Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 349 (2018) 42–52

FUZZY sets and systems

www.elsevier.com/locate/fss

Nontransferable utility games with fuzzy coalition restrictions

J.M. Gallardo, N. Jiménez, A. Jiménez-Losada *

Matemática Aplicada II, Escuela Técnica Superior de Ingeniería, Universidad de Sevilla, Camino de los Descubrimientos, 41092 Sevilla, Spain

Received 18 October 2016; received in revised form 17 July 2017; accepted 14 August 2017
Available online 18 August 2017

Abstract

A value for nontransferable utility (NTU) games with fuzzy coalition restrictions is introduced and characterized. In a similar way as the Shapley value for transferable utility (TU) games has been extended in the literature to study games with restricted cooperation, we extend the Shapley NTU value to deal with NTU games in situations in which there are fuzzy dependency relationships among the players.

Keywords: Nontransferable utility games; Shapley NTU value; Restricted cooperation; Fuzzy coalition

1. Introduction

Cooperative game theory deals with groups of players that aim to share the benefits derived from their cooperation. The cooperative model abstracts away from some details of the interaction among the players and describes only the outcomes that result when players cooperate within different coalitions. At this point, two different approaches can be considered. In some situations, the outcome of each coalition is described by a real number. The games used in these cases are called transferable utility (TU) games. The adjective transferable refers to the assumption that a player can losslessly transfer any part of his utility to another player, usually through money, and that the players' utilities are linear in money with the same scale for all players. If there is no possibility of transferring the utility between players by using money or, if there is, the scales between utilities and money are different, then nontransferable utility (NTU) games are used.

Given a cooperative game, it is often assumed that the players are free to participate in any coalition, but in some situations there are dependency relationships among the players that restrict their capacity to cooperate within some coalitions. Those relationships must be taken into account if we want to distribute the profits fairly. In this regard, different kinds of limitations on cooperation among players have been studied in the literature, and various structures have been used, such as the permission structures introduced by Gilles, Owen and van den Brink [5]. In Gallardo et al. [4] one of these models for games with restricted cooperation was introduced. This model is more general than others

* Corresponding author.
E-mail address: ajlosada@us.es (A. Jiménez-Losada).

http://dx.doi.org/10.1016/j.fss.2017.08.004

in several ways. For instance, it allows to deal with non-hierarchical or non-transitive dependency relationships. But the most important advantage of this model is that it allows to deal with fuzzy dependency relationships, which arise in situations in which each player has a degree of freedom to cooperate within a coalition. Our goal is to extend this model to NTU games. To this end, we will use the value for NTU games introduced by Shapley [7].

The paper is organized as follows. In Section 2, several basic definitions and results concerning cooperative games are recalled. In Section 3, fuzzy authorization structures are introduced. They will be used to model situations in which some players depend partially on other players. In Section 4, NTU games with fuzzy authorization structure are defined, and the Shapley fuzzy authorization NTU correspondence is defined and characterized. Finally, in Section 5, we draw some conclusions.

2. Preliminaries

2.1. Transferable utility games

A *transferable utility game* or *TU game* is a pair (N, v), where N is a set of cardinality n with $n \in \mathbb{N}$ and $v : 2^N \to \mathbb{R}$ is a function satisfying that $v(\emptyset) = 0$. The elements of N are called players, the subsets $E \subseteq N$ are called coalitions and the number $v(E)$ is the worth of E. Often, the TU game (N, v) is identified with the function v. The set of all TU games on N is denoted by $\mathcal{G}^N$.

Given a TU game (N, v), a problem that arises is how to assign a payoff to each player in a fair way. An allocation rule or value assigns to each game (N, v) a payoff vector $\psi(v) \in \mathbb{R}^N$. Many allocation rules have been defined in literature. The best known of them is the Shapley value, introduced by Shapley [6] in 1953. Given $v \in \mathcal{G}^N$, the *Shapley value of* v, denoted by $\phi(v)$, is defined as

$$\phi_i(v) = \sum_{\{E \subseteq N : i \in E\}} p_E \left[v(E) - v(E \setminus \{i\}) \right] \quad \text{for all } i \in N,$$

where $p_E = \dfrac{(n - |E|)!\,(|E| - 1)!}{n!}$ and $|E|$ denotes the cardinality of E.

2.2. Nontransferable utility games

A *cooperative game with nontransferable utility* or NTU game is a pair (N, V) where N is a set of cardinality $n \in \mathbb{N}$ and V is a correspondence that assigns to each nonempty $E \subseteq N$ a nonempty subset $V(E) \subseteq \mathbb{R}^E$. The set-valued function V is the characteristic function of the NTU game (N, V). Often, the NTU game (N, V) is identified with the function V.

If V and W are NTU games, the NTU game $V + W$ is defined by

$$(V + W)(E) = V(E) + W(E) = \{x + y : x \in V(E), y \in W(E)\}$$

for every $E \in 2^N \setminus \{\emptyset\}$.

