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The Banzhaf power index for ternary bicooperative games

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ABSTRACT

In this paper we analyze ternary bicooperative games, which are a refinement of the concept of a ternary voting game introduced by Felsenthal and Machover. Furthermore, majority voting rules based on the difference of votes are simple bicooperative games. First, we define the concepts of the defender and detractor swings for a player. Next, we introduce the Banzhaf power index and the normalized Banzhaf power index. The main result of the paper is an axiomatization of the Banzhaf power index for the class of ternary bicooperative games. Moreover, we study ternary bicooperative games with two lists of weights and compute the Banzhaf power index using generating functions.

1. Introduction

A cooperative game with a transferable utility is given by a finite set of players and a real-valued worth function defined on the set of all the subsets, or *coalitions*, of players such that the worth of the empty set is zero. For each coalition, the worth can be interpreted as the maximal gain or minimal cost that the players in this coalition can achieve by themselves against the best offensive threat by the complementary coalition. Classical market games for economies with private goods are examples of cooperative games. We say that such a game has *orthogonal coalitions* (see [18, Chapter 9]).

Games with non-orthogonal coalitions are games in which the worth of a coalition depends on the actions of its complementary coalition. Clearly, social situations involving externalities and public goods are such cases. For instance, the joint owners of a building are considering hiring a gardener to work in the common areas of their residence. The garden is a public good. Each owner can decide to support the proposal or to veto it. However, some of them may decide not to take part in the decision making and would thus not necessarily be *defenders* or *detractors* of the project.

Situations of this kind may be modeled in the following manner. We consider ordered pairs of disjoint coalitions of players. Each such pair yields a partition of the set of all players in three groups. Players in the first coalition are in favor of the proposal, and players in the second coalition object to it. The remaining players are not convinced of its benefits, but they have no intention of objecting to it. This leads us in a natural way into the concept of a *bicooperative game* introduced by Bilbao [3].

The analysis of the distribution of power in voting systems is the main application of the concept of a simple game. Two power indices have received the most theoretical attention as well as application to political structures. The first such power index was proposed by Shapley and Shubik [19] and it depends on the number of permutations of the set of players in which each player is pivotal. The second power index was introduced by Banzhaf [2] and has been used in various legal proceedings. The Banzhaf power index depends on the number of ways in which each voter can effect a swing and the axiomatic characterization of the Banzhaf power index makes special use of a certain number that intuitively it represents the total power available in the game.

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A simple game allows a voter only two responses in any decision: voting 'yes' or 'no'. But in most real-life decision rules, abstention plays a key role. The simple games cannot take the possibility of abstention into account and they do not recognize abstention as a third option. Thus, only knowing who is in favor of the proposal is not enough to describe the situation. Simple games with abstention have been studied by Felsenthal and Machover [10] under the name of *ternary voting games*. More recently, several works by Freixas [11,12] and Freixas and Zwicker [13], have been devoted to the study of voting systems with several ordered levels of approval in the input and in the output. Chua and Huang [8] have studied the Shapley–Shubik index for ternary voting games. An axiomatic characterization of the Shapley value for bicooperative games was introduced by Bilbao et al. [5], by using the approach by Weber [22].

Let us briefly outline the contents of this paper. Section 2 deals with notation, definitions and the formal description of the bicooperative games. In Section 3 we introduce ternary bicooperative games and we observe some properties of these games. In Section 4 the notions of defender swing and detractor swing are introduced and the Banzhaf power index and its normalization are defined for these games. We obtain an axiomatization of the Banzhaf power index in this context. Some of these axioms are extensions of the classical axioms for the Banzhaf power index in the cooperative case. In the last section we introduce a new model of the voting system by using ternary bicooperative games. This model takes into account situations in which the power of a player to block a decision is not equal to the power of this player to approve it. Finally we compute the swings for a ternary bicooperative game with two lists of weights using generating functions.

Let S be a finite set. As it became common practice, given $i \in S$ we will for simplicity write $S \setminus i$ instead of $S \setminus \{i\}$, and given $i \notin S$ we will write $S \cup i$ instead of $S \cup \{i\}$. The number of players in S is denoted by $|S|$ or s.

2. Bicooperative games

Let $N = \{1, \dots, n\}$ be a finite set and we define the set

$$3^N = \{(A, B) : A, B \subseteq N,\ A \cap B = \emptyset\}.$$

Grabisch and Labreuche [14] proposed a relation in 3^N given by

$$(A, B) \sqsubseteq (C, D) \Longleftrightarrow A \subseteq C,\ B \supseteq D.$$

We denote by $\sqsubset$ the relation defined by means of the weak strict inclusion, that is, $(A, B) \sqsubset (C, D)$ if and only if ($A \subset C,\ B \supseteq D$) or ($A \subseteq C,\ B \supset D$). Let us consider the following ordered 3-partitions defined by

$$X = (A, N \setminus (A \cup B), B) \quad \text{and} \quad Y = (C, N \setminus (C \cup D), D).$$

For two ordered 3-partitions X, Y, Freixas and Zwicker [13, Section 2] write $X\ {}^3\!\subseteq Y$ to mean that either $X = Y$ or X may be transformed into Y by shifting 1 or more voters to higher levels of approval. Then $(A, B) \sqsubseteq (C, D) \Longleftrightarrow X\ {}^3\!\subseteq Y$ and so the relation $\sqsubseteq$ coincides with the inclusion ${}^3\!\subseteq$ for 3-partitions.

The set $(3^N, \sqsubseteq)$ is a partially ordered set (poset) with the following properties:

1. $(\emptyset, N)$ is the first element: $(\emptyset, N) \sqsubseteq (A, B)$ for all $(A, B) \in 3^N$.
2. $(N, \emptyset)$ is the last element: $(A, B) \sqsubseteq (N, \emptyset)$ for all $(A, B) \in 3^N$.
3. Every pair of elements of 3^N has a join $(A, B) \vee (C, D) = (A \cup C, B \cap D)$ and a meet $(A, B) \wedge (C, D) = (A \cap C, B \cup D)$.

Moreover, $(3^N, \sqsubseteq)$ is a finite distributive lattice. Two pairs (A, B) and (C, D) are comparable if $(A, B) \sqsubseteq (C, D)$ or $(C, D) \sqsubseteq (A, B)$. Otherwise, (A, B) and (C, D) are incomparable. A chain of 3^N is an induced subposet of 3^N in which any two elements are comparable. In $(3^N, \sqsubseteq)$, all maximal chains have the same number of elements and this number is $2n + 1$. Thus, the *rank function* $\rho : 3^N \to \{0, 1, \dots, 2n\}$ can be considered such that $\rho[(\emptyset, N)] = 0$ and $\rho[(S, T)] = \rho[(A, B)] + 1$ if (S, T) covers (A, B), that is, $(A, B) \sqsubset (S, T)$ but there is no $(H, J) \in 3^N$ such that $(A, B) \sqsubset (H, J) \sqsubset (S, T)$.

The following results were proved in [5], by using several lattice properties (see [20, Section 3.5]). These results will be used to justify one of the axioms in the axiomatic characterization of the Banzhaf power index in Section 4.

Proposition 1. *The number of maximal chains of* 3^N *is* $(2n)!/2^n$*, where* $n = |N|$.

Proposition 2. *For all* $(A, B) \in 3^N$*, the number of maximal chains of the sublattice* $[(\emptyset, N), (A, B)]$ *is* $(n + a - b)!/2^a$*, where* $a = |A|$ *and* $b = |B|$.

Proposition 3. *Let* $(A, B),\ (C, D) \in 3^N$ *with* $(A, B) \sqsubseteq (C, D)$*. The number of maximal chains of the sublattice* $[(A, B), (C, D)]$ *is equal to the number of maximal chains of the sublattice* $[(D, C), (B, A)]$.

Hereafter we denote by $c(3^N)$ the number of maximal chains in 3^N and by $c\,([(A, B), (C, D)])$ the number of maximal chains in the sublattice $[(A, B), (C, D)]$.

