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# The selectope for bicooperative games

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ABSTRACT

A bicooperative game is defined by a worth function on the set of ordered pairs of disjoint coalitions of players. The aim of this paper is to analyze the selectope for bicooperative games. This solution concept was introduced by Hammer et al. (1977) [20] and studied by Derks et al. (2000) [10] for cooperative games. We show the relations between the selectope, the core and the Weber set and obtain a characterization of almost positive bicooperative games as bicooperative games such that the core, the Weber set and the selectope coincide. Moreover, an axiomatic characterization of the elements of the selectope is obtained.



## 1. Introduction

A cooperative game with transferable utility is given by a finite set of players and a real-valued worth function defined on the set of all the subsets, or *coalitions*, of players such that the worth of the empty set is zero. For each coalition, the worth can be interpreted as the maximal gain or minimal cost that the players in this coalition can achieve by themselves against the best offensive threat by the complementary coalition. Classical market games for economies with private goods are examples of cooperative games. Here, we say that such a game has *orthogonal coalitions* (see Myerson [25, Chapter 9]).

Games with non-orthogonal coalitions are games in which the worth of a coalition depends on the actions of its complementary coalition. Clearly, social situations involving externalities and public goods are such cases. For instance, the joint owners of a building are considering hiring a gardener to work in the common areas of their residence. The garden is a public good. Each owner can decide to support the proposal or to veto it. However, some of them may decide not to take part in the decision making and would thus not necessarily be *defenders* or *detractors* of the project. This is the case with multicriteria decision making when underlying scales are bipolar, i.e. a central value exists on each scale and it is considered a neutral value.

Situations of this kind may be modeled in the following manner. We consider ordered pairs of disjoint coalitions of players. Each such pair yields a partition of the set of all players in three groups. Players in the first coalition are in favor of the proposal, and players in the second coalition object to it. The remaining players are not convinced of its benefits, but they have no intention of objecting to it. This leads us in a natural way into the concept of *bicooperative game* introduced by Bilbao [1] . Bicooperative games generalize classical cooperative games in the sense that a player is allowed to play in favor or against some aim, besides non participation.

An application provided by one of the referees is the following example of a Chinese postman problem [7]. Each player is associated to one and only one edge of an undirected graph. In addition, each edge has a cost. A particular node, denoted by $v^0$, represents the post office. Then a $S$-tour associated with coalition $S$ is defined by a closed walk starting at $v^0$, visiting all players in $S$ and coming back to $v^0$. In the usual definition of a Chinese postman problem, a $S$-tour can also use edges associated to players outside $S$. Each player first pays the cost of his edge and a worth for coalition $S$ is then measured by the minimal total cost of edges. However, it can happen that a subset $T \subseteq N \setminus S$ of the players outside $S$ is not in favor of the $S$-tour. Then, one can imagine an alternative definition of a $S$-tour in which the closed walk can also use edges associated to the neutral players in $N \setminus (S \cup T)$ but not edges associated to players in $T$, i.e. the roads owned by the coalition of detractors $T$ are not open for the $S$-tour. Bicooperative games incorporate this type of extra feature.

A central question in game theory is to define a solution concept for a game, or a class of games. A solution concept for a class of games is a function which assigns to every game a set of real-valued vectors, each one of them represents a payoff distribution among the players. The *core* [19] is one of the most studied solution concepts and it consists of all payoff vectors which distribute the worth of the grand

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coalition under the condition that the players in each coalition receive at least the amount that they can obtain by cooperating. However, if a game is not convex [28], then the core can be empty. This leads us to considering other solution concepts. In 1978, Weber [30] proposed a set that is always nonempty. It is now called the *Weber set*. Weber showed that the core of any cooperative game is a subset of the Weber set, and Ichiishi [23] proved that if the Weber set is a subset of the core of a game, then the game is convex. Another solution concept, the *selectope*, was introduced by Hammer et al. [20] and independently by Vasil'ev [29] and investigated by Derks et al. [10] for cooperative games. This concept is the main objective of our paper. Briefly, the selectope contains all ways of distributing the so-called *dividends* of the coalitions between their members and it is also called Harsanyi set in [12]. The objective of this paper have been to generalize the concepts and prove the results in [10] for bicooperative games.

Let us briefly outline the contents of our work. We begin by introducing bicooperative games and, in this context, different solution concepts are considered: imputations, core and Weber set. In [2], some relations between them are established. In the second part of this paper, we introduce the selectope for bicooperative game and we prove that the core and the Weber set are always contained in the selectope. Moreover, we give some conditions for which these solution concepts coincide. Finally, the elements of the selectope are considered from a value-theoretic point of view and we include a section of concluding remarks. Throughout this paper, for $S \subseteq N$, we use $S \setminus i$ instead of $S \setminus \{i\}$, the number of elements in $S$ is $|S|$ and $u(S) = \sum_{i \in S} u_i$ for $u \in \mathbb{R}^n$, with $u(\emptyset) = 0$.

## 2. Bicooperative games

Let $N = \{1, \ldots, n\}$ be a finite set and $3^N = \{(A, B) : A, B \subseteq N, \ A \cap B = \emptyset\}$. Grabisch and Labreuche [17] proposed the partial order in $3^N$ given by

$$(A, B) \sqsubseteq (C, D) \Longleftrightarrow A \subseteq C, \quad B \supseteq D.$$

We denote by $\sqsubset$ the relation defined by means of the weak strict inclusion, that is, $(A, B) \sqsubset (C, D)$ if and only if $(A \subset C, \ B \supseteq D)$ or $(A \subseteq C, \ B \supset D)$.

The set $(3^N, \sqsubseteq)$ is a partially ordered set (poset) with the following properties:

1. $(\emptyset, N)$ is the first element: $(\emptyset, N) \sqsubseteq (A, B)$ for all $(A, B) \in 3^N$.
2. $(N, \emptyset)$ is the last element: $(A, B) \sqsubseteq (N, \emptyset)$ for all $(A, B) \in 3^N$.
3. Every pair $\{(A, B), (C, D)\}$ of elements of $3^N$ has a join

$$(A, B) \vee (C, D) = (A \cup C, B \cap D),$$

and a meet

$$(A, B) \wedge (C, D) = (A \cap C, B \cup D).$$

Moreover, $(3^N, \sqsubseteq)$ is a finite distributive lattice.