If $\alpha \in \mathbb{R}^N$, the NTU game $\alpha * V$ is defined by

$$(\alpha * V)(E) = \left\{ \alpha^E * x : x \in V(E) \right\}$$

for every $E \in 2^N \setminus \{\emptyset\}$, where α^E is the restriction of α to E (i.e., $\alpha^E \in \mathbb{R}^E$ and $\alpha_i^E = \alpha_i$ for every $i \in E$) and $*$ denotes the Hadamard product (i.e., if $x, y \in \mathbb{R}^E$, then $x * y$ is the element in $\mathbb{R}^E$ defined as $(x * y)_i = x_i y_i$ for every $i \in E$).

Given $v \in \mathcal{G}^N$ the NTU game corresponding to v is defined as

$$V_v(E) = \left\{ x \in \mathbb{R}^E : \sum_{k \in E} x_k \leqslant v(E) \right\} \quad \text{for every nonempty } E \subseteq N.$$

Suppose C is a convex set in $\mathbb{R}^N$. We denote by ∂C the boundary of C and by $\overline{C} = C \cup \partial C$ the closure of C. Set C is comprehensive if $y \leqslant x$ and $x \in C$ imply $y \in C$. We recall that C is smooth if it has a unique supporting hyperplane at each point of ∂C.

Let us consider NTU games V satisfying some conditions:

(i) $V(E)$ is convex, comprehensive and a proper subset of $\mathbb{R}^E$ for all nonempty $E \subseteq N$.
(ii) $V(N)$ is smooth.
(iii) If $x \in \partial(V(N))$ then $\{y \in \mathbb{R}^N : y \geqslant x\} \cap V(N) = \{x\}$.
(iv) There exists $x \in \mathbb{R}^N$ such that $V(E) \times \{0\}^{N \setminus E} \subseteq x + V(N)$ for every nonempty $E \subseteq N$.

The set of NTU games satisfying (i), (ii), (iii) and (iv) is denoted by $\hat{\Gamma}^N$.

The *Shapley NTU correspondence* was introduced by Shapley [7] and it was characterized by Aumann [2] for games in $\hat{\Gamma}^N$. Given $V \in \hat{\Gamma}^N$, a vector $x \in \mathbb{R}^N$ is a *Shapley NTU payoff vector* of V if there exists $\lambda \in \mathbb{R}^N_{++}$ such that

(a) $x \in \overline{V(N)}$,
(b) the set $\left\{\lambda^E \cdot z : z \in V(E)\right\}$ is bounded above for every nonempty $E \subseteq N$,
(c) $\lambda * x = \phi(v_\lambda)$ where v_λ is the TU game given by

$$v_\lambda(E) = \sup\left\{\lambda^E \cdot z : z \in V(E)\right\} \qquad \text{for every nonempty } E \subseteq N.$$

Let $SH : \hat{\Gamma}^N \to 2^{\mathbb{R}^N}$ be the mapping that assigns to each $V \in \hat{\Gamma}^N$ the set of Shapley NTU payoff vectors of V. The correspondence SH is called the *Shapley NTU correspondence* (*on* N). Aumann [2] proved that the Shapley NTU correspondence satisfies the following properties.

Efficiency. If $V \in \hat{\Gamma}^N$, then

$$SH(V) \subseteq \partial(V(N)).$$

Conditional additivity. Let $V, W \in \hat{\Gamma}^N$ be such that $V + W = U \in \hat{\Gamma}^N$. Then

$$(SH(V) + SH(W)) \cap \partial(U(N)) \subseteq SH(U).$$

Scale covariance. If $V \in \hat{\Gamma}^N$ and $\alpha \in \mathbb{R}^N_{++}$, then

$$SH(\alpha * V) = \alpha * SH(V).$$

Independence of irrelevant alternatives. Let $V, W \in \hat{\Gamma}^N$ be such that $V(N) \subseteq W(N)$ and $V(E) = W(E)$ for every nonempty $E \subsetneqq N$. Then

$$SH(W) \cap V(N) \subseteq SH(V).$$

3. Games with fuzzy authorization structure

Many different structures have been used to model games with restricted cooperation. In the majority of these structures, the dependency relationships are considered to be complete, in the sense that, when a coalition is formed, a player in the coalition either can fully cooperate within the coalition or cannot cooperate at all. Nevertheless, in some situations it is possible to consider another option: that a player has a degree of freedom to cooperate within the coalition. In order to deal with these situations, a kind of structure that was introduced by Gallardo et al. [4] will be used. Firstly, we need to recall the concept of fuzzy coalition.

Aubin [1] defined a *fuzzy coalition* in N to be a fuzzy subset $e \in [0,1]^N$, where for all $i \in N$ the number $e_i \in [0,1]$ is regarded as the degree of participation of player i in e. Every coalition $E \subseteq N$ can be identified with the fuzzy coalition $1_E \in [0,1]^N$ defined as

$$(1_E)_i = \begin{cases} 1 & \text{if } i \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Now, fuzzy authorization operators will be presented. These operators will allow to model games with fuzzy coalition restrictions.

Definition 1. A fuzzy authorization operator on N is a mapping $a : 2^N \to [0,1]^N$ that satisfies the following conditions:

1. $a(E) \subseteq 1_E$ for any $E \subseteq N$,
2. if $E \subset F$ then $a(E) \subseteq a(F)$.