We model above mentioned class of non-orthogonal situations by means of the set of all ordered pairs of disjoint coalitions, that is, the set 3^N and a worth function $b : 3^N \to \mathbb{R}$. For each $(S, T) \in 3^N$, the number $b(S, T)$ can be interpreted as the gain (whenever $b(S, T) > 0$) or loss (whenever $b(S, T) < 0$) that S can achieve when T is the opposer coalition and $N \setminus (S \cup T)$ is the neutral coalition. The pair $(\emptyset, N)$ represents the situation if all the players object to the change and $(N, \emptyset)$ represents the situation where all the players wish the change.

Definition 1. A bicooperative game is a pair (N, b), where N a finite set of players and $b : 3^N \to \mathbb{R}$ is a function such that $b(\emptyset, \emptyset) = 0$.

A bicooperative game $b \in \mathcal{BG}^N$ is *monotonic* if for any $(S_1, T_1), (S_2, T_2)$ in 3^N with $(S_1, T_1) \sqsubseteq (S_2, T_2)$, we have $b(S_1, T_1) \leq b(S_2, T_2)$.

Grabisch and Labreuche [14,15] study the *bicapacities*, which coincide with our definition of monotonic bicooperative games although both concepts were proposed independently and for different domains. A bicapacity is a function $v : 3^N \to \mathbb{R}$ such that $v(\emptyset, \emptyset) = 0$ and $A \subseteq B \subseteq N$ implies $v(A, \cdot) \leq v(B, \cdot)$ and $v(\cdot, A) \geq v(\cdot, B)$. The bicapacities can be considered as bicooperative games if the monotonicity assumption is no required.

We denote by $\mathcal{BG}^N$ the set of all bicooperative games on N, that is, $\mathcal{BG}^N = \{b : 3^N \to \mathbb{R},\ b(\emptyset, \emptyset) = 0\}$. With respect to the addition and multiplication by real numbers, the set $\mathcal{BG}^N$ is a vector space. There are some special collections of games in $\mathcal{BG}^N$ taking values in $\{-1, 0, 1\}$, the *superior unanimity games* and the *inferior unanimity games* which are defined, for any $(S, T) \in 3^N$, with $(S, T) \neq (\emptyset, \emptyset)$ as follows.

The superior unanimity game $\overline{u}_{(S,T)} : 3^N \to \mathbb{R}$ is given by

$$\overline{u}_{(S,T)}(A, B) = \begin{cases} 1 & \text{if } (S, T) \sqsubseteq (A, B),\ (A, B) \neq (\emptyset, \emptyset), \\ 0 & \text{otherwise.} \end{cases}$$

The inferior unanimity game $\underline{u}_{(S,T)} : 3^N \to \mathbb{R}$ is defined by

$$\underline{u}_{(S,T)}(A, B) = \begin{cases} -1 & \text{if } (A, B) \sqsubseteq (S, T),\ (A, B) \neq (\emptyset, \emptyset), \\ 0 & \text{otherwise.} \end{cases}$$

The relevance of these collections of games is made clear in the following result (see [6]).

Proposition 4. *The collections* $\left\{\overline{u}_{(S,T)} : (S, T) \in 3^N, (S, T) \neq (\emptyset, \emptyset)\right\}$ *and* $\left\{\underline{u}_{(S,T)} : (S, T) \in 3^N, (S, T) \neq (\emptyset, \emptyset)\right\}$ *are basis of* $\mathcal{BG}^N$. *The dimension of* $\mathcal{BG}^N$ *is* $3^n - 1$.

3. Ternary bicooperative games

Similarly to the cooperative case in which each coalition $S \in 2^N$ can be identified with a $\{0, 1\}$-vector, each coalition $(S, T) \in 3^N$ can be identified with the $\{-1, 0, 1\}$-vector $\mathbf{1}_{(S,T)}$ defined, for all $i \in N$, by

$$\mathbf{1}_{(S,T)}(i) = \begin{cases} 1 & \text{if } i \in S, \\ -1 & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

In voting games, each voter has three choices: voting for a proposal, voting against it, and abstaining. Thus, only knowing who is in favor of the proposal is not enough to describe the situation. Felsenthal and Machover [10] generalize the standard voting games by recognizing abstention as an option alongside *yes* and *no* votes. Ternary voting games are formally described by the mappings $u : 3^N \to \{-1, 1\}$ satisfying the following three conditions: $u(N, \emptyset) = 1, u(\emptyset, N) = -1$, and $\mathbf{1}_{(S,T)}(i) \leq \mathbf{1}_{(S',T')}(i)$ for all $i \in N$, implies $u(S, T) \leq u\left(S', T'\right)$. A negative outcome, -1, is interpreted as a defeat and a positive outcome, 1, as a victory, the passing of a bill. The proposal of Felsenthal and Machover could be refined by introducing a third output for u, which is 0, and represents the 'no decision' situation. Thus, our definition of ternary bicooperative game is as follows.

Definition 2. A game $b \in \mathcal{BG}^N$ is called a ternary bicooperative game if it satisfies the following conditions:

1. For every bicoalition $(S, T) \in 3^N$, its worth $b(S, T) \in \{-1, 0, 1\}$.
2. If $(S, T), \left(S', T'\right) \in 3^N$ with $(S, T) \sqsubseteq \left(S', T'\right)$, then $b(S, T) \leq b\left(S', T'\right)$.

Freixas [11,12] and Freixas and Zwicker [13] have generalized the ternary voting games by the definition of the so-called (j, k) *simple games*. In the (j, k) simple games, each individual voter expresses one of j possible levels of input support, and the output consists of one of k possible levels of collective support. Standard simple games are $(2, 2)$ simple games, while a ternary bicooperative game is a $(3, 3)$ simple game such that $b(\emptyset, N) = -1$, $b(\emptyset, \emptyset) = 0$, and $b(N, \emptyset) = 1$.

Let N be a set of n voters that has to choose between a pair of two alternatives x and y. Every voter $i \in N$ has a preference $R_i \in \{-1, 0, 1\}$ over the two alternatives x and y, where $R_i = 1$ means that voter i prefers x to y, $R_i = 0$ means that voter i is indifferent, and $R_i = -1$ means that voter i prefers y to x. Thus, the set of preferences is 3^N. The aggregate preference is obtained by means a *social welfare function* $F : 3^N \to \{-1, 0, 1\}$ (see [17]). For instance, we can consider the following majority rules based on the difference of votes (see [16]). Given $k \in \{0, 1, \ldots, n-1\}$, the M_k majority rule is the social welfare function $M_k : 3^N \to \{-1, 0, 1\}$, defined by

$$M_k(S, T) = \begin{cases} 1 & \text{if } |S| - |T| > k, \\ -1 & \text{if } |T| - |S| > k, \\ 0 & \text{if } -k \leq |S| - |T| \leq k. \end{cases}$$

It is easy to check that M_k is a ternary bicooperative game for any $k \in \{0, 1, \ldots, n-1\}$.

Note in particular that M_0 is the majority rule defined by $M_0(S,T) = \text{sgn}(|S| - |T|)$, where sgn (x) is the standard sign-function for real numbers x with sgn $(x) = 1$ for $x > 0$, sgn $(x) = -1$ for $x < 0$, and sgn $(x) = 0$ for $x = 0$. Notice also that for $k = n-1$ we obtain the unanimous rule defined by $M_{n-1}(N, \emptyset) = 1$, $M_{n-1}(\emptyset, N) = -1$, and $M_{n-1}(S,T) = 0$ otherwise.

We denote by $\mathcal{TBG}^N$ the class of all ternary bicooperative games and define the internal operation meet and join by

$$(b_1 \vee b_2)(S,T) = \max\{b_1(S,T), b_2(S,T)\},$$
$$(b_1 \wedge b_2)(S,T) = \min\{b_1(S,T), b_2(S,T)\}.$$

It is easy to check that $(b_1 \vee b_2) + (b_1 \wedge b_2) = b_1 + b_2$.