Two pairs $(A, B)$ and $(C, D)$ in $3^N$ are *comparable* if $(A, B) \sqsubseteq (C, D)$ or $(C, D) \sqsubseteq (A, B)$; otherwise, $(A, B)$ and $(C, D)$ are *non-comparable*. A *chain* in $3^N$ is an induced subposet of $3^N$ in which any two elements are comparable. Moreover, if two pairs are comparable, there exists at least one chain that contains them. In $(3^N, \sqsubseteq)$, all maximal chains have the same number of elements and this number is $2n + 1$.

We model above mentioned class of non-orthogonal situations by mean of the set of all ordered pairs of disjoint coalitions, that is, the set $3^N$ and a worth function $b : 3^N \to \mathbb{R}$. For each $(S, T) \in 3^N$, the number $b(S, T)$ can be interpreted as the gain (whenever $b(S, T) > 0$) or loss (whenever $b(S, T) < 0$) that $S$ can achieve when $T$ is the opposer coalition and $N \setminus (S \cup T)$ is the neutral coalition. The pair $(\emptyset, N)$ represents the situation if all the players object to the change and $(N, \emptyset)$ represents the situation where all the players wish the change.

**Definition 1.** A bicooperative game is a pair $(N, b)$, where $N$ a finite set of players and $b : 3^N \to \mathbb{R}$ is a function such that $b(\emptyset, \emptyset) = 0$.

A bicooperative game $(N, b)$ is *monotonic*if for all pairs $(S_1, T_1), (S_2, T_2)$ in $3^N$ with $(S_1, T_1) \sqsubseteq (S_2, T_2)$, we have $b(S_1, T_1) \leqslant b(S_2, T_2)$, that is, the addition of players to the *defender coalition* and the desertion of players from the *detractor coalition* does not decrease the worth. A *bicapacity* [17,18] is a function $v : 3^N \to \mathbb{R}$ such that $v(\emptyset, \emptyset) = 0$ and $A \subseteq B \subseteq N$ implies $v(A, \cdot) \leqslant v(B, \cdot)$ and $v(\cdot, A) \geqslant v(\cdot, B)$. Although bicooperative games and bicapacities were proposed independently and for different domains, bicapacities are monotonic bicooperative games.

As for standard cooperative games, where each coalition $S \in 2^N$ can be identified with a $\{0,1\}$-vector, each $(S, T) \in 3^N$ can be identified with the $\{-1, 0, 1\}$-vector $\mathbf{1}_{(S,T)}$ defined, for all $i \in N$, by

$$\mathbf{1}_{(S,T)}(i) = \begin{cases} 1 & \text{if } i \in S, \\ -1 & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

More generally, one may imagine that each player can choose one alternative and hence bicooperative games can be seen as a particular case of games with $n$ players and $r$ alternatives (for $r = 3$), introduced by Bolger in [5,6]. However, the $r$ possible input alternatives analyzed by Bolger are not ordered and hence the lattice structures of the domains of bicooperative games and games with $n$ players and three alternatives are different. For instance, the element $(\emptyset, \emptyset)$ is central in our structure $(3^N, \sqsubseteq)$ and $(0,0,0)$ is the least element in $(3^N, \preceq)$, where $\preceq$ is the coordinatewise order. Note that in a bicooperative game, the value 0 is central, and $-1$, 1 are symmetric extremes. This suggests that bicooperative games are a symmetrization of classical cooperative games and this is the main reason to choose $b(\emptyset, \emptyset) = 0$. Also should be noted that bicooperative games *with ordered finite output* are a particular class of the $(3, k)$ hypergraphs introduced by Freixas and Zwicker in [16].

In voting games, each voter has three choices: voting for a proposal, voting against it, and abstaining. Thus, only knowing who is in favor of the proposal is not enough to describe the situation. These games have been studied by Felsenthal and Machover [13] under the name of *ternary voting games*. They generalize the standard voting games by recognizing abstention as an option alongside *yes* and *no* votes. They are described by mappings $u : 3^N \to \{-1, 1\}$ satisfying the following three conditions: $u(N, \emptyset) = 1, u(\emptyset, N) = -1$, and $\mathbf{1}_{(S,T)}(i) \leqslant \mathbf{1}_{(S',T')}(i)$ for all $i \in N$, implies $u(S,T) \leqslant u(S',T')$. A negative outcome, $-1$, is interpreted as defeat and a positive outcome, 1, as victory, the passing of a bill. The proposal of Felsenthal and Machover could be refine by introducing a third output for $u$, which is 0, and represents the 'no decision' situation. More recently, several works by Freixas [14,15] and Freixas and Zwicker [16] have been devoted to the study of voting systems with several ordered levels of approval in the input and in the output. In their model, the abstention is a level of input approval intermediate between *yes* and *no* votes. These authors have been generalized the ternary voting games by the definition of the so-called $(j, k)$ *simple games*. Thus, a bicooperative simple game $b : 3^N \to \{-1, 0, 1\}$ is a (3,3) simple game such that $b(\emptyset, N) = -1, b(\emptyset, \emptyset) = 0$, and $b(N, \emptyset) = 1$.

An special case of simple games are the weighted voting games. In the context of cooperative games, the power of a player to block a decision is the same as the player has in order to approve it. However, let us consider the quota $q$ to take decisions and another quota $p$ to block decisions. These situations can be represented by a simple bicooperative game $b : 3^N \to \{-1, 0, 1\}$, defined by

$$b(S,T) = \begin{cases} 1 & \text{if } w(S) \geqslant q \quad \text{and} \quad p(T) < p, \\ -1 & \text{if } w(S) < q \quad \text{and} \quad p(T) \geqslant p, \\ 0 & \text{otherwise}, \end{cases}$$

where each player $i \in N$ has a number of votes $w_i > 0$ in order to approve a decision and a number of votes $p_i > 0$ in order to block it. That is, a weighted voting bicooperative game is represented by the voting scheme $b = [[q; w_1, \ldots, w_n], [p; p_1, \ldots, p_n]]$.