The pair (N, a) is called a fuzzy authorization structure. The set of all fuzzy authorization operators on N is denoted by $\mathcal{FA}^N$.

Given $E \subseteq N$ and $i \in N$, we interpret $a_i(E)$ as the proportion of the whole operating capacity of player i that he is allowed to use within coalition E.

Definition 2. A fuzzy authorization operator a is said to be normal if $a(N) = 1_N$. The set of normal fuzzy authorization operators is denoted by $\widetilde{\mathcal{FA}}^N$.

Definition 3. A game with fuzzy authorization structure on N is a pair (v, a) where $v \in \mathcal{G}^N$ and $a \in \mathcal{FA}^N$.

Given a game with fuzzy authorization structure, a characteristic function that gathers the information from the game and the structure in a reasonable way is defined. To do this, we follow the approach described by Tsurumi, Tanino and Inuiguchi [8], using the Choquet integral [3] to define a new auxiliary game that combines the information from the original game and from the fuzzy dependency relationships.

Definition 4. Let $v \in \mathcal{G}^N$ and $a \in \mathcal{FA}^N$. The restricted game of (v, a) is the game $v^a \in \mathcal{G}^N$ defined as

$$v^a(E) = \int a(E)\, dv \qquad \text{for all } E \subseteq N,$$

where $\int a(E)\, dv$ denotes the Choquet integral of $a(E)$ with respect to v.

The number $v^a(E)$ is the worth of E in the game with fuzzy authorization structure (v, a).

An allocation rule for games with fuzzy authorization structure assigns to every game with fuzzy authorization structure a payoff vector.

Definition 5. The Shapley fuzzy authorization value is defined as

$$\varphi(v, a) = \phi\left(v^a\right) \qquad \text{for all } v \in \mathcal{G}^N \text{ and } a \in \mathcal{FA}^N.$$

The Shapley fuzzy authorization value has been studied by Gallardo et al. [4].

4. The Shapley fuzzy authorization NTU correspondence

In this section we use a well known solution for NTU games, the Shapley correspondence, to introduce solutions for NTU games with authorization structure. We only consider the NTU games V satisfying the following conditions:

(i) $V(E)$ is convex and comprehensive for all nonempty $E \subseteq N$.
(ii) $V(N)$ is a proper, closed and smooth subset of $\mathbb{R}^N$.
(iii) For every $x \in \partial(V(N))$, $\{y \in \mathbb{R}^N : y \geqslant x\} \cap V(N) = \{x\}$.
(iv) There exists $x \in \mathbb{R}^N$ such that $V(E) \times \{0\}^{N \setminus E} \subseteq x + V(N)$ for every nonempty $E \subseteq N$.
(v) For every nonempty $E \subseteq N$ and $\lambda \in \mathbb{R}^E_{++}$, the set $\{\lambda \cdot x : x \in V(E)\}$ is closed.

We denote Γ^N the set of NTU games satisfying (i), (ii), (iii), (iv) and (v).

Now, we introduce NTU games with fuzzy authorization structure.

Definition 6. An NTU game with fuzzy authorization structure is a pair (V, a) where V is an NTU game on N and $a \in \widetilde{\mathcal{FA}}^N$.

Definition 7. Let $V \in \Gamma^N$, $a \in \widetilde{\mathcal{FA}}^N$ and $\{t_l : l = 0, \ldots, r\} = \{a_k(F) : F \subseteq N, k \in N\}$ with $0 = t_0 < \ldots < t_r = 1$. The restricted game of (V, a) is the NTU game V^a defined as

$$V^a(E) = \sum_{l=1}^{r} (t_l - t_{l-1})\, V^{a^{t_l}}(E), \quad \text{for all } E \in 2^N \setminus \{\emptyset\},$$

where, for every $t \in (0, 1]$, $a^t(E) = \{k \in E : a_k(E) \geqslant t\}$ and

$$V^{a^t}(E) = \begin{cases} V(E) & \text{if } a^t(E) = E, \\ (-\infty, 0]^E & \text{if } a^t(E) = \emptyset, \\ V(a^t(E)) \times (-\infty, 0]^{E \setminus a^t(E)} & \text{otherwise.} \end{cases}$$

We aim to define a correspondence that assigns to each NTU game with fuzzy authorization structure (V, a) with $V \in \Gamma^N$ and $a \in \tilde{\mathcal{FA}}^N$ a set of payoff vectors. We need a previous result.

Proposition 1. *Let $V \in \Gamma^N$ and $a \in \widetilde{\mathcal{FA}}^N$. Then,*

a) $V^a(N) = V(N)$,
b) $V^a \in \Gamma^N$.