In a ternary bicooperative game, a bicoalition $(S,T) \in 3^N$ is a *defender bicoalition* if $b(S,T) = 1$, and a bicoalition $(S,T) \in 3^N$ is a *detractor bicoalition* if $b(S,T) = -1$. We denote by $\mathcal{W}^D$ the set of all defender bicoalitions and by $\mathcal{W}_D$ the set of all detractor bicoalitions. We say that a bicoalition (S,T) is a *minimal defender* if it is a defender, and there does not exist any defender bicoalition contained in (S,T). A bicoalition (S,T) is a *maximal detractor* if it is a detractor, and there does not exist any detractor bicoalition such that (S,T) is contained in it.

We denote by $\mathcal{MW}^D$ the set of all minimal defender bicoalitions. In general, if there exists at least one defender bicoalition, we write

$$\mathcal{MW}^D = \{(S_1, T_1), \ldots, (S_r, T_r)\},$$

where $r \geq 1$. We denote by $\mathcal{MW}_D$ the set of all maximal detractor bicoalitions. If there exists at least one detractor bicoalition, we write

$$\mathcal{MW}_D = \left\{\left(S_1', T_1'\right), \ldots, (S_k', T_k')\right\}$$

where $k \geq 1$. Note that it could be $\mathcal{MW}^D = \emptyset$ or $\mathcal{MW}_D = \emptyset$, but both sets are empty only if the bicooperative game is the null game.

In the following result we establish a decomposition of each $b \in \mathcal{TBG}^N$ in terms of the inferior and superior unanimity bicooperative games corresponding to the maximal detractor and minimal defender bicoalitions.

Notice that the collections $\left\{\overline{u}_{(S,T)} : (S,T) \in 3^N, (S,T) \neq (\emptyset,\emptyset)\right\}$ and $\left\{\underline{u}_{(S,T)} : (S,T) \in 3^N, (S,T) \neq (\emptyset,\emptyset)\right\}$ include non monotonic bicooperative games. Notice also that if $(S,T) \in \mathcal{MW}^D$ and $(S',T') \in \mathcal{MW}_D$, then $\overline{u}_{(S,T)}$ and $\underline{u}_{(S',T')}$ are monotonic games.

Proposition 5. *Let $b \in \mathcal{TBG}^N$ such that $\mathcal{MW}^D = \{(S_1,T_1), \ldots, (S_r,T_r)\}$ and $\mathcal{MW}_D = \left\{\left(S_1',T_1'\right), \ldots, \left(S_k',T_k'\right)\right\}$ are the sets of minimal defender and maximal detractor bicoalitions. Then b can be written as $b = v + v'$, where*

$$v = \overline{u}_{(S_1,T_1)} \vee \cdots \vee \overline{u}_{(S_r,T_r)} \quad \textit{and} \quad v' = \underline{u}_{(S_1',T_1')} \wedge \cdots \wedge \underline{u}_{(S_k',T_k')}.$$

Proof. Let $b \in \mathcal{TBG}^N$ satisfying the hypothesis. We consider the following three cases:
1. Let $(S,T) \in 3^N$ such that $b(S,T) = -1$. Since b is monotonic, $(S_l,T_l) \not\sqsubseteq (S,T)$ for all $(S_l,T_l) \in \mathcal{MW}^D$, and hence $\overline{u}_{(S_l,T_l)}(S,T) = 0$ for all $1 \leq l \leq r$. From the definition of v it follows that

$$\begin{aligned} v(S,T) &= \left(\overline{u}_{(S_1,T_1)} \vee \cdots \vee \overline{u}_{(S_r,T_r)}\right)(S,T) \\ &= \max\left\{\overline{u}_{(S_1,T_1)}(S,T), \ldots, \overline{u}_{(S_r,T_r)}(S,T)\right\} = 0. \end{aligned}$$

Since $(S,T) \in 3^N$ is a detractor bicoalition, there exists $(S_j',T_j') \in \mathcal{MW}_D$ with $(S,T) \sqsubseteq \left(S_j',T_j'\right)$, and hence $\underline{u}_{\left(S_j',T_j'\right)}(S,T) = -1$. Then

$$\begin{aligned} v'(S,T) &= \left(\underline{u}_{(S_1',T_1')} \wedge \cdots \wedge \underline{u}_{(S_k',T_k')}\right)(S,T) \\ &= \min\left\{\underline{u}_{(S_1',T_1')}(S,T), \ldots, \underline{u}_{(S_k',T_k')}(S,T)\right\} = -1. \end{aligned}$$

2. Let $(S,T) \in 3^N$ such that $b(S,T) = 0$. Since b is monotonic, we have that $(S_l,T_l) \not\sqsubseteq (S,T)$ for all $1 \leq l \leq r$, and also that $(S,T) \not\sqsubseteq \left(S_j',T_j'\right)$ for all $1 \leq j \leq k$. Hence $\overline{u}_{(S_l,T_l)}(S,T) = 0$ for all $1 \leq l \leq r$, and $\underline{u}_{\left(S_j',T_j'\right)}(S,T) = 0$ for all $1 \leq j \leq k$. Then

$$v(S,T) = \left(\overline{u}_{(S_1,T_1)} \vee \cdots \vee \overline{u}_{(S_r,T_r)}\right)(S,T) = 0,$$
$$v'(S,T) = \left(\underline{u}_{(S_1',T_1')} \wedge \cdots \wedge \underline{u}_{(S_k',T_k')}\right)(S,T) = 0.$$

3. Let $(S,T) \in 3^N$ such that $b(S,T) = 1$. Since $(S,T) \in 3^N$ is a defender bicoalition, there exists $(S_l,T_l) \in \mathcal{MW}^D$ such that $(S_l,T_l) \sqsubseteq (S,T)$, and hence $\overline{u}_{(S_l,T_l)}(S,T) = 1$. Since b is monotonic, this implies $(S,T) \not\sqsubseteq \left(S_j',T_j'\right)$ and hence $\underline{u}_{\left(S_j',T_j'\right)}(S,T) = 0$ for all $1 \leq j \leq k$. Then

$$v(S,T) = \left(\overline{u}_{(S_1,T_1)} \vee \cdots \vee \overline{u}_{(S_r,T_r)}\right)(S,T) = 1,$$
$$v'(S,T) = \underline{u}_{(S_1',T_1')} \wedge \cdots \wedge \underline{u}_{(S_k',T_k')}(S,T) = 0.$$

Therefore, the decomposition $b = v + v'$ is proved. ■

4. Banzhaf power index for ternary bicooperative games

A *value* on $\mathcal{BG}^N$ is a function $\Psi : \mathcal{BG}^N \to \mathbb{R}^n$, which associates to each bicooperative game b a vector $(\Psi_1(b), \ldots, \Psi_n(b))$ which represents the 'a priori' value that every player has in the game b. The value $\sum_{i\in N} \Psi_i(b)$ represents the total power available in the game. In order to define a reasonable value for ternary bicooperative games, we consider that a player i estimates his participation in game b, evaluating his marginal contributions $b(S \cup i, T) - b(S, T)$ in those bicoalitions $(S \cup i, T)$ that are formed from (S, T) when i joins S, and also his marginal contributions $b(S, T) - b(S, T \cup i)$ in those (S, T) that are formed when i leaves $T \cup i$. The possible values of the sum of these marginal contributions lead us to define the concepts of a defender swing and a detractor swing.

Let $i \in N$. We say that a bicoalition $(S, T) \in 3^{N\setminus i}$ is a *defender swing* for player i if $b\,(S \cup i, T) = 1$ and $b\,(S, T \cup i) \neq 1$, and we say that a bicoalition $(S, T) \in 3^{N\setminus i}$ is a *detractor swing* for player i if $b\,(S, T \cup i) = -1$ and $b(S\cup i, T) \neq -1$. In both cases, the sum of the marginal contributions $b(S \cup i, T) - b(S, T)$ and $b(S, T) - b(S, T \cup i)$ is greater that or equal to 1. Note that a bicoalition $(S, T) \in 3^{N\setminus i}$ can be a defender swing and a detractor swing for a player i. In this case $b(S\cup i, T) - b(S, T \cup i) = 2$.

We denote by $\overline{\eta}_i(b)$ the number of defender swings for player i and by $\underline{\eta}_i(b)$ to the number of detractor swings for player i. Let $\eta_i(b) = \overline{\eta}_i(b) + \underline{\eta}_i(b)$ be the number of swings for player i and let $\eta(b) = \sum_{i\in N} \eta_i(b)$.