Another model suppose that each player can "participate" at different levels, ranging from no participation to full participation. If there is a finite number of such participation levels, we have a *multi-choice game* (see Hsiao and Raghavan [22], Nouweland et al. [27]). In a multi-choice game, each player has at his/her disposal a linear ordered set of levels of participation labelled $0, 1, \ldots, m$ where 0 indicates no participation, and $m$ full participation. A multi-choice game is a function $v : \{0, 1, \ldots, m\}^N \to \mathbb{R}$ such that $v(0, \ldots, 0) = 0$. The number $v(x)$ is the amount for profile of participation $x \in \{0, 1, \ldots, m\}^N$. Since $\{0, 1, 2\}^N$ is isomorphic to the set $3^N$, the domains of bicooperative games and multi-choice games with $m = 2$ coincide. But the lattice structures of these sets are different. In a 2-choice game, the levels of participation are 0 (non participation), 1 (mild participation), and 2 (full participation). However, in a bicooperative game, the value 0 is central, and $-1$, 1 are extreme values.

We denote by $\mathcal{BG}^N$ the set of all bicooperative games on $N$, that is

$$\mathcal{BG}^N = \{b : 3^N \to \mathbb{R} \text{ such that } b(\emptyset, \emptyset) = 0\}.$$

With the addition and multiplication by real numbers, the set $\mathcal{BG}^N$ is a vector space. There are some special collections of games in $\mathcal{BG}^N$ taking values in $\{-1, 0, 1\}$, the *superior unanimity games* and the *inferior unanimity games* which are defined, for any $(S,T) \in 3^N, \ (S,T) \neq (\emptyset, \emptyset)$ as follows.

The superior unanimity game $\bar{u}_{(S,T)} : 3^N \to \mathbb{R}$ is given by

$$\bar{u}_{(S,T)}(A,B) = \begin{cases} 1 & \text{if } (S,T) \sqsubseteq (A,B), \quad (A,B) \neq (\emptyset, \emptyset), \\ 0 & \text{otherwise}. \end{cases}$$

The inferior unanimity game $\underline{u}_{(S,T)} : 3^N \to \mathbb{R}$ is defined by

$$\bar{u}_{(S,T)}(A,B) = \begin{cases} -1 & \text{if } (A,B) \sqsubseteq (S,T), \quad (A,B) \neq (\emptyset, \emptyset), \\ 0 & \text{otherwise}. \end{cases}$$

The relevance of these collections of games is made clear in the following result (see [3,4]).

**Proposition 1.** *The two collections* $\{\bar{u}_{(S,T)} : (S,T) \in 3^N, (S,T) \neq (\emptyset, \emptyset)\}$ *and* $\{\underline{u}_{(S,T)} : (S,T) \in 3^N, (S,T) \neq (\emptyset, \emptyset)\}$ *are bases of* $\mathcal{BG}^{\mathcal{N}}$ *and the dimension of* $\mathcal{BG}^N$ *is* $3^n - 1$.

## 3. Solution concepts for bicooperative games

Since in a bicooperative game, $b(\emptyset, N)$ is the cost (or expense) incurred when all the players object to a proposal and $b(N, \emptyset)$ is the gain obtained when all players are in its favor, then the net profit is given by $b(N, \emptyset) - b(\emptyset, N)$. A solution concept for bicooperative games is a function that assigns to every bicooperative game a set of payoff vectors that distribute the net profit among the players. In [2], we introduce two solution concepts for bicooperative games: the core and the Weber set.

A vector $x \in \mathbb{R}^n$ which satisfies $\sum_{i \in N} x_i = b(N, \emptyset) - b(\emptyset, N)$ is an *efficient vector* and the set of all efficient vectors, denoted by $I^*(N, b)$, is the *preimputation set*. The *imputations* for a bicooperative game $b$ are the *preimputations* that satisfy the *individual rationality principle* for all players, that is, each player gets at least the difference between the amount that he can attain by himself against the rest of the players and the worth of the pair $(\emptyset, N)$. Thus, the *imputation set* is

$$I(N,b) = \{x \in I^*(N,b) : x_i \geqslant b(i, N \setminus i) - b(\emptyset, N) \quad \text{for all } i \in N\}.$$

Another distribution criterion is the following: every pair $(S,T) \in 3^N$ receives at least the amount it contributes to the pair $(\emptyset, N)$, the difference $b(S,T) - b(\emptyset, N)$. Now, two different sets of players contribute to the formation of each $(S,T) \in 3^N$. On the one hand, the players who are in $N \setminus T$ do not act against the players of $S$ and so, they must receive a payoff (represented by a vector $z \in R^n$). On the other hand, those players in $N \setminus T$ who also are in $S$ must get an additional payoff (represented by a vector $y \in R^n$). This leads us to the following definition of the core of a bicooperative game $b$ which is an extension of the classical core of cooperative games (see [2]),

$$C(N,b) = \left\{ \begin{array}{l} x \in I^*(N,b) : \quad \text{there exist } y, z \in \mathbb{R}^n \text{ such that } x = y + z, \text{ and} \\ y(S) + z(N \setminus T) \geqslant b(S,T) - b(\emptyset, N), \quad \text{for all } (S,T) \in 3^N \end{array} \right\}.$$

In order to extend the Weber set to a bicooperative game $(N, b)$, it is assumed that $(N, \emptyset)$ is formed by a sequential process where at each step a player joins the defender coalition or a player leaves the detractor coalition. These sequential processes are obtained for each chain from $(\emptyset, N)$ to $(N, \emptyset)$. For each chain, a player can evaluate his contribution when he joins the defenders or when he leaves the detractors. These contributions are given as vectors in $\mathbb{R}^n$, called *superior marginal worth vectors* and *inferior marginal worth vectors*. To formalize this idea, we introduce the following notation.

For $N = \{1, \ldots, n\}$, let $\overline{N} = \{-n, \ldots, -1, 1, \ldots, n\}$. Let $\Lambda : 3^N \to 2^{\overline{N}}$ be the injective map defined by $\Lambda(S, T) = S \cup \{-i : i \in N \setminus T\} \in 2^{\overline{N}}$, for each $(S, T) \in 3^N$. For instance, $\Lambda(\emptyset, N) = \emptyset$ and $\Lambda(N, \emptyset) = \overline{N}$. As $S \subseteq N \setminus T$, we see that $i \in \Lambda(S, T)$ and $i > 0$ imply $-i \in \Lambda(S, T)$.