Proof. Let $V \in \Gamma^N$, $a \in \widetilde{\mathcal{FA}}^N$ and $\{t_l : l = 0, \ldots, r\} = \{a_k(F) : F \subseteq N, k \in N\}$ with $0 = t_0 < \ldots < t_r = 1$.

a) We have

$$V^a(N) = \sum_{l=1}^{r} (t_l - t_{l-1})\, V^{a^{t_l}}(N) = \sum_{l=1}^{r} (t_l - t_{l-1})\, V(N)$$

which, taking into consideration that $V(N)$ is convex, is equal to $V(N)$.

b) We must prove that V^a satisfies the five properties that characterize the NTU games in Γ^N. From $V \in \Gamma^N$ and $V^a(N) = V(N)$ it follows that V^a satisfies (ii) and (iii).
Let E be a nonempty subset of N. From the fact that $V^{a^{t_l}}(E)$ is convex and comprehensive for all $l = 1, \ldots, r$ it can be derived that $V^a(E)$ is convex and comprehensive. Hence V^a satisfies (i).
Since $V \in \Gamma^N$ there exists $x \in \mathbb{R}^N$ such that $V(F) \times \{0\}^{N \setminus F} \subseteq x + V(N)$ for every nonempty $F \subseteq N$. Since $V(N)$ is comprehensive, it is clear that we can assume that $0 \in x + V(N)$. In these conditions, it is easy to check, making use of the comprehensiveness of $V(N)$, that $V^{a^{t_l}}(E) \times \{0\}^{N \setminus E} \subseteq x + V(N)$ for all $l = 1, \ldots, r$. We have

$$\begin{aligned} V^a(E) \times \{0\}^{N \setminus E} &= \left(\sum_{l=1}^{r} (t_l - t_{l-1})\, V^{a^{t_l}}(E) \right) \times \{0\}^{N \setminus E} \\ &= \sum_{l=1}^{r} (t_l - t_{l-1}) \left(V^{a^{t_l}}(E) \times \{0\}^{N \setminus E} \right) \\ &\subseteq \sum_{l=1}^{r} (t_l - t_{l-1})\, (x + V(N)) \end{aligned}$$

which, using the convexity of $V(N)$, is equal to $x + V(N)$. Therefore, V^a satisfies (iv).

Let $\lambda \in \mathbb{R}^E_{++}$. We have

$$\lambda \cdot V^a(E) = \lambda \cdot \left(\sum_{l=1}^{r} (t_l - t_{l-1}) \, V^{a^{t_l}}(E) \right) = \sum_{l=1}^{r} (t_l - t_{l-1}) \left(\lambda \cdot V^{a^{t_l}}(E) \right)$$

which is closed, since it is a sum of closed intervals in the real line. So V^a satisfies (v). □

Definition 8. The Shapley fuzzy authorization NTU correspondence is given by

$$\theta(V, a) = SH\left(V^a\right) \quad \text{for every } V \in \Gamma^N \text{ and } a \in \widetilde{\mathcal{FA}}^N,$$

where SH denotes the Shapley NTU correspondence.

We aim to give a characterization of the Shapley fuzzy authorization NTU correspondence. To this end, we consider the properties that we state below. In the statement of these properties, ψ is a correspondence that assigns to each $V \in \Gamma^N$ and $a \in \widetilde{\mathcal{FA}}^N$ a subset $\psi(V, a) \subseteq \mathbb{R}^N$.

- **Non-emptiness for transferable total profit.** Let $V \in \Gamma^N$ be such that $\partial(V(N))$ is a hyperplane and $a \in \widetilde{\mathcal{FA}}^N$. Then

  $$\psi(V, a) \neq \emptyset.$$

- **Efficiency.** If $V \in \Gamma^N$ and $a \in \widetilde{\mathcal{FA}}^N$, then

  $$\psi(V, a) \subseteq \partial(V(N)).$$

- **Conditional additivity.** If $a \in \widetilde{\mathcal{FA}}^N$ and $V, W \in \Gamma^N$ are such that $V + W \in \Gamma^N$, then

  $$(\psi(V, a) + \psi(W, a)) \cap \partial\left((V + W)(N)\right) \subseteq \psi(V + W, a).$$

- **Scale covariance.** If $V \in \Gamma^N$, $a \in \widetilde{\mathcal{FA}}^N$ and $\alpha \in \mathbb{R}^N_{++}$, then

  $$\psi(\alpha * V, a) = \alpha * \psi(V, a).$$

- **Independence of irrelevant alternatives.** If $a \in \widetilde{\mathcal{FA}}^N$ and $V, W \in \Gamma^N$ are such that $V(N) \subseteq W(N)$ and $V(E) = W(E)$ for every $E \neq N$, then

  $$\psi(W, a) \cap V(N) \subseteq \psi(V, a).$$

- **Consistency with the Shapley fuzzy authorization value.** If $v \in \mathcal{G}^N$ and $a \in \widetilde{\mathcal{FA}}^N$, then

  $$\psi(V_v, a) = \{\varphi(v, a)\},$$

  where φ denotes the Shapley fuzzy authorization value.

In the following theorem it is proved that these properties characterize the Shapley fuzzy authorization NTU correspondence.

Theorem 2. *A mapping $\psi : \Gamma^N \times \widetilde{\mathcal{FA}}^N \to 2^{\mathbb{R}^N}$ is equal to the Shapley fuzzy authorization NTU correspondence if and only if it satisfies the properties of non-emptiness for transferable total profit, efficiency, conditional additivity, scale covariance, independence of irrelevant alternatives and consistency with the Shapley fuzzy authorization value.*

Proof. Firstly, we prove that the Shapley fuzzy authorization NTU correspondence satisfies such properties.