Now, we define the collections

$$\overline{\mathcal{W}}^D(i) = \left\{(S, T) \in \mathcal{W}^D : i \in S\right\}, \qquad \underline{\mathcal{W}}^D(i) = \left\{(S, T) \in \mathcal{W}^D : i \in T\right\},$$
$$\overline{\mathcal{W}}_D(i) = \{(S, T) \in \mathcal{W}_D : i \in S\}, \qquad \underline{\mathcal{W}}_D(i) = \{(S, T) \in \mathcal{W}_D : i \in T\}.$$

The following result provides an expression of the swings for a player in terms of the following sets of defender and detractor bicoalitions.

Proposition 6. *If $b \in \mathcal{TBG}^N$ and $i \in N$, then*

$$\overline{\eta}_i(b) = \left|\overline{\mathcal{W}}^D(i)\right| - \left|\underline{\mathcal{W}}^D(i)\right| \quad \text{and} \quad \underline{\eta}_i(b) = \left|\underline{\mathcal{W}}_D(i)\right| - \left|\overline{\mathcal{W}}_D(i)\right|.$$

Proof. For every $b \in \mathcal{TBG}^N$, we obtain

$$\begin{aligned}\overline{\eta}_i(b) &= \left|(S, T) \in 3^{N\setminus i} : b(S \cup i, T) = 1 \text{ and } b\,(S, T \cup i) \neq 1\right| \\ &= \left|(S, T) \in \mathcal{W}^D : i \in S\right| - \left|(S, T) \in \mathcal{W}^D : i \in T\right| \\ &= \left|\overline{\mathcal{W}}^D(i)\right| - \left|\underline{\mathcal{W}}^D(i)\right|.\end{aligned}$$

By the same argument

$$\begin{aligned}\underline{\eta}_i(b) &= \left|(S, T) \in 3^{N\setminus i} : b\,(S, T \cup i) = -1 \text{ and } b\,(S \cup i, T) \neq -1\right| \\ &= |(S, T) \in \mathcal{W}_D : i \in T| - |(S, T) \in \mathcal{W}_D : i \in S| \\ &= \left|\underline{\mathcal{W}}_D(i)\right| - \left|\overline{\mathcal{W}}_D(i)\right|.\end{aligned}$$

The proof is now completed. ■

Since the principal interest in these numbers, $\overline{\eta}_i(b)$, $\underline{\eta}_i(b)$, lies in their ratios rather than their magnitudes, we can define two Banzhaf power indices. The first of them consists of a normalization to add up to 1.

Definition 3. The normalized Banzhaf power index for the ternary bicooperative game $b \in \mathcal{TBG}^N$ is given, for each $i \in N$, by

$$\beta_i(b) = \frac{\eta_i(b)}{\eta(b)}.$$

If we now consider, for each player $i \in N$, the number of all bicoalitions $(S, T) \in 3^{N\setminus i}$ which can be defender swings or detractor swings for the player, we can introduce another Banzhaf power index for a ternary bicooperative game.

Definition 4. The probabilistic Banzhaf power index for the ternary bicooperative game $b \in \mathcal{TBG}^N$ is given, for each $i \in N$, by

$$\beta'_i(b) = \frac{1}{3^{n-1}}\eta_i(b).$$

In Bilbao et al. [6], the biprobabilistic values are introduced. A biprobabilistic value for a player $i \in N$ is defined by

$$\varphi_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} \left[\overline{p}^i_{(S,T)}\,(b(S \cup i, T) - b(S, T)) + \underline{p}^i_{(S,T)}\,(b(S, T) - b\,(S, T \cup i))\right],$$

where for every (S,T), the coefficient $\overline{p}^i_{(S,T)}$ can be interpreted as the probability that player i joins S and $\underline{p}^i_{(S,T)}$ as the probability that player i leaves $T \cup i$. Therefore, $\varphi_i(b)$ is the value that player i can expect in the game b. If we suppose that all the probabilities are equal, then $\overline{p}^i_{(S,T)} = \underline{p}^i_{(S,T)} = 1/3^{n-1}$, the probabilistic Banzhaf power index for a ternary bicooperative game is obtained.

Theorem 1. *The Banzhaf power index for the ternary bicooperative game $b \in \mathcal{TBG}^N$ satisfies*

$$\beta'_i(b) = \frac{1}{3^{n-1}} \sum_{(S,T)\in 3^{N\setminus i}} [b(S \cup i, T) - b(S, T \cup i)].$$

Proof. Let $b \in \mathcal{TBG}^N$. By using Proposition 6 we obtain

$$\begin{aligned}
\beta'_i(b) &= \frac{1}{3^{n-1}} \eta_i(b) \\
&= \frac{1}{3^{n-1}} \left(\overline{\eta}_i(b) + \underline{\eta}_i(b)\right) \\
&= \frac{1}{3^{n-1}} \left(\left|\overline{\mathcal{W}}^D(i)\right| - \left|\underline{\mathcal{W}}^D(i)\right| + \left|\underline{\mathcal{W}}_D(i)\right| - \left|\overline{\mathcal{W}}_D(i)\right|\right) \\
&= \frac{1}{3^{n-1}} \left(\left|\overline{\mathcal{W}}^D(i)\right| - \left|\overline{\mathcal{W}}_D(i)\right|\right) - \frac{1}{3^{n-1}} \left(\left|\underline{\mathcal{W}}^D(i)\right| - \left|\underline{\mathcal{W}}_D(i)\right|\right) \\
&= \frac{1}{3^{n-1}} \sum_{(S,T)\in 3^{N\setminus i}} b(S \cup i, T) - \frac{1}{3^{n-1}} \sum_{(S,T)\in 3^{N\setminus i}} b(S, T \cup i) \\
&= \frac{1}{3^{n-1}} \sum_{(S,T)\in 3^{N\setminus i}} [b(S \cup i, T) - b(S, T \cup i)]. \quad \blacksquare
\end{aligned}$$

In order to give an axiomatic characterization of these power indices, let us introduce a set of axioms for a value $\Psi : \mathcal{BG}^N \to \mathbb{R}^n$ and we will obtain a characterization of the vector of swings $(\eta_i(b))_{i\in N}$ by using the method provided by Dubey and Shapley [9]. First, we introduce the concept of a *null player* as follows.

Definition 5. A player $i \in N$ is a null player in the bicooperative game $b \in \mathcal{BG}^N$ if $b(S \cup i, T) - b(S, T \cup i) = 0$ for all $(S,T) \in 3^{N\setminus i}$.

Proposition 7. *Let $(S,T) \in 3^N$ be such that $(S,T) \neq (\emptyset, \emptyset)$. Then*
1. *Every player $i \in T$ is a null player in the bicooperative game $\overline{u}_{(S,T)}$.*
2. *Every player $i \in S$ is a null player in the bicooperative game $\underline{u}_{(S,T)}$.*

Proof. 1. First note that if $i \in T$ then $i \notin S$. Thus,

$$S \subseteq A \cup i \quad \text{and} \quad T \supseteq B \iff S \subseteq A \quad \text{and} \quad T \supseteq B \cup i,$$

and hence $\overline{u}_{(S,T)}(A \cup i, B) = \overline{u}_{(S,T)}(A, B \cup i)$ for all $(A,B) \in 3^{N\setminus i}$.
2. Since $i \in S$ implies $i \notin T$, we have

$$A \cup i \subseteq S \quad \text{and} \quad B \supseteq T \iff A \subseteq S \quad \text{and} \quad B \cup i \supseteq T.$$

Then clearly $\underline{u}_{(S,T)}(A \cup i, B) = \underline{u}_{(S,T)}(A, B \cup i)$ for all $(A,B) \in 3^{N\setminus i}$. $\blacksquare$

Axiom 1 (*Null Player*). If $i \in N$ is a null player in $b \in \mathcal{BG}^N$, then $\Psi_i(b) = 0$.

Axiom 2 (*Total Swings*). If $b \in \mathcal{TBG}^N$, then $\sum_{i\in N} \Psi_i(b) = \eta(b)$.