In the lattice $(3^N, \sqsubseteq)$, let $\Theta(3^N)$ denote the set of all maximal chains going from $(\emptyset, N)$ to $(N, \emptyset)$. We identify a maximal chain $\theta \in \Theta(3^N)$ given by

$$(\emptyset, N) \sqsubset (S_1, T_1) \sqsubset \cdots \sqsubset (S_j, T_j) \sqsubset \cdots \sqsubset (S_{2n-1}, T_{2n-1}) \sqsubset (N, \emptyset),$$

with an ordering $\theta = (i_1, \ldots, i_{2n})$ on $\overline{N}$ in such a way that $\Lambda(S_j, T_j) = \theta(i_j)$ for all $j = 1, \ldots, 2n$, where $\theta(i_j) = \{i_1, \ldots, i_j\}$ is the set of predecessors of $i_j$ in the order $\theta$ and its elements are written following the order of incorporation in the defender coalitions or desertion from the detractor coalitions. That is, if $i_j > 0, i_j$ is the last player who joins $S_j (i_j \in S_j$ and $i_j \notin S_{j-1})$ and, if $i_j < 0, -i_j$ is the last player who leaves $T_{j-1} (-i_j \notin T_j$ and $-i_j \in T_{j-1})$. In particular $\Lambda^{-1}[\theta(i_{2n})] = (N, \emptyset)$ and $\Lambda^{-1}[\theta(i_1) \setminus i_1] = (\emptyset, N)$.

For instance, let $N = \{1, 2, 3\}$and let $\theta \in \Theta(3^N)$ be given by

$$(\emptyset, N) \sqsubset (\emptyset,\ \{1, 3\}) \sqsubset (\{2\}\ , \{1, 3\}) \sqsubset (\{2\},\ \{1\}) \sqsubset (\{2\},\ \emptyset) \sqsubset (\{2, 3\},\ \emptyset) \sqsubset (N,\ \emptyset).$$

Its associated chain of sets in $2^{\overline{N}}$ is

$$\emptyset \subset \{-2\} \subset \{-2, 2\} \subset \{-2, 2, -3\} \subset \{-2, 2, -3, -1\} \subset \{-2, 2, -3, -1, 3\} \subset \overline{N},$$

and the maximal chain can also be represented by $\theta = (-2, 2, -3, -1, 3, 1)$.

Let $\theta \in \Theta(3^N)$and $b \in \mathcal{BG}^N$. The inferior and superior marginal worth vectors with respect to $\theta$are the vectors $m^\theta(b),\ M^\theta(b) \in \mathbb{R}^n$ given by

$$m_i^\theta(b) = b(\Lambda^{-1}[\theta(-i)]) - b(\Lambda^{-1}[\theta(-i) \setminus -i]),$$
$$M_i^\theta(b) = b(\Lambda^{-1}[\theta(i)]) - b(\Lambda^{-1}[\theta(i) \setminus i]),$$

for all. The vector $a^\theta(b) = m^\theta(b) + M^\theta(b)$ is called the marginal worth vector with respect to $\theta$.

The Weber set of $b$ is the convex hull of the marginal worth vectors of $b$ , that is,

$$W(N, b) = \mathrm{conv}\{a^\theta(b) : \theta \in \Theta(3^N)\}.$$

These vectors are extensions of the marginal worth vectors in the cooperative case. Its definition and properties have allowed to prove that the core of a bicooperative game is always included in its Weber set (see [2]). It is easy to see that the vectors $a^\theta(b)$ satisfy efficiency, that is $\sum_{j \in N} a_j^\theta(b) = b(N, \emptyset) - b(\emptyset, N)$. These vectors are extensions of the marginal worth vectors in the cooperative case. Its definition and properties have allowed to prove that the core of a bicooperative game is always included in its Weber set (see [2]).

**Theorem 1.** *If $b \in \mathcal{BG}^{\mathcal{N}}$, then $C(N, b) \subseteq W(N, b)$.*

## 4. The selectope for bicooperative games

The above solution concepts are subsets of the imputation set and hence they are defined for games with nonempty imputation sets. We now introduce a new solution concept, the *selectope*, for which $I(N, b)$ could be empty. This solution concept was introduced by Hammer et al. [20] and investigated by Derks et al. [10] in the case of cooperative games. Now, we extend this concept for games $b \in \mathcal{BG}^N$. The definition of the selectope is based in the dividends of Harsanyi [21]. We define the *dividend* of $(S, T) \in 3^N$ for the game $b \in \mathcal{BG}^N$, by the recursive formula

$$d_b(S, T) = \begin{cases} b(S, T) - \sum\limits_{(A, B) \sqsubset (S, T)} d_b(A, B) & \text{if } (S, T) \neq (\emptyset, \emptyset), \\ 0 & \text{if } (S, T) = (\emptyset, \emptyset). \end{cases}$$

Note that $d_b(\emptyset, N) = b(\emptyset, N)$. Moreover, for every $b \in \mathcal{BG}^N$ and $(S, T) \in 3^N$,

$$\sum_{(A, B) \sqsubseteq (S, T)} d_b(A, B) = b(S, T) \tag{1}$$

and hence every $b \in \mathcal{BG}^N$ can be uniquely represented by

$$b = \sum_{(A, B) \in 3^N \setminus \{(\emptyset, \emptyset)\}} d_b(A, B) \bar{u}_{(A, B)},$$

where $\bar{u}_{(A, B)} \in \mathcal{BG}^N$ is the superior unanimity game for $(A, B) \neq (\emptyset, \emptyset)$.

Applying the Möbius function to formula (1) (see [17]), these coefficients satisfy

$$d_b(S, T) = \sum_{\substack{A \subseteq S \\ T \subseteq B \subseteq N \setminus S}} (-1)^{|S \setminus A| + |B \setminus T|} b(A, B). \tag{2}$$

**Definition 2.** A selector on $3^N$ is a function $\alpha : 3^N \setminus \{(\emptyset, N)\} \to N$ such that $\alpha(S,T) \in N \setminus T$ for every $(S,T) \neq (\emptyset, N)$.

