a permission structure (see van den Brink, 2000).

For antimatroid $\mathcal{A}$ on N and coalition $E \in \mathcal{A}$, let $au_{\mathcal{A}}(E)$ be the set of all augmentation points of E in $\mathcal{A}$. The new *strong fairness* axiom states that deleting the feasible coalition E from the set of feasible coalitions (as long as $\mathcal{A} \backslash \{E\}$ is still an antimatroid) changes the payoffs of all players in E by the same amount and, moreover, the payoffs of the augmentation points of E in $\mathcal{A}$ also change by this same amount but in opposite direction with respect to the elements in E.

Axiom 6 (*Strong fairness*). *For every cooperative game (N, v), antimatroid $\mathcal{A}$ on N, and $E \in \mathcal{A}$ with $|E| \geq 2$ such that $\mathcal{A} \backslash \{E\}$ is an antimatroid on N, it holds that for all $i \in E$ and $j \in au_{\mathcal{A}}(E)$*

$$f_i(N, v, \mathcal{A}) - f_i(N, v, \mathcal{A} \backslash \{E\}) = f_j(N, v, \mathcal{A} \backslash \{E\}) - f_j(N, v, \mathcal{A}).$$

The following result (given in Algaba et al. (2003)) establishes under what conditions a coalition E can be deleted, $\mathcal{A}\backslash\{E\}$ being still an antimatroid.

Lemma 2. *Let $\mathcal{A}$ be an antimatroid on N and $E \in \mathcal{A}$. Then, $\mathcal{A}\backslash\{E\}$ is an antimatroid on N if and only if E is a path, $E \notin \{\emptyset, N\}$ and every $F \in \mathcal{A}$ satisfying $E \subseteq F$ and $|F| = |E| + 1$* is not a path.

The last axiom is based on the proxy neutrality and amalgamation properties of the Banzhaf value as considered in Lehrer (1988) and Haller (1994). For antimatroid $\mathcal{A}$ on N and player $j \in N$ define $\mathcal{A}_j$ on N by $\mathcal{A}_j = \{E \in \mathcal{A}: j \in E\}$, i.e., $\mathcal{A}_j$ is the set of coalitions in $\mathcal{A}$ that contain player j. Further, for player $h \in \mathbb{N}\backslash N$ we define $\mathcal{A}_{hj}$ by

$$\mathcal{A}_{hj} = \{E \cup \{h\}: E \in \mathcal{A}_j\} \cup \{E\backslash\{j\} \cup \{h\}: E \in \mathcal{A}_j\} \cup (\mathcal{A}\backslash\mathcal{A}_j).$$

Note that $\mathcal{A}\backslash\mathcal{A}_j$ need not be equal to $\mathcal{A}_{-j}$. In some sense player $h \in \mathbb{N}\backslash N$ is controlling the player j when going from antimatroid $\mathcal{A}$ to $\mathcal{A}_{hj}$. Feasible coalitions containing player j now should also contain player h. Moreover, all feasible coalitions in $\mathcal{A}$ that contain player j are still feasible if we replace player j by player h. Finally, all feasible coalitions in $\mathcal{A}$ that do not contain player j stay feasible. We first establish that $\mathcal{A}_{hj}$ is an antimatroid.

Lemma 3. *If $\mathcal{A}$ is an antimatroid on N, then for every $j \in N$ and $h \in \mathbb{N}\backslash N$, $\mathcal{A}_{hj}$ is an antimatroid on $N \cup \{h\}$.*

Proof. (i) If $\emptyset \in \mathcal{A}$ then $\emptyset \in \mathcal{A}\backslash\mathcal{A}_j$, and thus $\emptyset \in \mathcal{A}_{hj}$.

(ii) For $E \in \mathcal{A}_{hj}$ we establish that: (a) if $h \notin$ then $j \notin E$, and thus $E \in \mathcal{A}\backslash\mathcal{A}_j$; (b) if $h \in E$ and $j \notin E$ then $E\backslash\{h\} \cup \{j\} \in \mathcal{A}$; (c) if $h \in E$ and $j \in E$ then $E\backslash\{h\} \cup \{j\} = E\backslash\{h\} \in \mathcal{A}$.
Suppose that $E, F \in \mathcal{A}_{hj}$. With the cases (a), (b) and (c) it follows that: (1) if $h \notin \cup F$ then $E, F \in \mathcal{A}\backslash\mathcal{A}_j$, and thus $E \cup F \in \mathcal{A}\backslash\mathcal{A}_j \subseteq \mathcal{A}_{hj}$; (2) if $h \in E \cup F$ then $[E\backslash\{h\} \cup \{j\}, F\backslash\{h\} \cup \{j\} \in \mathcal{A}]$ or $[E\backslash\{h\} \in \mathcal{A}\backslash\mathcal{A}_j$ and $F\backslash\{h\} \cup \{j\} \in \mathcal{A}]$ or $[E\backslash\{h\} \cup \{j\} \in \mathcal{A}$ and $F\backslash\{h\} \in \mathcal{A}\backslash\mathcal{A}_j]$. In all these cases $(E \cup F \cup \{j\})\backslash\{h\} \in \mathcal{A}$. But then $E \cup F \in \mathcal{A}_{hj}$. So, $\mathcal{A}_{hj}$ is closed under union.

(iii) For $E \in \mathcal{A}_{hj}$ we have that: (a) if $j \in E$ then $h \in E$, and thus $E\backslash\{j\} \in \mathcal{A}_{hj}$; (b) if $j \notin E$ and $h \notin E$ then $E \in \mathcal{A}\backslash\mathcal{A}_j$, and thus there exists $i \in E$ such that $E\backslash\{i\} \in \mathcal{A}\backslash\mathcal{A}_j \subseteq \mathcal{A}_{hj}$; (c) if $j \notin E$ and $h \in E$ then $E \cup \{j\}\backslash\{h\} \in \mathcal{A}$ and thus there exists $k \in E \cup \{j\}\backslash\{h\}$ such that $E \cup \{j\}\backslash\{h, k\} \in \mathcal{A}$. If $k = j$ then $E \cup \{j\}\backslash\{h, k\} = E\backslash\{h\} \in \mathcal{A}\backslash\mathcal{A}_j \subseteq \mathcal{A}_{hj}$. Otherwise $k \neq j$ and $E \cup \{j\}\backslash\{h, k\} \in \mathcal{A}$, implying that $E\backslash\{k\} \in \mathcal{A}_{hj}$. So, $\mathcal{A}_{hj}$ satisfies accessibility.