Axiom 3 (*Transfer Property*). For any $b, w \in \mathcal{TBG}^N$, we have that $\Psi(b) + \Psi(w) = \Psi(b \vee w) + \Psi(b \wedge w)$.

Axiom 4 (*Simple Additivity*). For any $b \in \mathcal{TBG}^N$ such that $b = v + v'$, where $v = \bigvee_{(S,T)\in \mathcal{MW}^D} \overline{u}_{(S,T)}$ and $v' = \bigwedge_{(S,T)\in \mathcal{MW}_D} \underline{u}_{(S,T)}$, we have that $\Psi(b) = \Psi(v) + \Psi(v')$.

Axiom 5 (*Structural Property*). For all $j \in S$, $i \notin S \cup T$ and $k \in T$,

$$\frac{\Psi_j\left(\overline{u}_{(S,T)}\right)}{\Psi_i\left(\overline{u}_{(S,T)}\right)} = \frac{c\left([(\emptyset, N), (S \setminus j, T)]\right)}{c\left([(\emptyset, N), (S, T \cup i)]\right)},$$

$$\frac{\Psi_k\left(\underline{u}_{(S,T)}\right)}{\Psi_i\left(\underline{u}_{(S,T)}\right)} = \frac{c\left([(S, T \setminus k), (N, \emptyset)]\right)}{c\left([(S \cup i, T), (N, \emptyset)]\right)}.$$

Taking into account that Proposition 2 implies that

$$c\left([(\emptyset, N), (S \setminus j, T)]\right) = \frac{(n+s-t-1)!}{2^{s-1}},$$
$$c\left([(\emptyset, N), (S, T \cup i)]\right) = \frac{(n+s-t-1)!}{2^{s}},$$

then Axiom 5 can be written as follows.

Axiom 5. For all $j \in S$, $i \notin S \cup T$ and $k \in T$,

$$\Psi_j\left(\overline{u}_{(S,T)}\right) = 2\Psi_i\left(\overline{u}_{(S,T)}\right),$$
$$\Psi_k\left(\underline{u}_{(S,T)}\right) = 2\Psi_i\left(\underline{u}_{(S,T)}\right).$$

The interpretation of Axiom 5 is as follows. First of all, note that the bicoalitions $(S \setminus j, T)$ and $(S, T \cup i)$ where $j \in S$ and $i \notin S \cup T$ have the same rank $\rho\left[(S \setminus j, T)\right] = \rho\left[(S, T \cup i)\right] = n+s-t-1$ (see Fig. 1). However, the number of maximal chains in the sublattice $[(\emptyset, N), (S \setminus j, T)]$ is not the same as the number of maximal chains in $[(\emptyset, N), (S, T \cup i)]$.

Hence, beginning from the bicoalition $(\emptyset, N)$, the probability of formation of the bicoalition (S, T) when player j joins $(S \setminus j, T)$ is the double of the probability when player i leaves $(S, T \cup i)$ because the number of maximal chains in $[(\emptyset, N), (S \setminus j, T)]$ is the double of the number of chains in $[(\emptyset, N), (S, T \cup i)]$. In analogous form, if we consider $(S, T \setminus k)$ with $k \in T$ and $(S \cup i, T)$ which have the same rank, the number of maximal chains in $[(S, T \setminus k), (N, \emptyset)]$ is the double of the number of maximal chains in $[(S \cup i, T), (N, \emptyset)]$. Therefore, the probability of the formation of (S, T) beginning from $(S, T \setminus k)$ when player k joins T must be distinct from the probability when player i leaves $(S \cup i, T)$.

Taking into account these considerations, the values that one player must obtain in the unanimity games must be proportional to the number of maximal chains in the corresponding sublattices. A similar axiom has been used in the axiomatic characterization of the Shapley value for bicooperative games (see [5]).

Theorem 2. *There is a unique function $\Psi : \mathcal{TBG}^N \to \mathbb{R}^n$ that satisfies the Axioms 1–5. Moreover, $\Psi(b) = (\eta_i(b))_{i \in N}$ for all $b \in \mathcal{TBG}^N$.*

Proof. Let $(S, T) \in 3^N$ be such that $(S, T) \neq (\emptyset, \emptyset)$. Then Proposition 7 implies that every $k \in T$ is a null player in game $\overline{u}_{(S,T)}$, and every $j \in S$ is a null player in game $\underline{u}_{(S,T)}$. Axiom 1 implies that $\Psi_k(\overline{u}_{(S,T)}) = 0$ and $\Psi_j(\underline{u}_{(S,T)}) = 0$ for all $k \in T$ and $j \in S$. Applying Axiom 2, we have that

$$\begin{aligned}
\eta\left(\overline{u}_{(S,T)}\right) &= \sum_{i \in N} \Psi_i(\overline{u}_{(S,T)}) = \sum_{i \in N \setminus T} \Psi_i(\overline{u}_{(S,T)}) \\
&= \sum_{j \in S} \Psi_j(\overline{u}_{(S,T)}) + \sum_{l \in N \setminus (T \cup S)} \Psi_l(\overline{u}_{(S,T)}), \\
\eta\left(\underline{u}_{(S,T)}\right) &= \sum_{i \in N} \Psi_i(\underline{u}_{(S,T)}) = \sum_{i \in N \setminus S} \Psi_i(\underline{u}_{(S,T)}) \\
&= \sum_{k \in T} \Psi_k(\underline{u}_{(S,T)}) + \sum_{l \in N \setminus (T \cup S)} \Psi_l(\underline{u}_{(S,T)}).
\end{aligned}$$

By using Axiom 5, we obtain

$$\Psi_j(\overline{u}_{(S,T)}) = 2\Psi_l(\overline{u}_{(S,T)}) \quad \text{and} \quad \Psi_k(\underline{u}_{(S,T)}) = 2\Psi_l(\underline{u}_{(S,T)}),$$

for all $j \in S$, $l \notin S \cup T$, and $k \in T$. As a consequence of the above equations,

$$\begin{aligned}
\eta\left(\overline{u}_{(S,T)}\right) &= 2s\Psi_l(\overline{u}_{(S,T)}) + (n-s-t)\,\Psi_l(\overline{u}_{(S,T)}) \\
&= (n+s-t)\,\Psi_l(\overline{u}_{(S,T)}), \\
\eta\left(\underline{u}_{(S,T)}\right) &= 2t\Psi_l(\underline{u}_{(S,T)}) + (n-s-t)\,\Psi_l(\underline{u}_{(S,T)}) \\
&= (n+t-s)\,\Psi_l(\underline{u}_{(S,T)}),
\end{aligned}$$

for all $l \notin S \cup T$. Hence we obtain

$$\Psi_i(\overline{u}_{(S,T)}) = \begin{cases} 0 & \text{if } i \in T, \\ \dfrac{\eta\left(\overline{u}_{(S,T)}\right)}{n+s-t} & \text{if } i \notin S \cup T, \\ \dfrac{2\eta\left(\overline{u}_{(S,T)}\right)}{n+s-t} & \text{if } i \in S \end{cases}$$

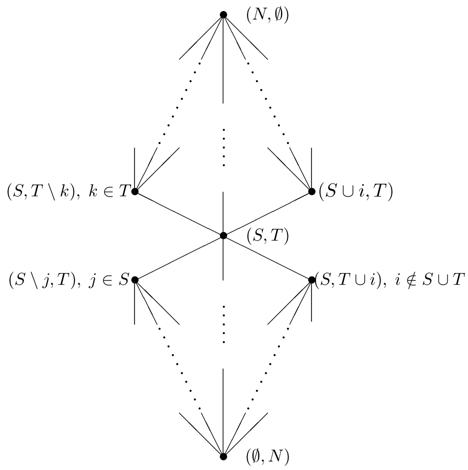

Fig. 1. Axiom 5.

and also

$$\Psi_i(\underline{u}_{(S,T)}) = \begin{cases} 0 & \text{if } i \in S, \\ \dfrac{\eta\left(\underline{u}_{(S,T)}\right)}{n+t-s} & \text{if } i \notin S \cup T, \\ \dfrac{2\eta\left(\underline{u}_{(S,T)}\right)}{n+t-s} & \text{if } i \in T. \end{cases}$$