We denote by $\mathscr{A}(3^N)$ the set of all selectors on $3^N$. Note that the number of selectors on $3^N$ is equal to

$$\prod_{(S,T)\in 3^N\setminus\{(\emptyset,N)\}} |N \setminus T|.$$

**Definition 3.** Let $b \in \mathscr{BG}^N$. The selector value or selection given by the selector $\alpha \in \mathscr{A}(3^N)$ is the vector $p^\alpha(b) = (p_i^\alpha(b))_{i\in N}$ defined by

$$p_i^\alpha(b) = \sum_{\{(S,T)\in 3^N\setminus\{(\emptyset,N)\}:\alpha(S,T)=i\}} d_b(S,T).$$

The selectope for a game $b \in \mathscr{BG}^N$ is defined by

$$Sel(N,b) = \mathrm{conv}\{p^\alpha(b) \in \mathbb{R}^n : \alpha \in \mathscr{A}(3^N)\}.$$

For every game $b \in \mathscr{BG}^N$ and $\alpha \in \mathscr{A}(3^N)$, we have that $p^\alpha(b) \in I^*(N,b)$ because of

$$\sum_{i\in N} p_i^\alpha(b) = \sum_{i\in N} \sum_{\{(S,T)\in 3^N\setminus\{(\emptyset,N)\}:\alpha(S,T)=i\}} d_b(S,T) = \sum_{(S,T)\in 3^N\setminus\{(\emptyset,N)\}} d_b(S,T) = b(N,\emptyset) - b(\emptyset,N).$$

The *preimputation* set is convex and hence $Sel(N,b) \subseteq I^*(N,b)$.

Now, with the objective of establishing the relation between the selectope and the Weber set, we show a relation between the selections corresponding to the selectors on $3^N$ and the marginal worth vectors associated to the maximal chains in the structure $3^N$.

**Proposition 2.** *Let $\theta \in \Theta(3^N)$ be a maximal chain of $3^N$ given by*

$$(\emptyset,N) \sqsubset (S_1,T_1) \sqsubset \cdots \sqsubset (S_j,T_j) \sqsubset \cdots \sqsubset (S_{2n-1},T_{2n-1}) \sqsubset (N,\emptyset),$$

*where the ordering $\theta = (i_1,\ldots,i_{2n})$ on $\overline{N}$ satisfies $\Lambda(S_j,T_j) = \theta(i_j)$ for all $j = 1,\ldots,2n$. Let $\alpha : 3^N \setminus \{(\emptyset,N)\} \to N$ be a selector defined by*

$$\alpha(S,T) = \begin{cases} i & \text{if } i \in S \quad \text{and} \quad (S,T) \sqsubseteq \Lambda^{-1}[\theta(i)], \\ j & \text{if } j \in N \setminus (S\cup T) \quad \text{and} \quad (S,T) \sqsubseteq \Lambda^{-1}[\theta(-j)]. \end{cases}$$

*Then $\alpha$ is a selector and $p^\alpha(b) = a^\theta(b)$, for every $b \in \mathscr{BG}^N$.*

**Proof.** First of all, we prove that the selector $\alpha$ is well-defined. Indeed, for every $(S,T) \in 3^N \setminus \{(\emptyset,N)\}$ one and only one of the following cases hold:

*Case 1:* There exists a unique $i \in S$ such that $(S,T) \sqsubseteq \Lambda^{-1}[\theta(i)]$, where $i$ is the last player in $N \setminus T$ who joins $S$ in the maximal chain $\theta \in \Theta(3^N)$. Note that $i \in N \setminus T$ satisfies $\Lambda^{-1}[\theta(k)] \sqsubseteq \Lambda^{-1}[\theta(i)]$ for all $k \in S$.

*Case 2:* There exists a unique $j \in N \setminus (S\cup T)$ such that $(S,T) \sqsubseteq \Lambda^{-1}[\theta(-j)]$ where $j \in N \setminus T$ is the last player who leaves $T$ in the maximal chain $\theta \in \Theta(3^N)$. This player $j \in N \setminus T$ is such that $\Lambda^{-1}[\theta(-k)] \sqsubseteq \Lambda^{-1}[\theta(-j)]$ for all $k \in N \setminus (S \cup T)$.

Therefore, $\alpha$ is well-defined and it is the selector that associates, to every $(S,T) \in 3^N \setminus \{(\emptyset,N)\}$, the last player who joins the defender coalition or leaves the detractor coalition. Now, for every $i \in N$ and $b \in \mathscr{BG}^N$, we get

$$\begin{aligned} a_i^\theta(b) &= M_i^\theta(b) + m_i^\theta(b) = b(\Lambda^{-1}[\theta(i)]) - b(\Lambda^{-1}[\theta(i)\setminus i]) + b(\Lambda^{-1}[\theta(-i)]) - b(\Lambda^{-1}[\theta(-i)\setminus -i]) \\ &= \sum_{\{(S,T)\sqsubseteq\Lambda^{-1}[\theta(i)]\}} d_b(S,T) - \sum_{\{(S,T)\sqsubseteq\Lambda^{-1}[\theta(i)\setminus i]\}} d_b(S,T) + \sum_{\{(S,T)\sqsubseteq\Lambda^{-1}[\theta(-i)]\}} d_b(S,T) - \sum_{\{(S,T)\sqsubseteq\Lambda^{-1}[\theta(-i)\setminus -i]\}} d_b(S,T) \\ &= \sum_{\{(S,T)\sqsubseteq\Lambda^{-1}[\theta(i)]:i\in S\}} d_b(S,T) + \sum_{\{(S,T)\sqsubseteq\Lambda^{-1}[\theta(-i)]:i\in N\setminus(S\cup T)\}} d_b(S,T) = p_i^\alpha(b). \quad \square \end{aligned}$$

As the selectope for a bicooperative game $b \in \mathscr{BG}^N$ is a convex set, one direct consequence of this proposition is the following result.

**Theorem 2.** *If $b \in \mathscr{BG}^N$ then $W(N,b) \subseteq Sel(N,b)$.*

Although the next result is not necessary to establish relations between different solution concepts, the converse of Proposition 2 can be obtained if we define a consistency condition.

**Definition 4.** A selector on $3^N$ is called consistent if for all $(A,B) \sqsubseteq (S,T), (A,B) \neq (\emptyset,N)$ with $\alpha(S,T) \in N \setminus B$, the following conditions hold:

1. If $\alpha(S,T) \in S$ and $\alpha(S,T) \in A$, then $\alpha(A,B) = \alpha(S,T)$.
2. If $\alpha(S,T) \notin S \cup T$ and $\alpha(S,T) \notin A \cup B$, then $\alpha(A,B) = \alpha(S,T)$.