NON-EMPTINESS FOR TRANSFERABLE TOTAL PROFIT. Let $V \in \Gamma^N$ be such that $\partial(V(N))$ is a hyperplane. From properties (i) and (iii) of the games in Γ^N we can derive that there exist $\lambda \in \mathbb{R}^N_{++}$ and $\alpha \in \mathbb{R}$ such that $\partial(V(N)) = \{y \in \mathbb{R}^N : \lambda \cdot y = \alpha\}$. Therefore, $V(N) = \left\{y \in \mathbb{R}^N : \lambda \cdot y \leqslant \alpha\right\}$. From property (iv) it follows that

$\sup\left\{\lambda^E \cdot z : z \in V^a(E)\right\} < +\infty$ for all nonempty $E \subseteq N$. Let $w \in \mathcal{G}^N$ given by $w(E) = \sup\left\{\lambda^E \cdot z : z \in V^a(E)\right\}$ for all nonempty $E \subseteq N$ and $w(\emptyset) = 0$, and take $x \in \mathbb{R}^N$ defined by $x_i = \dfrac{\phi_i(w)}{\lambda_i}$. It is clear that $x \in \theta(V, a)$.

EFFICIENCY. Let $V \in \Gamma^N$ and $a \in \widetilde{\mathcal{FA}}^N$. Using the efficiency property of the Shapley NTU correspondence we can write

$$\theta(V, a) = SH\left(V^a\right) \subseteq \partial\left(V^a(N)\right) = \partial(V(N)).$$

CONDITIONAL ADDITIVITY. Let $V, W \in \Gamma^N$ be such that $V + W \in \Gamma^N$. Let $a \in \widetilde{\mathcal{FA}}^N$. We have

$$(\theta(V, a) + \theta(W, a)) \cap \partial((V + W)(N)) = \left(SH\left(V^a\right) + SH\left(W^a\right)\right) \cap \partial\left((V + W)^a(N)\right)$$

which, taking into account that $(V + W)^a = V^a + W^a$, is equal to

$$\left(SH\left(V^a\right) + SH\left(W^a\right)\right) \cap \partial\left((V^a + W^a)(N)\right) \subseteq SH\left(V^a + W^a\right)$$

where we have used the fact that the Shapley NTU correspondence satisfies conditional additivity. Finally it suffices to notice that

$$SH\left(V^a + W^a\right) = SH\left((V + W)^a\right) = \theta(V + W, a).$$

SCALE COVARIANCE. Let $V \in \Gamma^N$, $a \in \widetilde{\mathcal{FA}}^N$ and $\alpha \in \mathbb{R}^N_{++}$. We have

$$\theta(\alpha * V, a) = SH\left((\alpha * V)^a\right) = SH\left(\alpha * V^a\right)$$

which, taking into account that the Shapley NTU correspondence satisfies scale covariance, is equal to

$$\alpha * SH\left(V^a\right) = \alpha * \theta(V, a).$$

INDEPENDENCE OF IRRELEVANT ALTERNATIVES. Let $V, W \in \Gamma^N$ be such that $V(N) \subseteq W(N)$ and $V(E) = W(E)$ for every $E \neq N$. Let $a \in \widetilde{\mathcal{FA}}^N$. We have

$$\theta(W, a) \cap V(N) = SH\left(W^a\right) \cap V^a(N) \subseteq SH\left(V^a\right) = \theta(V, a)$$

where the inclusion follows from the fact that the Shapley NTU correspondence satisfies the property of independence of irrelevant alternatives.

CONSISTENCY WITH THE SHAPLEY FUZZY AUTHORIZATION VALUE. Let $v \in \mathcal{G}^N$, $a \in \widetilde{\mathcal{FA}}^N$ and $\{t_l : l = 0, \dots, r\} = \{a_k(F) : F \subseteq N, k \in N\}$ with $0 = t_0 < \ldots < t_r = 1$. Since $\theta(V_v, a) = SH((V_v)^a)$, we will calculate $SH((V_v)^a)$. A vector $x \in \mathbb{R}^N$ belongs to $SH((V_v)^a)$ if and only if there exists $\lambda \in \mathbb{R}^N_{++}$ such that

1. $x \in (V_v)^a(N)$,
2. $\sup\left\{\lambda^E \cdot z : z \in (V_v)^a(E)\right\} < +\infty$ for all nonempty $E \subseteq N$,
3. $\lambda * x = \phi(w_\lambda)$ where w_λ is the TU game defined by

$$w_\lambda(E) = \sup\left\{\lambda^E \cdot z : z \in (V_v)^a(E)\right\} \text{ for all nonempty } E \subseteq N.$$

Taking into consideration that

$$(V_v)^a(N) = V_v(N) = \left\{z \in \mathbb{R}^N : \sum_{k \in N} z_k \leqslant v(N)\right\},$$

it is easily verified that

$$\sup\left\{\lambda \cdot z : z \in (V_v)^a(N)\right\} = +\infty \quad \text{for all } \lambda \in \mathbb{R}^N_{++} \setminus \{s \cdot 1_N : s > 0\}.$$