Each ternary bicooperative game $b \in \mathcal{TBG}^N$ has a finite number of minimal defender bicoalitions $\mathcal{MW}^D = \{(S_1,T_1),\ldots,(S_r,T_r)\}$ and a finite number of maximal detractor bicoalitions $\mathcal{MW}_D = \{(S_1',T_1'),\ldots,(S_k',T_k')\}$. Since Proposition 5 provides us a decomposition $b = v + v'$, where

$$v = \overline{u}_{(S_1,T_1)} \vee \cdots \vee \overline{u}_{(S_r,T_r)} \quad \text{and} \quad v' = \underline{u}_{(S_1',T_1')} \wedge \cdots \wedge \underline{u}_{(S_k',T_k')}.$$

Axiom 4 implies that

$$\begin{aligned} \Psi(b) &= \Psi(v) + \Psi\left(v'\right) \\ &= \Psi\left(\left[\bigvee_{(S_j,T_j)\in\mathcal{MW}^D} \overline{u}_{(S_j,T_j)}\right]\right) + \Psi\left(\left[\bigwedge_{\left(S_j',T_j'\right)\in\mathcal{MW}_D} \underline{u}_{\left(S_j',T_j'\right)}\right]\right). \end{aligned}$$

If we consider the game v and apply Axiom 3,

$$\Psi(v) = \Psi\left(\overline{u}_{(S_1,T_1)}\right) + \Psi\left(\overline{u}_{(S_2,T_2)} \vee \cdots \vee \overline{u}_{(S_r,T_r)}\right) - \Psi\left(\overline{u}_{(S_1,T_1)} \wedge \left(\overline{u}_{(S_2,T_2)} \vee \cdots \vee \overline{u}_{(S_r,T_r)}\right)\right).$$

Each game that appears in the second member is a game with fewer minimal defender bicoalitions than v. So, we can perform an induction on the number of minimal defender bicoalitions and it follows that $\Psi(v)$ is uniquely determined. By using the same Axiom 3,

$$\Psi\left(v'\right) = \Psi\left(\underline{u}_{(S_1',T_1')}\right) + \Psi\left(\underline{u}_{(S_2',T_2')} \wedge \cdots \wedge \underline{u}_{(S_k',T_k')}\right) - \Psi\left(\underline{u}_{(S_1',T_1')} \vee \left(\underline{u}_{(S_2',T_2')} \wedge \cdots \wedge \underline{u}_{(S_k',T_k')}\right)\right).$$

Each game that appears in the second member is a game with fewer maximal detractor bicoalitions than v'. So, we can perform an induction on the number of maximal detractor bicoalitions and it follows that $\Psi\left(v'\right)$ is uniquely determined. If b is the null game, that is $b(S,T) = 0$ for all $(S,T) \in 3^N$, then all the players are null players and hence $\Psi_i(b) = 0$ for all $i \in N$.

We have proved the uniqueness and we must establish the existence. Thus, is suffices to check directly that the vector $\Psi(b) = (\eta_i(b))_{i\in N}$ satisfies the five axioms. Since

$$\eta_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} \left[b(S\cup i,T) - b(S,T\cup i)\right],$$

for all $i \in N$, it follows that $(\eta_i(b))_{i\in N}$ satisfies Axioms 1–4. Let $(S, T) \in 3^N$ be such that $(S, T) \neq (\emptyset, \emptyset)$ and let $i \notin S \cup T$. Then

$$\eta_i\left(\overline{u}_{(S,T)}\right) = \sum_{(A,B)\in 3^{N\setminus i}} \left[\overline{u}_{(S,T)}(A\cup i, B) - \overline{u}_{(S,T)}(A, B\cup i)\right] = \sum_{(A,B)\in 3^{N\setminus i}} \overline{u}_{(S,T)}(A\cup i, B),$$

since $i \notin T$ implies $B \cup i \not\subseteq T$. Notice also that $i \notin S$ implies $S \subseteq A \cup i \Leftrightarrow S \subseteq A$, and hence

$$\eta_i\left(\overline{u}_{(S,T)}\right) = \left|\left\{(A, B) \in 3^{N\setminus i} : A \supseteq S \text{ and } B \subseteq T\right\}\right| = 2^{n-s+t}.$$

If we choose $j \in S$ we have $j \notin T$ and therefore

$$\eta_j\left(\overline{u}_{(S,T)}\right) = \sum_{(A,B)\in 3^{N\setminus j}} \left[\overline{u}_{(S,T)}(A\cup j, B) - \overline{u}_{(S,T)}(A, B\cup j)\right] = \sum_{(A,B)\in 3^{N\setminus j}} \overline{u}_{(S,T)}(A\cup j, B).$$

Since $j \in S$ implies $S \subseteq A \cup j \Leftrightarrow S \setminus j \subseteq A$, we obtain

$$\eta_j\left(\overline{u}_{(S,T)}\right) = \left|\left\{(A, B) \in 3^{N\setminus i} : A \supseteq S \setminus j \text{ and } B \subseteq T\right\}\right| = 2^{n-s+1+t}.$$

Therefore, $\eta_j\left(\overline{u}_{(S,T)}\right) = 2\eta_i\left(\overline{u}_{(S,T)}\right)$ for all $j \in S$ and $i \notin S \cup T$, which implies that $(\eta_i(b))_{i\in N}$ satisfies the first equation of Axiom 5. The second equation $\eta_k\left(\underline{u}_{(S,T)}\right) = 2\eta_i\left(\underline{u}_{(S,T)}\right)$ for all $k \in T$ and $i \notin S \cup T$, is proved analogously. ■

5. The Banzhaf power index for bicooperative games with two lists of weights

A *weighted voting game* is defined on a finite set N of players, which can be people, companies, political parties or countries. Each player $i \in N$ has a number of votes $w_i > 0$, so each coalition of players $S \subseteq N$, has the sum of votes of its components $w(S) = \sum_{i\in S} w_i$. Fixed a quota q to take decisions, a coalition S is *winning* if $w(S) \geq q$, and is *losing* if $w(S) < q$. As there are exactly two possibilities for each coalition of players, a weighted voting game is modeled with the simple game $v : 2^N \to \{0, 1\}$, defined by

$$v(S) = \begin{cases} 1, & \text{if } w(S) \geq q, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, a weighted voting game is represented by the following scheme $v \equiv [q; w_1, \ldots, w_n]$. The power of a player is an '*a priori*' measure of his/her influencing capacity, based on computing the capacity of each player to participate in winning coalitions. There are two well-known power indices, the Banzhaf index [2] and the Shapley–Shubik index [19]. Both of them give a more precise measure of the power of a player that the number of votes assigned to each player.

Another aspect to be mentioned in a weighted voting game is the following: How can we measure the power of a player to block a decision? The answer to this question is that the power of a player to block a decision is the same as the one player has in order to approve it. So, these above indices measure both the capacity of a player to adopt a proposal and to block it (see [9,21]).

However, we can concur that the power of a player to block a decision is not equal to the power of this player to approve it. Fixed the quota q to take decisions and the quota m to block decisions, these situations can be represented by a ternary bicooperative game $b : 3^N \to \{-1, 0, 1\}$, defined by

$$b(S, T) = \begin{cases} 1, & \text{if } w(S) \geq q \text{ and } p(T) < m \\ -1, & \text{if } w(S) < q \text{ and } p(T) \geq m \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where each player $i \in N$ has a number of votes $w_i > 0$ in order to approve a decision and a number of votes $p_i > 0$ in order to block it and $0 < q \leq w(N)$, $0 < m \leq p(N)$. It is represented by the scheme $b \equiv [[q; w_1, \ldots, w_n], [m; p_1, \ldots, p_n]]$ and is called *bi-weighted ternary bicooperative game*.

It is easy to check that the following ternary bicooperative game, for $k > 0$,

$$b(S, T) = \begin{cases} 1, & \text{if } w(S) - w(T) > k \\ -1, & \text{if } w(S) - w(T) < k \\ 0, & \text{otherwise} \end{cases}$$

and hence, the majority rule in [16] is the bicooperative game (1).