Note that if we take a maximal chain $\theta \in \Theta(3^N)$ and define the selector $\alpha$ as in Proposition 2, it is easy to check that $\alpha$ is consistent. Moreover, different chains correspond to different selectors.

**Theorem 3.** *Let $\alpha \in \mathscr{A}(3^N)$. Then there exists a maximal chain $\theta \in \Theta(3^N)$ such that $p^\alpha(b) = a^\theta(b)$ for every $b \in \mathscr{BG}^N$ if and only if $\alpha$ is consistent. Moreover, the maximal chain*

$$(\emptyset, N) \sqsubset (S_1, T_1) \sqsubset \cdots \sqsubset (S_j, T_j) \sqsubset \cdots \sqsubset (S_{2n-1}, T_{2n-1}) \sqsubset (N, \emptyset),$$

*is unique and it is associated to the ordering $\theta = (i_1, \ldots, i_{2n})$ on $\overline{N}$ in such a way that $\Lambda(S_j, T_j) = \theta(i_j)$ for all $j = 1, \ldots, 2n$, where the ordering is recursively defined by*

$$i_{2n} = \alpha(N, \emptyset) = \alpha(\Lambda^{-1}[\theta(i_{2n})]),$$
$$i_k = \begin{cases} \alpha(\Lambda^{-1}[\theta(i_{2n}) \setminus \{i_{2n-1}, \ldots, i_{k+1}\}]) & \text{if } i_k \in S_k, \\ -\alpha(\Lambda^{-1}[\theta(i_{2n}) \setminus \{i_{2n-1}, \ldots, i_{k+1}\}]) & \text{if } -i_k \in N \setminus (S_k \cup T_k), \end{cases}$$

*for $1 \leqslant k < 2n$.*

Since the proof of this result is rather long and it is very close to that of cooperative game in [10], this proof is included in the Appendix A. This result permit us to affirm that there is a one-to-one correspondence between permutations and consistent selectors, and between marginal values and selector values corresponding to consistent selectors.

The relation between the core and the selectope for bicooperative games, is obtained as a direct consequence of Theorems 1 and 2, because of $C(N,b) \subseteq W(N,b) \subseteq Sel(N,b)$. This result can also be proved with an alternative proof which is closely related to the proof of the inclusion of the core in the Weber set for cooperative games given by Derks [9] and it is similar to the proof of the analogous result in [4].

We now prove that the equality $Sel(N,b) = C(N,b)$ characterizes a class of games called *almost positive*.

**Definition 5.** A game $b \in \mathscr{BG}^N$ is called almost positive if $d_b(S,T) \geqslant 0$ for all $(S,T) \in 3^N$ with $|T| < n-1$.

**Theorem 4.** *Let $b \in \mathscr{BG}^N$ be a bicooperative game. The following statements are equivalent:*

(a) *$Sel(N,b) = C(N,b)$.*
(b) *$Sel(N,b) \subseteq I(N,b)$.*
(c) *The bicooperative game b is almost positive.*

**Proof.** Since $C(N,b) \subseteq I(N,b)$ we obtain that $(a)$ implies $(b)$. Now we will prove that $(b)$ implies $(c)$, that is, the bicooperative game $b$ is almost positive. Assume to the contrary that there exists a pair $(S,T) \in 3^N$ with $|T| < n-1$ and $d_b(S,T) < 0$. Next we define $\alpha \in \mathscr{A}(3^N)$, where $\alpha(S,T) = i$ and $\alpha(A,B) \neq i$ for all $(A,B) \in 3^N$ such that $(A,B) \notin \{(S,T), (\{i\}, N \setminus i), (\emptyset, N \setminus i)\}$. Thus

$$p_i^\alpha(b) = d_b(S,T) + d_b(\{i\}, N \setminus i) + d_b(\emptyset, N \setminus i) < d_b(\{i\}, N \setminus i) + d_b(\emptyset, N \setminus i) = b(\{i\}, N \setminus i) - b(\emptyset, N),$$

which is a contradiction with $(b)$.

Finally, if $b$ is an almost positive game, we show that $p^\alpha(b) \in C(N,b)$ for every selector $\alpha$. Indeed, for all $i \in N$,

$$p_i^\alpha(b) = \sum_{\{(S,T) \in 3^N \setminus \{(\emptyset,N)\}: \alpha(S,T)=i\}} d_b(S,T) = \sum_{\{(S,T) \in 3^N \setminus \{(\emptyset,N)\}: \alpha(S,T)=i,\ i \in S\}} d_b(S,T) + \sum_{\{(S,T) \in 3^N \setminus \{(\emptyset,N)\}: \alpha(S,T)=i,\ i \notin S \cup T\}} d_b(S,T).$$

We consider the vectors $y, z \in \mathbb{R}^n$ defined by

$$y_i = \sum_{\{(S,T) \in 3^N \setminus \{(\emptyset,N)\}: \alpha(S,T)=i,\ i \in S\}} d_b(S,T), \quad z_i = \sum_{\{(S,T) \in 3^N \setminus \{(\emptyset,N)\}: \alpha(S,T)=i,\ i \notin S \cup T\}} d_b(S,T),$$

and hence $p^\alpha(b) = y + z$. Moreover, we have

$$y(A) + z(N \setminus B) = \sum_{i \in A} \sum_{\{(S,T) \in 3^N \setminus \{(\emptyset,N)\}: \alpha(S,T)=i,\ i \in S\}} d_b(S,T) + \sum_{i \in N \setminus B} \sum_{\{(S,T) \in 3^N \setminus \{(\emptyset,N)\}: \alpha(S,T)=i,\ i \notin S \cup T\}} d_b(S,T) \geqslant \sum_{\{(S,T) \sqsubseteq (A,B): (S,T) \neq (\emptyset,N)\}} d_b(S,T)$$
$$= b(A,B) - b(\emptyset, N),$$

for all $(A,B) \in 3^N$, where the inequality follows because of the dividends of game $b$ are nonnegative for all $(S,T) \in 3^N$ such that $|T| < n-1$. Since $C(N,b)$ is convex, we have $Sel(N,b) \subseteq C(N,b)$ and this implies $(a)$. □

As a consequence of this result, we deduce that every almost positive bicooperative game has nonempty core. Recall that every game $b \in \mathscr{BG}^N$ can be written as a linear combination of superior unanimity games, where the coefficients are the dividends of the pairs in $3^N$. Then we define the games

$$b^+ = \sum_{\{(A,B) \in 3^N \setminus \{(\emptyset,\emptyset)\}: d_b(A,B) > 0\}} d_b(A,B) \bar{u}_{(A,B)},$$
$$b^- = \sum_{\{(A,B) \in 3^N \setminus \{(\emptyset,\emptyset)\}: d_b(A,B) < 0\}} -d_b(A,B) \bar{u}_{(A,B)},$$

and therefore $b = b^+ - b^-$. Using this decomposition we can establish the next result.