(iv) Finally, it follows straightforward that $\mathcal{A}_{hj}$ satisfies normality, i.e., $N \cup \{h\} \in \mathcal{A}_{hj}$. □

Besides letting player h control player j in the antimatroid we also let h control j in the game by requiring the presence of h for any non-zero contribution of j. This is the only contribution of player h. This means that we consider the game $(N \cup \{h\}, v_{hj})$ with v_{hj} on $N \cup \{h\}$ given by $v_{hj}(E) = v(E\backslash\{j\})$ if $h \notin E$, and $v_{hj}(E) = v(E\backslash\{h\})$ otherwise.

The next axiom is required for a special class of antimatroids. Note that an antimatroid always has at least $|N| + 1$ elements. If $|\mathcal{A}| = |N| + 1$ then there is a unique coalition in $\mathcal{A}$ of cardinality i from $i = 1$ until $i = n$. So, there exists a unique i-path for every player i. We refer to such an antimatroid as a *chain* antimatroid.

For chain antimatroids we require that the sum of payoffs of players $h \in \mathbb{N} \backslash N$ and $j \in N$ after letting player h control player j is equal to the payoff of player j when h is not yet present.

Axiom 7 (*Proxy neutrality*). *For every cooperative game (N, v), chain antimatroid $\mathcal{A}$ on N, $j \in N$ and $h \in \mathbb{N} \backslash N$, it holds that*

$$f_h(N \cup \{h\}, v_{hj}, \mathcal{A}_{hj}) + f_j(N \cup \{h\}, v_{hj}, \mathcal{A}_{hj}) = f_j(N, v, \mathcal{A}).$$

Applying this axiom to games with a permission structure yields an axiom that is weaker than *vertical neutrality* and neither weaker nor stronger than *horizontal neutrality* for games with a permission structure (see van den Brink, 2000). A stronger version of proxy neutrality, namely the one which states that $f_h(N \cup \{h\}, v_{hj}, \mathcal{A}_{hj}) + f_j(N \cup \{h\}, v_{hj}, \mathcal{A}_{hj}) = f_j(N, v, \mathcal{A})$ for any antimatroid $\mathcal{A}$ on N and players $j \in N$ and $h \in \mathbb{N} \backslash N$, generalizes both vertical and horizontal neutrality for games with a permission structure. Although this stronger version is satisfied by the restricted Banzhaf value for games on antimatroids, for the characterization result presented in Theorem 2 it is sufficient to require this axiom just for chain antimatroids. Since we use the stronger version in our application in Section 5, we also formally state this version and prove that the restricted Banzhaf value satisfies this axiom.

Axiom 8 (*Strong proxy neutrality*). *For every cooperative game (N, v), antimatroid $\mathcal{A}$ on N, $j \in N$ and $h \in \mathbb{N} \backslash N$, it holds that*

$$f_h(N \cup \{h\}, v_{hj}, \mathcal{A}_{hj}) + f_j(N \cup \{h\}, v_{hj}, \mathcal{A}_{hj}) = f_j(N, v, \mathcal{A}).$$

The next result yields that the restricted Banzhaf value satisfies all axioms discussed above.

Theorem 1. *The restricted Banzhaf value $\bar{B}$ for cooperative games on antimatroids satisfies one-player efficiency, additivity, the necessary player property, structural monotonicity, the strong inessential player property, strong fairness and strong proxy neutrality.*

Proof. Let (N, v) be a cooperative game and $\mathcal{A}$ be an antimatroid on N.

1. If $N = \{i\}$ then $\mathcal{A} = \{\emptyset, N\}$, and thus $v = v_{\mathcal{A}}$. One-player efficiency then follows from one-player efficiency of the Banzhaf value for cooperative games.

2, 3 and 4. Additivity, the necessary player property and structural monotonicity of the restricted Banzhaf value follow in a similar way as for the restricted Shapley value in Algaba, Bilbao, van den Brink and Jiménez-Losada (2003). (To show the necessary player property and structural monotonicity take 'Banzhaf' weights $p_e = \frac{1}{2^{|N|-1}}$ instead of 'Shapley' weights $p'_e = \frac{(|N|-|E|)!(|E|-1)!}{|N|!}$).

5. If i is an inessential player for $\mathcal{A}$ in (N, v) then i is a null player in $(N, v_{\mathcal{A}})$. The null player property of the Banzhaf value then implies that $f_i(N, v, \mathcal{A}) = 0$. From the null player out property of the Banzhaf value

(see Derks and Haller, 1994) it follows that $f_j(N, v, \mathcal{A}) = f_j(N \backslash \{i\}, v_{-i}, \mathcal{A}_{-i})$ for $j \neq i$.

6. Let $E \in \mathcal{A}$ be such that $\mathcal{A} \backslash \{E\}$ is an antimatroid on N. For $i \in E$:

   (a) If $E \nsubseteq F$ then

$$int_{\mathcal{A}}(F) = \bigcup_{\{H \in \mathcal{A} : H \subseteq F\}} H = \bigcup_{\{H \in \mathcal{A} \backslash \{E\} : H \subseteq F\}} H = int_{\mathcal{A} \backslash \{E\}}(F).$$

   Hence, $v_{\mathcal{A}}(F) = v_{\mathcal{A} \backslash \{E\}}(F)$. In particular, as $i \in E$ it holds that $E \nsubseteq F \backslash \{i\}$ and therefore $v_{\mathcal{A}}(F \backslash \{i\}) = v_{\mathcal{A} \backslash \{E\}}(F \backslash \{i\})$.

   (b) If there exists $j \in au_{\mathcal{A}}(E)$ such that $E \cup \{j\} \subseteq F$ then $int_{\mathcal{A}}(F) = int_{\mathcal{A} \backslash \{E\}}(F)$. Hence also in this case, $v_{\mathcal{A}}(F) = v_{\mathcal{A} \backslash \{E\}}(F)$.