Table 1
Weights and worths.

(S, T)	$(w(S), p(T))$	$b(S, T)$
$(\emptyset, \{123\})$	$(0, 10)$	-1
$(\emptyset, \{12\})$	$(0, 5)$	0
$(\emptyset, \{13\})$	$(0, 7)$	-1
$(\emptyset, \{23\})$	$(0, 8)$	-1
$(\emptyset, \{3\})$	$(0, 5)$	0
$(\emptyset, \{2\})$	$(0, 3)$	0
$(\emptyset, \{1\})$	$(0, 1)$	0
$(\{1\}, \{23\})$	$(6, 8)$	-1
$(\{2\}, \{13\})$	$(1, 7)$	-1
$(\{3\}, \{12\})$	$(4, 5)$	0
$(\{1\}, \{3\})$	$(6, 5)$	0
$(\{1\}, \{2\})$	$(6, 3)$	0
$(\{2\}, \{3\})$	$(1, 5)$	0
$(\{2\}, \{1\})$	$(1, 2)$	0
$(\{3\}, \{2\})$	$(4, 3)$	0
$(\{3\}, \{1\})$	$(4, 2)$	0
$(\{1\}, \emptyset)$	$(6, 0)$	0
$(\{2\}, \emptyset)$	$(1, 0)$	0
$(\{3\}, \emptyset)$	$(4, 0)$	0
$(\{12\}, \{3\})$	$(7, 5)$	0
$(\{23\}, \{1\})$	$(5, 2)$	0
$(\{13\}, \{2\})$	$(10, 3)$	1
$(\{12\}, \emptyset)$	$(7, 0)$	0
$(\{23\}, \emptyset)$	$(5, 0)$	0
$(\{13\}, \emptyset)$	$(10, 0)$	1
$(\{123\}, \emptyset)$	$(11, 0)$	1

Table 2
Defender and detractor swings.

Player	Defender swings	Detractor swings
1	$(3, 2), (3, \emptyset), (23, \emptyset)$	$(\emptyset, 3), (2, 3)$
2		$(\emptyset, 3), (1, 3)$
3	$(1, 2), (1, \emptyset), (12, \emptyset)$	$(1, 2), (2, 1), (\emptyset, 1), (\emptyset, 2), (\emptyset, 12)$

Table 3
Number of swings.

Player	$\overline{\eta}_i(b)$	$\underline{\eta}_i(b)$	$\eta_i(b)$
1	3	2	5
2	0	2	2
3	3	5	8

Example 1. Let us consider the bi-weighted ternary bicooperative game given by $b \equiv [[8; 6, 1, 4], [6; 2, 3, 5]]$, that is, for each $(S, T) \in 3^N$,

$$b(S, T) = \begin{cases} 1, & \text{if } w(S) \geq 8 \text{ and } p(T) < 6 \\ -1, & \text{if } w(S) < 8 \text{ and } p(T) \geq 6 \\ 0, & \text{otherwise.} \end{cases}$$

We compute the normalized Banzhaf index. First we determine the weights and worths for b in Table 1.

As a consequence of the data given by Table 1, we obtain the detractor swings and defender swings for each player i (see Table 2).

The number of swings for each player is displayed in Table 3.

The data of Table 3 implies that the total number of swings is $\eta(b) = \sum_{i \in N} \eta_i(b) = 15$. Thus, the normalized Banzhaf index is

$$\beta(b) = \left(\frac{5}{15}, \frac{2}{15}, \frac{8}{15} \right).$$

Note that if we consider the indices

$$\overline{\beta}(b) = \left(\frac{\overline{\eta}_i(b)}{\sum\limits_{i \in N} \eta_i(b)} \right)_{i \in N} \quad \text{and} \quad \underline{\beta}(b) = \left(\frac{\underline{\eta}_i(b)}{\sum\limits_{i \in N} \eta_i(b)} \right)_{i \in N},$$

the power of the players for approving a decision or blocking it is given by

$$\overline{\beta}(b) = \left(\frac{3}{15}, 0, \frac{2}{15}\right) \quad \text{and} \quad \underline{\beta}(b) = \left(\frac{2}{15}, \frac{2}{15}, \frac{5}{15}\right).$$

The following result allows to compute the number of detractor swings and the number defender swings for each player without calculating the values of the bicoalitions. In the following expressions, if $w(N \setminus i) < q$ or $p(N \setminus i) < m$, for some $i \in N$, then the corresponding sum is not considered.

Proposition 8. *Let $b \equiv [[q; w_1, \dots, w_n], [m; p_1, \dots, p_n]]$ be a bi-weighted ternary bicooperative game. Then, the number of defender swings and the number of detractor swings are given by*

$$\overline{\eta}_i(b) = \sum_{k=q-w_i}^{w(N\setminus i)} \sum_{r=0}^{m-1} b^i_{kr} - \sum_{k=q}^{w(N\setminus i)} \sum_{r=0}^{m-1} b^i_{kr},$$

$$\underline{\eta}_i(b) = \sum_{k=0}^{q-1} \sum_{r=m-p_i}^{p(N\setminus i)} b^i_{kr} - \sum_{k=0}^{q-1} \sum_{r=m}^{p(N\setminus i)} b^i_{kr},$$

where b^i_{kr} is the number of bicoalitions (S, T) such that $i \notin S \cup T$ with $w(S) = k$ and $p(T) = r$.

Proof. First of all, consider the set of bicoalitions (S, T) in which player $i \notin S \cup T$ such that $w(S) \geq q - w_i$ and $p(T) \leq m - 1$. The cardinal of this set is given by

$$\overline{s}^i_1 = \sum_{k=q-w_i}^{w(N\setminus i)} \sum_{r=0}^{m-1} b^i_{kr}.$$

Since $w(S \cup i) \geq q$ and $p(T) < m$, this number $\overline{s}^i_1$ coincides with the cardinal of the set of all bicoalitions $(S, T) \in 3^{N\setminus i}$ such that $b(S \cup i, T) = 1$. On the other hand, inside this set, consider the subset of bicoalitions (S, T) where the presence of player i in S is not necessary for approving the decision. The cardinal of this subset is equal to the number of bicoalitions $(S, T) \in 3^{N\setminus i}$ such that $w(S) \geq q$ and $p(T) \leq m - 1$, that is,

$$\overline{s}^i_2 = \sum_{k=q-w_i}^{w(N\setminus i)} \sum_{r=0}^{m-1} b^i_{kr}.$$

To obtain the number of defender swings, it suffices to compute $\overline{s}^i_1 - \overline{s}^i_2$.

In analogous form, the number of detractor swings can be obtained from the set of bicoalitions (S, T) in which player $i \notin S \cup T$ such that $b(S, T \cup i) = -1$ and inside of this set, the subset of bicoalitions (S, T) where the presence of player i in T is not necessary for blocking the decision. ■

In Example 1, the application of Proposition 8 results

$$\overline{\eta}_1(b) = \sum_{k=2}^{5} \sum_{r=0}^{5} b^1_{kr} = 3,$$

$$\underline{\eta}_1(b) = \sum_{k=0}^{7} \sum_{r=4}^{8} b^1_{kr} - \sum_{k=0}^{7} \sum_{r=6}^{8} b^1_{kr} = 3 - 1 = 2,$$

$$\overline{\eta}_2(b) = \sum_{k=7}^{10} \sum_{r=0}^{3} b^2_{kr} - \sum_{k=8}^{10} \sum_{r=0}^{3} b^2_{kr} = 1 - 1 = 0,$$

$$\underline{\eta}_2(b) = \sum_{k=0}^{7} \sum_{r=4}^{7} b^2_{kr} - \sum_{k=0}^{7} \sum_{r=6}^{7} b^2_{kr} = 3 - 1 = 2,$$

$$\overline{\eta}_3(b) = \sum_{k=4}^{7} \sum_{r=0}^{5} b^3_{kr} = 3,$$

$$\underline{\eta}_3(b) = \sum_{k=0}^{7} \sum_{r=1}^{5} b^3_{kr} = 5.$$

Now, we focus on computing the Banzhaf index of the bi-weighted ternary bicooperative game by using generating functions.