**Theorem 5.** *Let $b \in \mathscr{BG}^N$. Then:*

(a) *$Sel(N,b^+) = C(N,b^+)$ and $Sel(N,b^-) = C(N,b^-)$.*
(b) *$Sel(N,b) = \{x \in \mathbb{R}^n : x = y - z, y \in C(N,b^+), z \in C(N,b^-)\}$.*

**Proof.** Since $b^+$ and $b^-$ are almost positive games, Theorem 4 implies $(a)$. In order to prove $(b)$, note that $p^\alpha(b) = p^\alpha(b^+) - p^\alpha(b^-)$, for all $\alpha \in \mathscr{A}(3^N)$. Applying part $(a)$, we obtain $p^\alpha(b^+) \in C(N, b^+)$and $p^\alpha(b^-) \in C(N, b^-)$, and hence

$$Sel(N,b) \subseteq \{x \in \mathbb{R}^n : x = y - z, y \in C(N,b^+), z \in C(N,b^-)\}.$$

In order to prove the reverse inclusion, in view of part $(a)$, it is sufficient to prove that for any $\alpha, \beta \in \mathscr{A}(3^N)$, we have that $p^\alpha(b^+) - p^\beta(b^-) \in Sel(N,b)$. Note that if we define $\gamma \in \mathscr{A}(3^N)$, for every $(S,T) \in 3^N \setminus \{(\emptyset, N)\}$, by

$$\gamma(S,T) = \begin{cases} \alpha(S,T) & \text{if } d_b(S,T) \geqslant 0, \\ \beta(S,T) & \text{if } d_b(S,T) < 0, \end{cases}$$

then $p^\gamma(b) = p^\alpha(b^+) - p^\beta(b^-)$ which implies $p^\gamma(b) \in Sel(N,b)$. □

We have showed that the almost positive bicooperative games are the unique games for which $Sel(N,b) = C(N,b)$ and in [2] is proved that the core of a bicooperative game is equal to the Weber set if and only if the bicooperative game $b \in \mathscr{BG}^N$ is *bisupermodular*, that is

$$b(S_1 \cup S_2, T_1 \cap T_2) + b(S_1 \cap S_2, T_1 \cup T_2) \geqslant b(S_1, T_1) + b(S_2, T_2),$$

for all $(S_1, T_1),\ (S_2, T_2) \in 3^N$. Thus, as always $W(N,b) \subseteq Sel(N,b)$, it is obvious that for the almost positive games, the Weber set is a subset of the core and so, these games are bisupermodular. Therefore, we can establish the following characterization.

**Theorem 6.** *A bicooperative game $b \in \mathscr{BG}^N$ is almost positive if and only if $C(N,b) = W(N,b) = Sel(N,b)$.*

In Crama et al. [8], a characterization of the class of almost positive cooperative games in terms of inequalities on the coalitional worths is given. A similar characterization can be obtained in the context of bicooperative games. For this purpose, we introduce the following concept.

**Definition 6.** A bicooperative game $b \in \mathscr{BG}^N$ is called bisupermodular of order $m$, where $m$ is an integer $\geqslant 2$, if it satisfies the following inequalities:

$$b\left(\bigcup_{i=1}^m S_i, \bigcap_{i=1}^m T_i\right) \geqslant \sum_{i=1}^m b(S_i, T_i) - \sum_{i=1}^{m-1} \sum_{j=i+1}^m b(S_i \cap S_j, T_i \cup T_j) + \cdots + (-1)^{m+1} b\left(\bigcap_{i=1}^m S_i, \bigcup_{i=1}^m T_i\right), \tag{3}$$

for all $(S_1, T_1), \ldots, (S_m, T_m) \in 3^N$. Note that the bisupermodularity of order $m$ implies bisupermodularity of order $p$ whenever $2 \leqslant p < m$.

**Theorem 7.** *Let $b \in \mathscr{BG}^N$ be a bicooperative game. The following statements are equivalent:*

(a) *The game $b$ is almost positive.*
(b) *The game $b$ is bisupermodular of order $m$ for all integers $m \geqslant 2$.*

**Proof.** $(a) \Rightarrow (b)$ Since the almost positive bicooperative games form a convex cone generated by $\bar{u}_{(S,T)}$ for all $(S,T) \in 3^N$ with $|T| < n-1, \pm\bar{u}_{(\emptyset, N\setminus k)}$ for all $k \in N$ and $\pm\bar{u}_{(\emptyset,N)}$, it is sufficient to show that the generators of the cone of almost positive games satisfy the inequalities of the bisupermodularity of order $m$. Let $\{(S_i, T_i)\}_{i=1}^m \in 3^N$ and consider a superior unanimity game $\bar{u}_{(S,T)}$ for $(S,T) \in 3^N$ with $|T| < n-1$. Assume that $(S_i, T_i) \sqsupseteq (S,T)$ holds true for exactly $p$ indices $i \in \{1, \ldots, m\}$. If $p = 0$ then the right hand side of (3) is zero, so we may assume $p \geqslant 1$. In this case (3) reduces to

$$1 \geqslant \sum_{l=1}^p (-1)^{l+1} \binom{p}{l},$$

which holds with equality because it is the expansion of $(1-1)^p$.

Next, consider $\pm\bar{u}_{(\emptyset,N\setminus k)}$ with $k \in N$ and assume that $k \in N \setminus T_i$ holds true for exactly $p$ of the indices $i \in \{1, \ldots, m\}$. If $p = 0$ then all terms in (3) are zero, while for $p \geqslant 1$ the above counting argument still works (a minus sign is no obstacle, since the inequality holds with equality). Finally, for $\pm\bar{u}_{(\emptyset,N)}$ the same argument with $p = m$ works.