   Since $i \in E$ and $E \subseteq F$ implies that $i \in F$, it follows for $i \in E$ and $j \in au_{\mathcal{A}}(E)$ that

$$\begin{aligned}
&\bar{B}_i(N, v, \mathcal{A}) - \bar{B}_i(N, v, \mathcal{A} \backslash \{E\}) \\
&\quad = B_i(N, v_{\mathcal{A}}) - B_i(N, v_{\mathcal{A} \backslash \{E\}}) \\
&\quad = \frac{1}{2^{|N|-1}} \sum_{\{F \subseteq N : i \in F\}} \big(v_{\mathcal{A}}(F) - v_{\mathcal{A}}(F \backslash \{i\}) - v_{\mathcal{A} \backslash \{E\}}(F) + v_{\mathcal{A} \backslash \{E\}}(F \backslash \{i\})\big) \\
&\quad = \frac{1}{2^{|N|-1}} \sum_{\{F \subseteq N : E \subseteq F, F \cap au_{\mathcal{A}}(E) = \emptyset\}} \big(v_{\mathcal{A}}(F) - v_{\mathcal{A} \backslash \{E\}}(F)\big) \\
&\quad = \frac{1}{2^{|N|-1}} \sum_{\{F \subseteq N : E \subseteq F, F \cap au_{\mathcal{A}}(E) = \{j\}\}} \big(v_{\mathcal{A}}(F \backslash \{j\}) - v_{\mathcal{A} \backslash \{E\}}(F \backslash \{j\})\big) \\
&\quad = \frac{1}{2^{|N|-1}} \sum_{\{F \subseteq N : j \in F\}} \big(v_{\mathcal{A}}(F \backslash \{j\}) - v_{\mathcal{A} \backslash \{E\}}(F \backslash \{j\}) + v_{\mathcal{A} \backslash \{E\}}(F) - v_{\mathcal{A}}(F)\big) \\
&\quad = B_j\big(N, v_{\mathcal{A} \backslash \{E\}}\big) - B_j(N, v_{\mathcal{A}}) \\
&\quad = \bar{B}_j(N, v, \mathcal{A} \backslash \{E\}) - \bar{B}_j(N, v, \mathcal{A}),
\end{aligned}$$

   showing that $\bar{B}$ satisfies strong fairness.

7. Let $j \in N$ and $h \in \mathbb{N} \backslash N$. We establish the following facts:

   (a) if $\{h, j\} \nsubseteq F$ then $v_{hj}(H) = v_{hj}(H \backslash \{h, j\}) = v(H \backslash \{h, j\})$ for all $H \subseteq F$.

   (b) since $F \in \mathcal{A} \backslash \mathcal{A}_j$ implies that $h \notin F$, and $F \subseteq \mathcal{A}_{hj} \backslash \mathcal{A}_j$ implies that $h \in F$, we have for $F \subseteq N \cup \{h\}$ and $h \in F$ that

$$(v_{hj})_{\mathcal{A}_{hj}}(F) = v_{hj}\big(int_{\mathcal{A}_{hj}}(F)\big) = v_{hj}\left(\bigcup_{\{H \in \mathcal{A}_{hj} : H \subseteq F\}} H\right)$$

$$= v_{hj}\left(\bigcup_{\substack{H \in \{E \cup \{h\} | E \in \mathcal{A}_j\} \\ H \subseteq F}} H \cup \bigcup_{\substack{H \in \{E \backslash \{j\} \cup \{h\} | E \in \mathcal{A}_j\} \\ H \subseteq F}} H \cup \bigcup_{\substack{H \in \mathcal{A} \backslash \mathcal{A}_j \\ H \subseteq F}} H\right)$$

$$
\begin{aligned}
&= v_{hj}\left(\bigcup_{\{H\in\mathcal{A}_j:H\subseteq F\}}(H\cup\{h\})\cup\bigcup_{\{H\in\mathcal{A}\setminus\mathcal{A}_j:H\subseteq F\}}H\right)\\
&= v\left(\bigcup_{\{H\in\mathcal{A}_j:H\subseteq F\setminus\{h\}\}}H\cup\bigcup_{\{H\in\mathcal{A}\setminus\mathcal{A}_j:H\subseteq F\setminus\{h\}\}}H\right)\\
&= v\left(\bigcup_{\{H\in\mathcal{A}:H\subseteq F\setminus\{h\}\}}H\right)\\
&= v(int_{\mathcal{A}}(F\setminus\{h\})) = v_{\mathcal{A}}(F\setminus\{h\}),
\end{aligned}
$$

and

$$
\begin{aligned}
(v_{hj})\mathcal{A}_{hj}(F\setminus\{h\}) &= v_{hj}\big(int_{\mathcal{A}_{hj}}(F\setminus\{h\})\big) = v_{hj}\left(\bigcup_{\{H\in\mathcal{A}_{hj}:H\subseteq F\setminus\{h\}\}}H\right)\\
&= v_{hj}\left(\bigcup_{\substack{H\in\{E\cup\{h\}|E\in\mathcal{A}_j\}\\ H\subseteq F\setminus\{h\}}}H\cup\bigcup_{\substack{H\in\{E\setminus\{j\}\cup\{h\}|E\in\mathcal{A}_j\}\\ H\subseteq F\setminus\{h\}}}H\cup\bigcup_{\substack{H\in\mathcal{A}\setminus\mathcal{A}_j\\ H\subseteq F\setminus\{h\}}}H\right)\\
&= v_{hj}\left(\bigcup_{\{H\in\mathcal{A}\setminus\mathcal{A}_j:H\subseteq F\setminus\{h\}\}}H\right) = v\left(\bigcup_{\{H\in\mathcal{A}\setminus\mathcal{A}_j:H\subseteq F\setminus\{h\}\}}H\right)\\
&= v\left(\bigcup_{\{H\in\mathcal{A}:H\subseteq F\setminus\{h,j\}\}}H\right)\\
&= v(int_{\mathcal{A}}(F\setminus\{h,j\})) = v_{\mathcal{A}}(F\setminus\{h,j\}).
\end{aligned}
$$

With this it follows that

$$
\begin{aligned}
&\bar{B}_h(N\cup\{h\},v_{hj},\mathcal{A}_{hj}) + \bar{B}_j(N\cup\{h\},v_{hj},\mathcal{A}_{hj})\\
&\quad = B_h(N\cup\{h\},(v_{hj})_{\mathcal{A}_{hj}}) + B_j(N\cup\{h\},(v_{hj})_{\mathcal{A}_{hj}})\\
&\quad = \frac{1}{2^{|N|}}\sum_{\{F\subseteq N\cup\{h\}:h\in F\}}\Big((v_{hj})_{\mathcal{A}_{hj}}(F) - (v_{hj})_{\mathcal{A}_{hj}}(F\setminus\{h\})\Big)\\
&\qquad + \frac{1}{2^{|N|}}\sum_{\{F\subseteq N\cup\{h\}:j\in F\}}\Big((v_{hj})_{\mathcal{A}_{hj}}(F) - (v_{hj})_{\mathcal{A}_{hj}}(F\setminus\{j\})\Big)\\
&\quad = \frac{1}{2^{|N|}}\sum_{\{F\subseteq N\cup\{h\}:\{h,j\}\subseteq F\}}\Big((v_{hj})_{\mathcal{A}_{hj}}(F) - (v_{hj})_{\mathcal{A}_{hj}}(F\setminus\{h\})\Big)\\
&\qquad + \frac{1}{2^{|N|}}\sum_{\{F\subseteq N\cup\{h\}:\{h,j\}\subseteq F\}}\Big((v_{hj})_{\mathcal{A}_{hj}}(F) - (v_{hj})_{\mathcal{A}_{hj}}(F\setminus\{j\})\Big)
\end{aligned}
$$