The most useful method for counting the number $f(n)$ of elements of finite sets S_n where $n \in \mathbb{N}$, is to obtain its *generating function* $F(x) = \sum_{n\geq 0} f(n)\, x^n$. These functions have been used for computing the number of swings in cooperative games (see [7,4,1]). We use generating functions of two variables to compute the numbers $\left\{b^i_{kr}\right\}_{k,r\geq 0}$, for every player $i \in N$.

Proposition 9. *Let* $b \equiv [[q; w_1, \ldots, w_n], [m; p_1, \ldots, p_n]]$ *be a bi-weighted ternary bicooperative game. Then, for* $i \in N$, *the generating function of numbers* $\{b^i_{kr}\}_{k\geq 0,\, r\geq 0}$, *where* b^i_{kr} *is the number of bicoalitions* (S, T) *such that* $i \notin S \cup T$ *with* $w(S) = k$ *and* $p(T) = r$ *is given by*

$$B_i(x, y) = \prod_{j=1,\ j\neq i}^{n} \left(1 + x^{w_j} + y^{p_j}\right).$$

Proof. Let $b \equiv [[q; w_1, \ldots, w_n], [m; p_1, \ldots, p_n]]$ be a bi-weighted ternary bicooperative game. Consider the function

$$B(x, y) = \prod_{j=1}^{n} \left(1 + x^{w_j} + y^{p_j}\right).$$

Expanding the above expression, it holds that

$$\begin{aligned} B(x, y) &= \prod_{j=1}^{n} \left(1 + x^{w_j} + y^{p_j}\right) \\ &= \sum_{S,T\subseteq N,T\subseteq N\setminus S} \left(x^{\sum_{i\in S} w_i}\right)\left(y^{\sum_{l\in T} p_l}\right) \\ &= \sum_{(S,T)\in 3^N} x^{w(S)} y^{p(T)} \\ &= \sum_{k=0}^{w(N)} \sum_{r=0}^{p(N)} b_{kr}\, x^k y^r. \end{aligned}$$

Hence, $B(x, y)$ is a generating function for computing the numbers $\{b_{kr}\}_{k\geq 0,\, r\geq 0}$ where each b_{kr} is the number of bicoalitions $(S, T) \in 3^N$ such that $w(S) = k$ and $p(T) = r$. Finally, to obtain the numbers $\{b^i_{kr}\}_{k\geq 0,\, r\geq 0}$, it suffices to remove the factor $(1 + x^{w_i} + y^{p_i})$ in the function $B(x, y)$ obtaining the generating function $B_i(x, y)$. ■

In Example 1, we have that

$$\begin{aligned} B_1(x, y) &= \left(1 + x^1 + y^3\right)\left(1 + x^4 + y^5\right) \\ &= 1 + x + x^4 + y^3 + x^5 + y^5 + y^8 + xy^5 + x^4y^3, \\ B_2(x, y) &= \left(1 + x^6 + y^2\right)\left(1 + x^4 + y^5\right) \\ &= 1 + y^2 + x^4 + x^6 + y^5 + y^7 + x^{10} + x^4y^2 + x^6y^5, \\ B_3(x, y) &= \left(1 + x^6 + y^2\right)\left(1 + x^1 + y^3\right) \\ &= 1 + x + y^2 + y^3 + x^6 + y^5 + x^7 + xy^2 + x^6y^3, \end{aligned}$$

and it is easy to check that the numbers of swings $(\overline{\eta}_i(b))_{i\in N}$ and $(\underline{\eta}_i(b))_{i\in N}$ are obtained using these functions.

Example 2. Consider an state with a regional structure, where a committee is made up of representatives of n regions and they have a number of votes proportional to its population, $\{w_1, \ldots, w_n\}$. When a proposal was put to a vote, it would be approved if the sum of favorable votes exceeding a quota q and the sum of opposed members is less than a quota m. In the case that the number of votes is less than the quota q, and the number of members against the proposal is greater than m, the proposal would be rejected; otherwise, the proposal would be deferred for consideration again at the reunion committee. This voting system can be modeled by the bicooperative game $b \equiv [[q; w_1, \ldots, w_n], [m; p_1, \ldots, p_n]]$, where $\{w_1, \ldots, w_n\}$ represents the number of votes of each region and $p_1 = \cdots = p_n = 1$.

If we consider the 19 autonomous communities of Spain and we apply the population data in 2008, we have the following bicooperative game:

$$b \equiv \Big[[501; 178, 159, 136, 109, 60, 55, 47, 45, 44, 31, 29, 24, 23, 23, 13, 13, 7, 2, 2],$$
$$[10; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]\Big].$$

The population data, number of votes and the Banzhaf power index for each community are given in Table 4.

The algorithm to compute the number of swings and Banzhaf power indices, written in the MATHEMATICA 7.0 computer system, with the package Combinatorica, is the following:

Table 4
The Banzhaf power of the Spanish communities.

	Communities	Population	Votes	Banzhaf power
1	Andalucía	8202 220	178	0.132148
2	Cataluña	7364 078	159	0.117892
3	Comunidad de Madrid	6271 638	136	0.101758
4	Comunidad deValencia	5029 601	109	0.085071
5	Galicia	2784 169	60	0.054824
6	Castilla y León	2557 330	55	0.052130
7	Pais Vasco	2157 112	47	0.047572
8	Canarias	2075 968	45	0.046532
9	Castilla La Mancha	2043 100	44	0.045978
10	Región de Murcia	1426 109	31	0.039087
11	Aragón	1326 918	29	0.038148
12	Extremadura	1097 744	24	0.035381
13	Principado de Asturias	1080 138	23	0.034750
14	Baleares	1072 844	23	0.034750
15	Navarra	620 377	13	0.029704
16	Cantabria	582 138	13	0.029704
17	La Rioja	317 501	7	0.026578
18	Ceuta	77 389	2	0.023996
19	Melilla	71 448	2	0.023996

```
banzhafBicoG[weights_List, pop_List] := Times @@ (1 +x^ weights+y^pop)
swingDefender[i_,weights_List,pop_List,q_Integer,m_Integer]:=
Module[{poly,coefi,delwe,delpo,s1,s2,sw,sp},
delwe=Delete[weights,i];
delpo=Delete[pop,i];
poly=banzhafBicoG[delwe,delpo];
sw=Apply[Plus,delwe]+1;
sp=Apply[Plus,delpo]+1;
coefi=CoefficientList[poly,{x,y}]/.{}->Table[0,{sp}];
s1=Apply[Plus,Flatten[coefi[[
Range[Max[1,q-weights[[i]]+1],sw],
Range[1,Min[sp,m]]]]]];
s2=If[((q+1) >sw),0,Apply[Plus,
Flatten[coefi[[Range[q+1,sw],Range[1,Min[sp,m]]]]]]];
s1-s2]
swingDetractor[i_, weights_List, pop_List, q_Integer, m_Integer] :=
Module[{poly, coefi, s1, s2, delwe, delpo, sw, sp},
delwe = Delete[weights, i];
delpo = Delete[pop, i];
poly = banzhafBicoG[delwe, delpo];
sw = Apply[Plus, delwe] +1;
sp = Apply[Plus, delpo] +1;
coefi = CoefficientList[poly, {x, y}] /. {} ->Table[0, {sp}];
s1 = Apply[Plus,
Flatten[coefi[[Range[1, Min[sw, q]],
Range[Max[1, m - pop[[i]] +1], sp]]]]];
s2 = If[((m +1) >sp), 0,
Apply[Plus, Flatten[coefi[[Range[1, Min[sw, q]], Range[m +1, sp]]]]]];
s1 - s2]
banzhafBicoPower[weights_List,pop_List,q_,m_]:=
#/(Plus @@ #)& @Table[
swingDefender[i,weights,pop,q,m]+swingDetractor[i,weights,pop,q,m],
{i,Length[weights]}]
```

The Banzhaf power indices displayed in Table 4 are obtained as follows:

```
votes= {178,159,136,109,60,55,47,45,44,31,29,24,23,23,13,13,7,2,2};
members=Table[1,{19}];
banzhafBicoPower[votes,members,501,10]//N
```

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