Therefore, it is clear that the only element in $SH((V_v)^a)$ is the one that is obtained with $\lambda = 1_N$. Hence, $\theta(V_v, a) = \{\phi(w_{1_N})\}$. If we prove that $w_{1_N} = v^a$ we will have finished, since $\phi(v^a) = \varphi(v, a)$. To this end, take E a nonempty subset of N. If $a(E) = 0$ it is clear that $w_{1_N}(E) = v^a(E) = 0$. If $a(E) \neq 0$, let $m = \max\left\{l : a^{t_l}(E) \neq \emptyset\right\}$. We have

$$\begin{aligned} w_{1_N}(E) &= \sup\left\{1_E \cdot z : z \in (V_v)^a(E)\right\} \\ &= \sup\left\{\sum_{l=1}^{r}(t_l - t_{l-1})\, 1_E \cdot z_l : z_l \in (V_v)^{a^{t_l}}(E) \text{ for all } l = 1, \dots, m\right\} \\ &= \sum_{l=1}^{r}(t_l - t_{l-1}) \sup\left\{ 1_E \cdot z : z \in (V_v)^{a^{t_l}}(E)\right\} \\ &= \sum_{l=1}^{m}(t_l - t_{l-1}) \sup\left\{ 1_{a^{t_l}(E)} \cdot y : y \in V_v(a^{t_l}(E))\right\} \\ &= \sum_{l=1}^{m}(t_l - t_{l-1})\, v\left(a^{t_l}(E)\right) \\ &= v^a(E). \end{aligned}$$

We have proved that θ satisfies the properties in the theorem. Now we aim to see that these properties uniquely determine the Shapley fuzzy authorization NTU correspondence.

Let $\psi : \Gamma^N \times \widetilde{\mathcal{FA}}^N \to 2^{\mathbb{R}^N}$ be a mapping satisfying the properties of non-emptiness for transferable total profit, efficiency, conditional additivity, scale covariance, independence of irrelevant alternatives and consistency with the Shapley fuzzy authorization value. Now, take $V \in \Gamma^N$ and $a \in \widetilde{\mathcal{FA}}^N$. We will prove that $\psi(V, a) = \theta(V, a)$.

Firstly, it will be proved that $\psi(V, a) \subseteq \theta(V, a)$. Let $x \in \psi(V, a)$. Since ψ satisfies efficiency, $x \in \partial(V(N))$. From properties (i), (ii) and (iii) of the games in Γ^N it follows that there exists $\lambda \in \mathbb{R}^N_{++}$ such that the supporting hyperplane of $V(N)$ at x is $\{z \in \mathbb{R}^N : \lambda \cdot z = \lambda \cdot x\}$. We have

$$V(N) \subseteq \{z \in \mathbb{R}^N : \lambda \cdot z \leqslant \lambda \cdot x\}, \tag{1}$$

and, hence,

$$\lambda * V(N) \subseteq \{y \in \mathbb{R}^N : 1_N \cdot y \leqslant 1_N \cdot (\lambda * x)\}. \tag{2}$$

Let V_0 be the NTU game corresponding to the TU game that is identically zero. From (2) it can be derived that

$$\lambda * V(N) + V_0(N) = \{y \in \mathbb{R}^N : 1_N \cdot y \leqslant 1_N \cdot (\lambda * x)\}.$$

Therefore,

$$\lambda * x \in \partial((\lambda * V + V_0)(N)). \tag{3}$$

On the one hand, since ψ satisfies scale covariance, $\lambda * x \in \psi(\lambda * V, a)$. On the other hand, taking into account that ψ satisfies the property of consistency with the Shapley fuzzy authorization value, we have $\psi(V_0, a) = \{0\}$. These two facts, together with (3) and the property of additivity, allow to conclude that

$$\lambda * x \in \psi\left(\lambda * V + V_0, a\right).$$

From (1) and property (iv) of the games in Γ^N it follows that

$$\sup\left\{\lambda^E \cdot z : z \in V(E)\right\} < +\infty \text{ for all nonempty } E \subseteq N.$$

Moreover, it can be easily verified, by using property (v) of the games in Γ^N, that $\lambda * V + V_0 = V_v$ where v is the TU game given by

$$v(E) = \sup\left\{\lambda^E \cdot z : z \in V(E)\right\} \text{ for all nonempty } E \subseteq N.$$

Since ψ satisfies the property of consistency with the Shapley fuzzy authorization value, we have

$$\lambda * x \in \psi\,(\lambda * V + V_0, a) = \psi\,(V_v, a) = \{\varphi(v, a)\},$$

hence, $\lambda * x = \varphi(v, a) = \phi(v^a)$. Therefore, we have

(a) $x \in V(N) = V^a(N)$,
(b) $\sup\left\{\lambda^E \cdot z : z \in V^a(E)\right\} < +\infty$ for all nonempty $E \subseteq N$,
(c) $\lambda * x = \phi(v^a)$.