$$=\frac{1}{2^{|N|}}\sum_{\{F\subseteq N\cup\{h\}:\{h,j\}\subseteq F\}}(v_{\mathcal{A}}(F\backslash\{h\})-v_{\mathcal{A}}(F\backslash\{h,j\}))$$
$$+\frac{1}{2^{|N|}}\sum_{\{F\subseteq N\cup\{h\}:\{h,j\}\subseteq F\}}(v_{\mathcal{A}}(F\backslash\{h\})-v_{\mathcal{A}}(F\backslash\{h,j\}))$$
$$=\frac{1}{2^{|N|}}\left(\sum_{\{F\subseteq N:j\in F\}}(v_{\mathcal{A}}(F)-v_{\mathcal{A}}(F\backslash\{j\}))+\sum_{\{F\subseteq N:j\in F\}}(v_{\mathcal{A}}(F)-v_{\mathcal{A}}(F\backslash\{j\}))\right)$$
$$=\frac{1}{2^{|N|-1}}\sum_{\{F\subseteq N:j\in F\}}((v_{\mathcal{A}}(F)-v_{\mathcal{A}}(F\backslash\{j\}))=B_j(N,v_{\mathcal{A}})=\bar{B}_j(N,v,\mathcal{A}),$$

showing that $\bar{B}$ satisfies strong proxy neutrality. □

To prove uniqueness we first prove the following lemma.

Lemma 4. *Let $\mathcal{A}$ be an antimatroid on N and let $i,j\in N$.* If $i\in P^j\backslash P_j$ then there is an $E\in\mathcal{A}$ *with $i,j\in E$ such that $\mathcal{A}\backslash\{E\}$ is an antimatroid.*

Proof. Let $i\in P^j\backslash P_j$, then j has at least two paths one of them contain i and the other path does not contain i. Let $E\in A(j)$, $E\neq N$ be a j-path containing i. By Lemma 2 one of the following two cases happen: $\mathcal{A}\backslash\{E\}$ is an antimatroid or there exists $k_1\in N\backslash E$ with $E\cup\{k_1\}$ a k_1-path. If $\mathcal{A}\backslash\{E\cup\{k_1\}\}$ is an antimatroid the proof is finished, otherwise applying again Lemma 2 there exists k_2 with $E\cup\{k_1,k_2\}$ a k_2-path. Repeating this reasoning and assuming that in each step the path obtained is not removable then we obtain a chain of paths from E to N (with at least one coalition different from E). By proving that this case is not possible, the lemma will be shown. Let F be a j-path such that $i\notin F$ and let k be the last element in F in incorporating to E in the chain above and H the path in this step of the chain. We have $H\neq E$ is a k-path, $E\cup F\subseteq H$ and $k\in F$. This is a contradiction because there exists a k-path H' contained in F and therefore in some other k-path, since H contains i and H' does not. □

The axioms discussed above characterize the restricted Banzhaf value for cooperative games on antimatroids.

Theorem 2. *A solution f for cooperative games on antimatroids is equal to the restricted Banzhaf value $\bar{B}$ if and only if it satisfies one-player efficiency, additivity, the necessary player property, structural monotonicity, the strong inessential player property, strong fairness and proxy neutrality.*

Proof. To prove uniqueness, suppose that solution f satisfies the seven axioms. Consider antimatroid $\mathcal{A}$ on N and the monotone game $w_T=c_Tu_T$, $c_T\geq 0$, where u_T is the unanimity game of $T\subseteq N$, i.e., $w_T(E)=c_T$ if $E\supseteq T$, and $w_T(E)=0$ otherwise. If $c_T=0$ then the strong inessential player property implies that $f_i(N,w_T,\mathcal{A})=0$ for all $i\in N$. Suppose that $c_T>0$. We show that $f(N,w_T,\mathcal{A})$ is uniquely determined by induction on $|N|$.

If $N=\{i\}$ then $f(N,w_T,\mathcal{A})$ is uniquely determined by one-player efficiency. Proceeding by induction assume that $f(N',w_T,\mathcal{A})$ is uniquely determined if $|N'|<|N|$.

We use a second induction step on $|\mathcal{A}|$. (Note that $|\mathcal{A}|\geq|N|+1$). Consider the sets $P^T=\bigcup_{i\in T}P^i$ and $P_T=\bigcup_{i\in T}P_i$.

1. If $|\mathcal{A}| = |N| + 1$ then there is a unique coalition in $\mathcal{A}$ of cardinality i from $i = 1$ until $i = n$. So, there exists a unique i-path for every player i. In this case, we have that $P^T = P_T$. We distinguish the following two cases:

   (i) Suppose that $P^T \neq N$. Then there exist inessential players. Take an inessential player j. The strong inessential player property implies that $f_j(N, w_T, \mathcal{A}) = 0$ and $f_i(N, w_T, \mathcal{A}) = f_i(N\backslash\{j\}, (w_T)_{-j}, \mathcal{A}_{-j})$ for every $i \in N\backslash\{j\}$. By the induction hypothesis $f_i(N, w_T, \mathcal{A})$ is uniquely determined for all $i \in N\backslash\{j\}$.
   (ii) Suppose that $P^T = N$. Then there exist $i_0, i_1 \in N$ such that $P^{i_0} = N$ and $P^{i_1} = P^{i_0}\backslash\{i_0\}$. Since $(\mathcal{A}_{-i_1})_{i_1 i_0} = \mathcal{A}$ and $\left(w_{T\backslash\{i_i\}}\right)_{i_1 i_0} = w_T$, proxy neutrality implies that

$$f_{i_1}(N, w_T, \mathcal{A}) + f_{i_0}(N, w_T, \mathcal{A}) = f_{i_0}\left(N\backslash\{i_1\}, w_{T\backslash\{i_1\}}, \mathcal{A}_{-i_1}\right). \qquad (1)$$

   Structural monotonicity and the necessary player property imply that there exists $c \in \mathbb{R}$ such that $f_i(N, w_T, \mathcal{A}) = c$ for all $i \in P_T = N$. With (1) it follows that $c = \frac{1}{2} f_{i_0}\left(N\backslash\{i_1\}, w_{T\backslash\{i_1\}}, \mathcal{A}_{-i_1}\right)$. With the induction hypothesis c is uniquely determined, and so is $f(N, w_T, \mathcal{A})$.

2. Now we suppose that $|\mathcal{A}| > |N| + 1$. Proceeding by induction assume that $f\left(N, w_T, \mathcal{A}'\right)$ is uniquely determined if $\left|\mathcal{A}'\right| < |\mathcal{A}|$. Again we distinguish the following two cases:

   (i) Suppose that $P^T \neq N$. Then $f(N,\ w_T,\ \mathcal{A})$ is uniquely determined by the strong inessential player property and the induction hypothesis similarly as shown above for the case $|\mathcal{A}| = |N| + 1$.
   (ii) Suppose that $P^T = N$. If $i \in P_T$ then structural monotonicity and the necessary player property imply that there exists $c \in \mathbb{R}$ such that $f_i(N, w_T, \mathcal{A}) = c$.

   If $i \in P^T\backslash P_T$ then there exists $j \in T$ with $i \in P^j\backslash P_j$. By Lemma 4 there exists an $E \in \mathcal{A}$ with $i, j \in E$ such that $\mathcal{A}\backslash\{E\}$ is an antimatroid. Applying strong fairness then yields that $f_i(N, w_T, \mathcal{A}) = c - c_i$ with $c_i = f_j\left(N, w_T, \mathcal{A}'\right) - f_i\left(N, w_T, \mathcal{A}'\right)$ for an $\mathcal{A}'$ with $\left|\mathcal{A}'\right| = |\mathcal{A}| - 1$.

   On the other hand, as $E \neq N$ there exists $k \in au_{\mathcal{A}}(E)$ and thus $k \notin E$. Applying strong fairness, it holds for $i \in E$ that

$$f_i(N, w_T, \mathcal{A}) - f_i\left(N, w_T, \mathcal{A}'\right) = f_k\left(N, w_T, \mathcal{A}'\right) - f_k(N, w_T, \mathcal{A}),$$

   where $\mathcal{A}' = \mathcal{A}\backslash\{E\}$. Since $f_k(N, w_T, \mathcal{A}) = c - c_k$, denoting by

$$\overline{c}_k = f_i\left(N, w_T, \mathcal{A}'\right) + f_k\left(N, w_T, \mathcal{A}'\right)$$

   it follows that $c - c_i - f_i(N, w_T, \mathcal{A}') = f_k(N, w_T, \mathcal{A}') - c + c_k$ implying that $c = \frac{1}{2}(\overline{c}_k + c_k + c_i)$ (like c_i, c_k and $\overline{c}_k$ are determined by the induction hypothesis). With cases (i) and (ii) above $f_i(N, w_T, \mathcal{A})$ is determined for all $i \in N$.

Above we showed that $f(N, w_T, \mathcal{A})$ is uniquely determined for all (monotone) games $w_T = c_T u_T$ with $c_T \geq 0$. Suppose that $w_T = c_T u_T$ with $c_T < 0$. (Then w_T is not monotone and the necessary player property and structural monotonicity cannot be applied.) Let $v_0 \in \mathcal{G}^N$ denote the *null game*, i.e., $v_0(E) = 0$ for all $E \subseteq N$. From the strong inessential player property it follows that $f_i(N, v_0, \mathcal{A}) = 0$ for all $i \in N$. Since $-w_T = -c_T u_T$ with $-c_T \geq 0$, and $(v_0)_{\mathcal{A}} = (w_T)_{\mathcal{A}} + \left(-(w_T)_{\mathcal{A}}\right)$, it follows from additivity of f that $f(N, w_T, \mathcal{A}) = f(N, v_0, \mathcal{A}) - f(N, -w_T, \mathcal{A}) = -f(N, -w_T, \mathcal{A})$ is uniquely determined because $-w_T$ is monotone. So, $f(N, c_T u_T, \mathcal{A})$ is uniquely determined for all $c_T \in \mathbb{R}$. Since every cooperative game v on N can be expressed as a linear combination of unanimity games it follows with additivity that $f(N, v, \mathcal{A})$ is uniquely determined. □

We end this section by showing logical independence of the axioms stated in Theorem 1.

1. The zero solution given by $f_i(N, v, \mathcal{A}) = 0$ for all $i \in N$ satisfies additivity, the necessary player property, structural monotonicity, the strong inessential player property, strong fairness and proxy neutrality. It does not satisfy one-player efficiency.
2. Let $\Delta(v) = \max\{d_v(T) : T \subseteq N\}$, where

$$d_v(T) = \sum_{F \subseteq T} (-1)^{|T|-|F|} v(F)$$

is the *Harsanyi dividend for T* and $D(v) = \{T \subseteq N : d_v(T) = \Delta(v)\}$. The solution given by $f(N, v, \mathcal{A}) = \bar{B}\left(N, \sum_{T \in D(v)} d_v(T) u_T, \mathcal{A}\right)$ satisfies one-player efficiency, the necessary player property, structural monotonicity, the strong inessential player property, strong fairness and proxy neutrality. It does not satisfy additivity.
3. Let $f_i(N, v, \mathcal{A}) = v(N) - v(N \backslash \{i\})$ if $\{i\} \in \mathcal{A}$ and $f_i(N, v, \mathcal{A}) = 0$ otherwise. This solution satisfies one-player efficiency, additivity, structural monotonicity, the strong inessential player property, strong fairness and proxy neutrality. It does not satisfy the necessary player property.
4. The solution given by $f(N, v, \mathcal{A}) = B(N, v)$ satisfies one-player efficiency, additivity, the necessary player property, the strong inessential player property, strong fairness and proxy neutrality. It does not satisfy structural monotonicity.
5. The solution

$$f_i(N, v, \mathcal{A}) = \frac{v(N)}{2^{|N|-1}}$$

for all $i \in N$, satisfies one-player efficiency, additivity, the necessary player property, structural monotonicity, strong fairness and proxy neutrality. It does not satisfy the strong inessential player property.
6. For antimatroid $\mathcal{A}$ on N and player $i \in N$, consider $P_i = \bigcap_{E \in \mathcal{A}(i)} E$. To every antimatroid $\mathcal{A}$ on N we associate the poset antimatroid $\mathcal{A}^P$ that is obtained by taking all coalitions that can be obtained as unions of coalitions in $\{P_i : i \in N\}$. The solution $f_i(N, v, \mathcal{A}) = \bar{B}(N, v, \mathcal{A}^P)$ satisfies one-player efficiency, additivity, the necessary player property, structural monotonicity, the strong inessential player property and proxy neutrality. It does not satisfy strong fairness.