If we prove that

$$v^a(E) = \sup\left\{\lambda^E \cdot z : z \in V^a(E)\right\} \quad \text{for all nonempty } E \subseteq N, \tag{4}$$

we will have finished, since in that case (a), (b) and (c) will mean, by definition, that $x \in SH(V^a) = \theta(V, a)$. In order to prove (4) take E a nonempty subset of N. If $a(E) = 0$ the equality is clear. So we assume $a(E) \neq 0$. Let $\{t_l : l = 0, \ldots, r\} = \{a_k(F) : F \subseteq N, k \in N\}$ with $0 = t_0 < \ldots < t_r = 1$. Let $m = \max\left\{l : a^{t_l}(E) \neq \emptyset\right\}$. We have

$$\begin{aligned}
v^a(E) &= \sum_{l=1}^{m} (t_l - t_{l-1})\, v\left(a^{t_l}(E)\right) \\
&= \sum_{l=1}^{m} (t_l - t_{l-1}) \,\sup\left\{\lambda^{a^{t_l}(E)} \cdot y : y \in V\left(a^{t_l}(E)\right)\right\} \\
&= \sum_{l=1}^{r} (t_l - t_{l-1}) \,\sup\left\{\lambda^E \cdot z : z \in V^{a^{t_l}}(E)\right\} \\
&= \sup\left\{\lambda^E \cdot \left(\sum_{l=1}^{r} (t_l - t_{l-1})\, z_l\right) : z_l \in V^{a^{t_l}}(E) \text{ for all } l = 1, \ldots, m\right\} \\
&= \sup\left\{\lambda^E \cdot z : z \in V^a(E)\right\}.
\end{aligned}$$

It remains to prove that $\theta(V, a) \subseteq \psi(V, a)$. Let $x \in \theta(V, a)$. By definition there exists $\lambda \in \mathbb{R}^N_{++}$ such that

(a) $x \in V^a(N)$,
(b) $\sup\left\{\lambda^E \cdot z : z \in V^a(E)\right\} < +\infty$ for all nonempty $E \subseteq N$,
(c) $\lambda * x = \phi(w_\lambda)$ where w_λ is the TU game defined by

$$w_\lambda(E) = \sup\left\{\lambda^E \cdot z : z \in V^a(E)\right\} \text{ for all nonempty } E \subseteq N.$$

Consider the NTU game $W \in \Gamma^N$ defined as

$$W(E) = \begin{cases} \left\{y \in \mathbb{R}^N : \lambda \cdot y \leqslant \lambda \cdot x\right\} & \text{if } E = N, \\ V(E) & \text{if } E \neq N. \end{cases}$$

It is easy to check that $\theta(W, a) = \{x\}$. We know that $\psi(W, a) \subseteq \theta(W, a)$. Besides, from the property of non-emptiness for transferable total profit, it follows that $\psi(W, a) \neq \emptyset$. Therefore, $\psi(W, a) = \{x\}$. Finally, by using the property of independence of irrelevant alternatives, we conclude that $x \in \psi(V, a)$. □

In practice, if we have $V \in \Gamma^N$ and $a \in \widetilde{\mathcal{FA}}^N$, we do not need to obtain V^a to calculate $\theta(V, a)$. Let $\lambda \in \mathbb{R}^N_{++}$ be such that $\sup\{\lambda \cdot z : z \in V(N)\} < +\infty$. Consider $w_\lambda, v_\lambda \in \mathcal{G}^N$ defined by

$$w_\lambda(E) = \sup\left\{\lambda^E \cdot z : z \in V^a(E)\right\},$$

$$v_\lambda(E) = \sup\left\{\lambda^E \cdot z : z \in V(E)\right\},$$

for every nonempty $E \subseteq N$. Let us see that

$$w_\lambda = v_\lambda^a. \tag{5}$$

Let $\{t_l : l = 0, \dots, r\} = \{a_k(F) : F \subseteq N, k \in N\}$ with $0 = t_0 < \dots < t_r = 1$ and let $E \in 2^N \setminus \{\emptyset\}$. We must prove that $w_\lambda(E) = v_\lambda^a(E)$. It is easy to check that

$$\sup\left\{\lambda^E \cdot z : z \in V^{a^{t_l}}(E)\right\} = v_\lambda^{a^{t_l}}(E) \quad \text{for every } l = 1, \dots, r. \tag{6}$$

We have that

$$\begin{aligned} w_\lambda(E) &= \sup\left\{\lambda^E \cdot z : z \in V^a(E)\right\} \\ &= \sup\left\{\sum_{l=1}^{r}(t_l - t_{l-1})\left(\lambda^E \cdot z_l\right) : z_l \in V^{a^{t_l}}(E) \text{ for all } l = 1, \dots, r\right\} \\ &= \sum_{l=1}^{r}(t_l - t_{l-1}) \sup\left\{\lambda^E \cdot z : z \in V^{a^{t_l}}(E)\right\} \end{aligned}$$

which, from (6), is equal to

$$\sum_{l=1}^{r}(t_l - t_{l-1})\, v_\lambda^{a^{t_l}}(E) = v_\lambda^a(E).$$

We can use (5) to give a definition of $\theta(V, a)$ that does not involve the restricted game V^a, as we see in the following remark.