7. The solution $f_i(N, v, \mathcal{A}) = v(\{i\})$ if $\{i\} \in \mathcal{A}$ and $f_i(N, v, \mathcal{A}) = 0$ otherwise satisfies one-player efficiency, additivity, the necessary player property, structural monotonicity, the strong inessential player property and strong fairness. It does not satisfy proxy neutrality.

4 Poset antimatroids

Deleting strong fairness from the set of axioms stated in Theorem 1 the remaining set of axioms characterizes the restricted Banzhaf value for games on poset antimatroids. Moreover, poset antimatroids are the unique antimatroids for which it is possible to delete the strong fairness axiom. In order to state this result we first have to show that the antimatroids $\mathcal{A}_{-j}$ and $\mathcal{A}_{hj}$, where $j \in N$, $h \in \mathbb{N}\backslash N$ are poset antimatroids whenever $\mathcal{A}$ is a poset antimatroid.

Lemma 5 *Let $\mathcal{A}$ be a poset antimatroid on N. If $j \in N$ and $h \in \mathbb{N}\backslash N$ then* (i) *$\mathcal{A}_{-j}$ is a poset antimatroid on $N\backslash\{j\}$, and* (ii) *$\mathcal{A}_{hj}$ is a poset antimatroid on $N \cup \{h\}$.*

Proof. From Lemma's 1 and 3 it already follows that $\mathcal{A}_{-j}$, and $\mathcal{A}_{hj}$ are antimatroids. So, we only have to show that they are closed under intersection.

(i) Suppose that $E, F \in \mathcal{A}_{-j}$. (a) if $E, F \in \mathcal{A}$ then $E \cap F \in \mathcal{A}$, and (since $j \notin E \cap F$), $E \cap F \in \mathcal{A}_{-j}$. (b) If $\{E, F\} \not\subseteq \mathcal{A}$ then assume without loss of generality that $E \cup \{j\} \in \mathcal{A}$. If also $F \cup \{j\} \in \mathcal{A}$ then $(E \cap F) \cup \{j\} = (E \cup \{j\}) \cap (F \cup \{j\}) \in \mathcal{A}$, and thus $E \cap F \in \mathcal{A}_{-j}$. Otherwise, if $F \cup \{j\} \notin \mathcal{A}$ then $j \notin F$ and $F \in \mathcal{A}$, which implies that $j \notin E \cap F = (E \cup \{j\}) \cap F \in \mathcal{A}$, and thus $E \cap F \in \mathcal{A}_{-j}$. So, $\mathcal{A}_{-j}$ is closed under intersection.

(ii) Suppose that $E, F \in \mathcal{A}_{hj}$. (a) If $h \in E \cap F$ then $E\backslash\{h\} \cup \{j\}, F\backslash\{h\} \cup \{j\} \in \mathcal{A}$, and thus $E \cap [F\backslash\{h\} \cup \{j\}] = [E\backslash\{h\} \cup \{j\}] \cap [F\backslash\{h\} \cup \{j\}] \in \mathcal{A}$. But then $E \cap F \in \mathcal{A}_{hj}$. (b) If $h \notin E \cap F$ then assume without loss of generality that $h \notin E$. Then $E \in \mathcal{A}\backslash\mathcal{A}_j$. If also $h \notin F$ then $E, F \in \mathcal{A}\backslash\mathcal{A}_j$, and thus $E \cap F \in \mathcal{A}\backslash\mathcal{A}_j \subseteq \mathcal{A}_{hj}$. Otherwise, if $h \in F$ then $F\backslash\{h\} \cup \{j\} \in \mathcal{A}$. But then $j \notin E \cap F = E \cap (F\backslash\{h\}) = E \cap [F\backslash\{h\} \cup \{j\}] \in \mathcal{A}$. But then $E \cap F \in \mathcal{A}_{hj}$. So, $\mathcal{A}_{hj}$ is closed under intersection. □

Note that poset antimatroids are the unique antimatroids such that every player has a unique path. In particular, we can conclude that given an antimatroid $\mathcal{A}$ on N, then $\mathcal{A}$ is a poset antimatroid if and only if $P^i = P_i$, for all $i \in N$.

Theorem 3. *A solution f for games on poset antimatroids is equal to the restricted Banzhaf value if and only if it satisfies one-player efficiency, additivity, the necessary player property, structural monotonicity, the strong inessential player property and proxy neutrality.*

Proof. Suppose that solution f satisfies the six axioms on poset antimatroids. Consider a poset antimatroid $\mathcal{A}$ on N and the game $w_T = c_T u_T$, $c_T \geq 0$. Taking into account that for a poset antimatroid $P^T = P_T$ the proof follows of the first part from Theorem 2. For arbitrary v it follows that $f(N, v, \mathcal{A})$ is uniquely determined in a similar way as in the proof of Theorem 2. □

Hence, on poset antimatroids the restricted Banzhaf value is uniquely determined without using strong fairness. The solutions 1, 2, 3, 4, 5 and 7 given at the end of the previous section show logical independence of the axioms stated in Theorem 3.

In Algaba et al. (2000) it is shown that for every $i \in N$,

$$\bar{B}_i(N, v, \mathcal{A}) = \sum_{E \in \mathcal{A}_i} \frac{d_{v_{\mathcal{A}}}(E)}{2^{|E|-1}}.$$

Next, we remark that for games on poset antimatroids the restricted Banzhaf value can be written using dividends of the original game as follows.

Proposition 1. *If $\mathcal{A}$ is a poset antimatroid on N then*

$$\bar{B}_i(N, v, \mathcal{A}) = \sum_{\{T \subseteq N : i \in P^T\}} \frac{d_v(T)}{2^{|P^T|-1}}.$$

Proof. Since $d_v(E) = 0$ for every $E \notin \mathcal{A}$ it follows that

$$\bar{B}_i(N, v, \mathcal{A}) = \sum_{E \in \mathcal{A}_i} \frac{d_{v_{\mathcal{A}}}(E)}{2^{|E|-1}} = \sum_{E \in \mathcal{A}_i} \frac{\sum_{\{T \subseteq E : E = P^T\}} d_v(T)}{2^{|E|-1}}$$

$$= \sum_{\{T \subseteq N : i \in P^T\}} \frac{d_v(T)}{2^{|P^T|-1}}. \qquad \square$$

We can characterize the class of cooperative games on poset antimatroids as the class of games on which the restricted Banzhaf value satisfies the six axioms of Theorem 3. Given an antimatroid $\mathcal{A}$ on N, let $\bar{B}(\cdot, \mathcal{A})$ be the function that assigns to every cooperative game (N, v) the restricted Banzhaf value $\bar{B}(N, v, \mathcal{A})$.

Theorem 4. *Suppose $\mathcal{F}$ is a set of antimatroids. Then $\mathcal{F}$ contains only poset antimatroids if and only if the restricted Banzhaf value is the unique solution on $\mathcal{F}$ satisfying one-player efficiency, additivity, the necessary player property, structural monotonicity, the strong inessential player property and proxy neutrality.*

Proof. In a similar way as the proof of Theorem 3 it can be shown that on a set of poset antimatroids $\mathcal{F}$ the restricted Banzhaf value is the unique solution satisfying one-player efficiency, additivity, the necessary player property, structural monotonicity, the strong inessential player property and proxy neutrality. Suppose that $\mathcal{A} \in \mathcal{F}$ is not a poset antimatroid. Define the solution g for cooperative games on antimatroids by

$$g_i(N, u_T, \mathcal{A}) = \begin{cases} \frac{1}{2^{|P_T|-1}} & \text{if } i \in P_T = \bigcup_{i \in T} P_i, \\ 0 & \text{otherwise,} \end{cases}$$

and for arbitrary game v

$$g_i(N, v, \mathcal{A}) = \sum_{T \subseteq N} d_v(T) g_i(N, u_T, \mathcal{A}) = \sum_{\{T \subseteq N : i \in P_T\}} \frac{d_v(T)}{2^{|P_T|-1}}.$$

This solution satisfies one-player efficiency, additivity, the necessary player property, structural monotonicity, the strong inessential player property and proxy neutrality.

To prove that $g \neq \bar{B}$ note that, if $\mathcal{A}$ is not a poset antimatroid then there exists $j \in N$ with $P^j \neq P_j$. By Proposition 1 it then follows that $g(N, u_T, \mathcal{A}) \neq \bar{B}(N, u_T, \mathcal{A})$ if $j \notin T$ and $(P^j \backslash P_j) \cap T \neq \emptyset$. □

An acyclic permission structure is a *permission forest structure* if every player has at most one predecessor. A permission forest structure is a *permission tree structure* if there is exactly one player i_0 which does not have any predecessor. Algaba et al. (2000, Lemma 2) showed that the permission forest structures are exactly those acyclic permission structures for which the sets of conjunctive and disjunctive feasible coalitions coincide. We also showed that the poset antimatroids satisfying the path property are exactly those antimatroids that can be obtained as the set of conjunctive or disjunctive feasible coalitions of some permission forest structure. From Theorem 4 we directly obtain a characterization of the Banzhaf value restricted to the class of poset antimatroids satisfying the path property, i.e., antimatroids that are obtained as the feasible coalitions for permission forest or tree structures. This result is interesting from an economic point of view since in economic theory we often encounter hierarchical structures that can be represented by *forests* or *trees* (see, e.g., a hierarchically structured firm).

Corollary 1. *Let $\mathcal{F}$ be a set of a poset antimatroids on N satisfying the path property. Then the restricted Banzhaf value is the unique solution on $\mathcal{F}$ satisfying one-player efficiency, additivity, the necessary player property, structural monotonicity, the strong inessential player property and proxy neutrality.*

5 An application: auction situations

We have already pointed out that an antimatroid can be the conjunctive feasible coalition set as well as the disjunctive feasible coalition set of some permission structure if and only if it is a poset antimatroid satisfying the path property (see Algaba et al. (2000)). A special class of such antimatroids are the feasible sets of *peer group situations* as considered in Branzei et al. (2002). In fact, they consider games with permission tree structure (N, v, S). The games v assign zero dividends to all coalitions that are not paths. The restricted *peer group game* then coincides with the (conjunctive or disjunctive) restricted game arising from this game with permission (tree) structure. Given that all coalitions that are not paths get a zero dividend, the restricted game is equal to the game itself. (Note that the same restricted game is obtained if we consider the game that assigns to every player i the dividend of its path P_i). Defining one-player efficiency, additivity, the necessary player property, structural monotonicity, the strong inessential player property and proxy neutrality restricted to this class we characterize the restricted Banzhaf value on this class.

As argued by Branzei et al. (2002) peer group situations generalize some other situations such as sealed bid second price auction situations (see Rasmusen, 1994). Consider a seller of an object who has a reservation value $r \geq 0$, and a set $N = \{1, \ldots, n\}$ of n bidders. Each bidder has a valuation

$w_i \geq r$ for the object. Assume that the bidders are labelled such that $w_1 > \cdots > w_n$. Using dominant bidding strategies for such auction situations Branzei, Fragnelli and Tijs (2002) define the corresponding peer group situation that can be represented as the game with permission (tree) structure (N, v, S) with $S(i) = \{i+1\}$ for $1 \leq i \leq n-1$, $S(n) = \emptyset$, and the game v determined by the dividends $d_v(\{1, \ldots, i\}) = w_i - w_{i+1}$ if $1 \leq i \leq n-1$ and $d_v(N) = w_n - r$. All other coalitions have a zero dividend. Clearly these games are determined by the set N of players and the vector of valuations $(w, r) \in \mathbb{R}_+^{n+1}$. Given such a valuation vector let $\bar{B}(N, w, r)$ denote the restricted Banzhaf value of the corresponding game on the antimatroid $\mathcal{A} = \{\emptyset, \{1\}, \{1,2\}, \ldots, \{1,2,\ldots,n-1\}, N\}$. Allowing the strict inequalities to be weak inequalities $w_1 \geq \cdots \geq w_n$, applying the axioms stated above to these situations yields that one-player efficiency straightforwardly says that $f_i(\{i\}, w, r) = w_i - r$. Structural monotonicity states that $f_i(N, w, r) \geq f_j(N, w, r)$ if $w_i \geq w_j$. The strong inessential player property states that if $w_i = r$, then $f_i(N, w, r) = 0$ and $f_j(N, w, r) = f_j(N \backslash \{i\}, w_{-i}, r)$ for all $j \neq i$, where $(w_{-i}, r) \in \mathbb{R}_+^n$ is given by $(w_{-i})_j = w_j$ for all $j \neq i$. Specifying additivity we must take care that the underlying permission structure does not change. So, we require additivity only for reservation value vectors that 'preserve the order of players', i.e., $f(N, w+z, r+s) = f(N, w, r) + f(N, z, s)$ if $w_i \geq w_j \Leftrightarrow z_i \geq z_j$. We refer to this as *additivity over order preserving valuations*.