Remark 1. Let $V \in \Gamma^N$ and $a \in \widetilde{\mathcal{FA}}^N$. A vector $x \in \mathbb{R}^N$ belongs to $\theta(V, a)$ if and only if there exists $\lambda \in \mathbb{R}_{++}^N$ such that

(a) $x \in V(N)$,
(b) $\sup\{\lambda \cdot z : z \in V(N)\} < +\infty$,
(c) $\lambda * x = \varphi(v_\lambda, a)$ where v_λ is the TU game defined by

$$v_\lambda(E) = \sup\left\{\lambda^E \cdot z : z \in V(E)\right\} \quad \text{for all nonempty } E \subseteq N.$$

Example 1. Let $N = \{1, 2, 3\}$. Let V be an NTU game defined by

$$\begin{aligned} V(\{i\}) &= \left\{z_i \in \mathbb{R}^{\{i\}} : z_i \leqslant 1\right\} \text{ for every } i \in N, \\ V(\{i, j\}) &= \left\{z \in \mathbb{R}^{\{i,j\}} : z \leqslant (2, 2)\right\} \text{ for every } i, j \in N \text{ with } i \neq j, \\ V(N) &= \left\{(z_1, z_2, z_3) \in \mathbb{R}^N : z_1 + z_2 + z_3 \leqslant 6\right\}. \end{aligned}$$

Let a be the fuzzy authorization operator on $\{1, 2, 3\}$ defined in the following table.

E	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$a(E)$	$(1,0,0)$	$(0,1,0)$	$(0,0,0.4)$	$(1,1,0)$	$(1,0,0.6)$	$(0,1,0.8)$	$(1,1,1)$

Consider the NTU game with fuzzy authorization structure (V, a).

Let us calculate $\theta(V, a)$ without using the expression of the restricted game. We use Remark 1. Since $\partial(V(N))$ is a hyperplane, it is plain to see that $\theta(V, a)$ is a singleton $\{x\}$. It is clear that x is associated to the comparison vector $\lambda = (1, 1, 1)$. We proceed to calculate v_λ^a,

$$v_\lambda^a(\{1\}) = v_\lambda(\{1\}) = 1$$
$$v_\lambda^a(\{2\}) = v_\lambda(\{2\}) = 1$$
$$v_\lambda^a(\{3\}) = 0.4\, v_\lambda(\{3\}) = 0.4$$
$$v_\lambda^a(\{1,2\}) = v_\lambda(\{1,2\}) = 4$$
$$v_\lambda^a(\{1,3\}) = 0.6\, v_\lambda(\{1,3\}) + 0.4\, v_\lambda(\{1\}) = 2.8$$
$$v_\lambda^a(\{2,3\}) = 0.8\, v_\lambda(\{2,3\}) + 0.2\, v_\lambda(\{2\}) = 3.4$$
$$v_\lambda^a(\{1,2,3\}) = v_\lambda(\{1,2,3\}) = 6$$

We have that

$$\varphi(v_\lambda, a) = \phi\left(v_\lambda^a\right) = (2.1, 2.4, 1.5)$$

and from Remark 1 we conclude that

$$x = (2.1, 2.4, 1.5).$$

So we have obtained that

$$\theta(V, a) = \{(2.1, 2.4, 1.5)\}.$$

5. Conclusions

We have defined and characterized a value for NTU games with fuzzy coalition restrictions. This value can be applied to a wide range of situations, since fuzzy authorization structures extend many of the structures considered in the literature to model cooperative games with restricted cooperation. The value introduced is an extension of the Shapley value for NTU games. The use of other values for NTU games, like the Harsanyi value, could be considered for future research.

Acknowledgement

This research has been supported by the Ministerio de Economía y Competitividad under grant ECO2013-17766-P.

References

[1] J.P. Aubin, Cooperative fuzzy games, Math. Oper. Res. 6 (1991) 1–13.
[2] R.J. Aumann, An axiomatization of the non-transferable utility value, Econometrica 53 (1985) 599–612.
[3] G. Choquet, Theory of capacities, Ann. Inst. Fourier 5 (1953) 131–295.
[4] J.M. Gallardo, N. Jiménez, A. Jiménez-Losada, E.A. Lebrón, Games with fuzzy authorization structure: a Shapley value, Fuzzy Sets Syst. 272 (2014) 115–125.
[5] R.P. Gilles, G. Owen, R. van den Brink, Games with permission structures: the conjunctive approach, Int. J. Game Theory 20 (1992) 277–293.
[6] L.S. Shapley, A value for n-person games, in: H.W. Kuhn, A.W. Tucker (Eds.), Contributions to the Theory of Games, vol. 2, Princeton University Press, New Jersey, 1953, pp. 307–317.
[7] L.S. Shapley, Utility comparison and theory of games, in: La Decision: Aggregation et Dynamique des Ordres de Preference, Editions du Centre National de la Recherche Scientifique, Paris, 1969, pp. 251–263.
[8] M. Tsurumi, T. Tanino, M. Inuiguchi, A Shapley function on a class of cooperative fuzzy games, Eur. J. Oper. Res. 129 (2001) 596–618.