The neutrality axiom that we use for auction situations establishes that given $h \in \mathbb{N} \backslash N$, $f_h(N \cup \{h\}, w', r) + f_j(N \cup \{h\}, w', r) = f_j(N, w, r)$ if $(w', r) \in \mathbb{R}_+^{n+2}$ is given by $w'_j = w_{j-1}$, $w'_h = w_j$ and $w'_i = w_i$ for all $i \neq j, h$. Note that this neutrality means that player j participates as representative of a hierarchical society formed by two players h, j where player h is superior to j. In this case, the value is neutral with respect to the negotiation with the superior h or with the society formed by h and j. Notice that although the auction follows the same above process where j, in some sense, is the representative of the society, the feasible coalitions do not form a chain in this case. Therefore the neutrality axiom described above is a special case of strong proxy neutrality, but not of proxy neutrality. For example, consider an auction situation with five players. In this case, the feasible coalitions are represented in Figure 1. Suppose that player 3 is the representative in the hierarchical society formed by 3 and 6 where 6 is superior to 3, in others words 6 controls 3. This information, leads to the rest of bidders think about the feasible coalitions of Figure 2, where besides negotiation with player 3 and therefore with the superior 6, also negotiations directly with 6 appear.

Finally, we can give a characterization of the restricted Banzhaf value for auction situations without using the necessary player property.

Theorem 5. *The restricted Banzhaf value* $\bar{B}(N, w, r)$ *is the unique solution for auction situations* $(w, r) \in \mathbb{R}_+^{n+1}$ *satisfying one-player efficiency, structural monotonicity, the strong inessential player property, additivity over order preserving valuations and strong proxy neutrality. Moreover,*

$$\bar{B}_i(N, w, r) = \frac{w_i}{2^{i-1}} - \sum_{h=i+1}^{n} \frac{w_h}{2^{h-1}} - \frac{r}{2^{n-1}}.$$

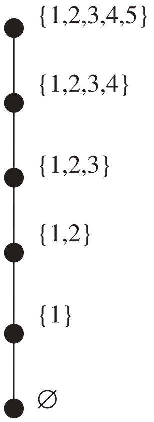

Figure. 1

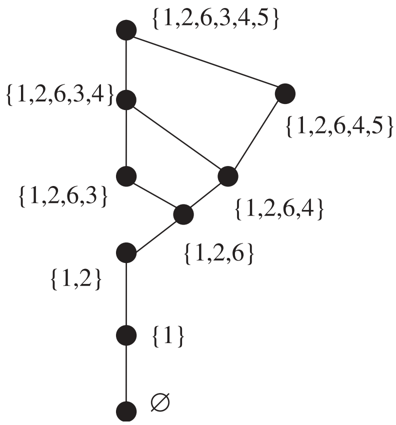

Figure. 2

Proof. For auction situations, $\bar{B}(N, w, r)$ is characterized by the axioms from Corollary 1 . However, we have to prove that we do not have to use the necessary player property and need additivity only over order preserving valuations. Suppose that f is a solution for auction situations that satisfies the axioms, and let $(w, r) \in \mathbb{R}_+^{n+1}$ be an auction situation. For $k = 1, \ldots, n-1$ define the auction situation $(w^k, 0)$ by $w_i^k = w_k - w_{k+1}$ for all $i \in \{1, \ldots, k\}$, $w_i^k = 0$ for all $i \in \{k+1, \ldots, n\}$, and define (w^n, r) by $w_i^n = w_n$ for all $i \in \{1, \ldots, n\}$. Let $k \in \{1, \ldots, n-1\}$. Structural monotonicity implies that all $f_i(N, w^k, 0)$ are equal for all $i \in \{1, \ldots, k\}$, i.e., $f_i(N, w^k, 0) = c_k,\ 1 \leq i \leq k$ for some $c_k \in \mathbb{R}$. On the other hand, structural monotonicity also implies that $f_i(N, w^n, r) = c_n$, $i \in N$, for some $c_n \in \mathbb{R}$. The strong inessential player property implies that $f_i(N, w^k, 0) = 0$ for all $i \in \{k+1, \ldots, n\}$.

If $N = \{i\}$ then $f(N, w, r)$ is uniquely determined by one-player efficiency.

Proceeding by induction assume that $f(N', w, r)$ is uniquely determined if $|N'| < |N|$.

(i) For $k = 1, \ldots, n-1$ consider the auction situation $(w^k, 0)$. Then there exist inessential players. Take an inessential player j. The strong inessential player property implies that $f_j(N, w^k, 0) = 0$ and $f_i(N, w^k, 0) = f_i\left(N \backslash \{j\}, w_{-j}^k, 0\right)$ for $i \in N \backslash \{j\}$. Thus by the induction hypothesis $f_i(N, w^k, 0)$ is uniquely determined for all $i \in N \backslash \{j\}$.
(ii) For $k = n$ consider (w^n, r). Then, strong proxy neutrality implies that for $h, j \in N$,

$$f_h(N, w^n, r) + f_j(N, w^n, r) = f_j\left(N \backslash \{h\}, w_{-h}^n, r\right). \tag{2}$$

With (2) it follows that $c_n = \frac{1}{2} f_j\left(N \backslash \{h\}, w_{-h}^n, r\right)$. With the induction hypothesis c_n is uniquely determined, and so is $f(N, w^n, r)$.

Since all $(w^k, 0)$, $k = 1, \ldots, n-1$, and (w^n, r) are order preserving, additivity over order preserving valuations determines $f(N, w, r)$.

Let $(w, r) \in \mathbb{R}_+^{n+1}$ be an auction situation. Then its corresponding poset antimatroid is $\mathcal{A} = \{\emptyset, \{1\}, \ldots, \{1, 2, \ldots, n-1\}, N\}$. It follows from Proposition 1 that

$$\bar{B}_i(w,r) = \sum_{\{T \subseteq N : i \in P^T\}} \frac{d_v(T)}{2^{|P^T|-1}}$$
$$= \frac{w_i - w_{i+1}}{2^{i-1}} + \sum_{k=i+1}^{n-1} \frac{w_k - w_{k+1}}{2^{k-1}} + \frac{w_n - r}{2^{n-1}}$$
$$= \frac{w_i}{2^{i-1}} - \sum_{k=i+1}^{n} \frac{w_k}{2^{k-1}} - \frac{r}{2^{n-1}}. \qquad \square$$

From the proof of the above theorem it follows that structural monotonicity could be replaced by *symmetry* stating that $f_i(N,w,r) = f_j(N,w,r)$ if $w_i = w_j$. Note that this cannot be done in more general cases as discussed earlier in the paper. In a similar way we can characterize solutions for other economic situations such as *airport games* or *hierarchically structured firms*.

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