$(b) \Rightarrow (a)$ Let $(S,T) \in 3^N$ with $|T| < n-1$. Then $|N \setminus T| \geqslant 2$ and $b$ bisupermodular of order $m$ for all integers $m \geqslant 2$ implies that $b$ is bisupermodular of order $|N \setminus T|$. Using this fact for the collection $\{(S_i, T_i)\}_{i \in N \setminus T}$ with

$$(S_i, T_i) = \begin{cases} (S \setminus i, T) & \text{if } i \in S, \\ (S, T \cup i) & \text{if } i \notin S, \end{cases}$$

for all $i \in N \setminus T$, and taking into account formula (2) we obtain $d_b(S,T) \geqslant 0$. □

## 5. An axiomatization of the values of the selectope

In this section the selectope will be considered from a value-theoretic point of view. A sharing system is a collection $q = (q^{(S,T)})_{(S,T) \in 3^N \setminus \{(\emptyset,N)\}}$ where the vectors $q^{(S,T)} \in \mathbb{R}^{N \setminus T}$ satisfy $q_i^{(S,T)} \geqslant 0$ for all $i \in N \setminus T$, and $\sum_{i \in N \setminus T} q_i^{(S,T)} = 1$ for every coalition $(S,T) \neq (\emptyset, N)$. Let $\mathscr{BS}^N$ denote the set of all sharing systems on $3^N$. For a bicooperative game $b \in \mathscr{BG}^N$ and a sharing system $q \in \mathscr{BS}^N$, a *sharing value* $s^q(b)$ is associated, defined by

$$s_i^q(b) := \sum_{\{(S,T): i \in N \setminus T\}} q_i^{(S,T)} d_b(S,T) \quad \text{for every } i \in N.$$

The vector $s^q(b)$ yields to any player $i \in N$ the sum over all coalitions $(S,T) \in 3^N$ such that $i \in N \setminus T$ of the share $q_i^{(S,T)}$ in the Harsanyi dividend $d_b(S,T)$ and therefore, in [12], they are called Harsanyi payoff vector. It is evident that these vectors are efficient

$$\sum_{i\in N} s_i^q(b) = \sum_{i\in N} \sum_{\{(S,T):i\in N\setminus T\}} q_i^{(S,T)} d_b(S,T) = \sum_{(S,T)\in 3^N\setminus\{(\emptyset,N)\}} d_b(S,T) = b(N,\emptyset) - b(\emptyset,N),$$

and all values $s^q$ are linear, that is, $s^q(\alpha b + \beta w) = \alpha s^q(b) + \beta s^q(w)$, for all $\alpha,\beta \in \mathbb{R}$, and $b,w \in \mathcal{BG}^N$.

Examples of sharing values are the marginal contribution vectors. For a maximal chain $\theta \in \Theta(3^N)$, consider the sharing system $q^\theta$ given, for all $(S,T) \neq (\emptyset,N)$ and $i \in N \setminus T$, by

$$(q^\theta)_i^{(S,T)} = \begin{cases} 1 & \text{if } i \in S \quad \text{and} \quad (S,T) \sqsubseteq \Lambda^{-1}[\theta(i)], \\ 1 & \text{if } i \in N \setminus (S \cup T) \quad \text{and} \quad (S,T) \sqsubseteq \Lambda^{-1}[\theta(-i)], \\ 0 & \text{otherwise}, \end{cases}$$

and it is easy to check that $s_i^{q^\theta}(b) = a_i^\theta(b)$ for all $i \in N$.

The following lemma relates sharing values with the selectope of a bicooperative game and its proof is analogous to that the cooperative case.

**Lemma 1.** *Let $b \in \mathcal{BG}^N$ be a bicooperative game.*

(i) *For any sharing system $q$,*

$$s^q(b) = \sum_{\alpha\in\mathcal{A}(3^N)} q_\alpha p^\alpha(b), \tag{4}$$

*where $q_\alpha = \prod_{(S,T)\in 3^N\setminus\{(\emptyset,N)\}} q_{\alpha(S,T)}^{(S,T)}$.*

(ii) *For any collection $(r_\alpha)_{\alpha\in\mathcal{A}(3^N)}$ with $r_\alpha \geqslant 0$ for all $\alpha \in \mathcal{A}(3^N)$ and $\sum_{\alpha\in\mathcal{A}(3^N)} r_\alpha = 1$,*

$$s^q(b) = \sum_{\alpha\in\mathcal{A}(3^N)} r_\alpha p^\alpha(b), \tag{5}$$

*where $q_i^{(S,T)} := \sum_{\{\alpha\in\mathcal{A}(3^N):\alpha(S,T)=i\}} r_\alpha$.*

(iii) *$Sel(N,b) = \{s^q(b) : q \in \mathcal{BS}^N\}$.*

**Proof.** By linearity of the values $s^q$ and $p^\alpha$ it is sufficient to show (i) and (ii) for a superior unanimity game $\bar{u}_{(S,T)}$. For a player $i \in T$, Eqs. (1) and (2) have zero on both sides because $d_{\bar{u}_{(S,T)}}(S,T) = 1$, and $d_{\bar{u}_{(S,T)}}(A,B) = 0$ for all $(A,B) \neq (S,T)$. For a player $i \in N \setminus T$ Eq. (2) reduces to $q_i^{(S,T)} := \sum_{\{\alpha\in\mathcal{A}(3^N):\alpha(S,T)=i\}} r_\alpha$ which is true by definition. For a player $i \in N \setminus T$ Eq. (1) reduces to

$$q_i^{(S,T)} = \sum_{\{\alpha\in\mathcal{A}(3^N):\alpha(S,T)=i\}} \prod_{(A,B)\in 3^N\setminus\{(\emptyset,N)\}} q_{\alpha(A,B)}^{(A,B)}.$$

This equality is a standard property that can be proved in an elementary way. (iii) Follows from (i) and (ii). □

Thus, sharing values are convex combinations of selector values and fill up the selectope. If in Eqs. (1) and (2) the summation is taken only over consistent selectors, then the corresponding sharing value is the compatible-order value [4]. These values fill up the Weber set.

Sharing values can be characterized by a set of axioms. Let us consider a value $\psi : \mathcal{BG}^N \to \mathbb{R}^N$. The axioms under consideration are the following.

(i) *Additivity*: $\psi_i$ satisfies $\psi_i(b+w) = \psi_i(b) + \psi_i(w)$, for all $b,w \in \mathcal{BG}^N$.
(ii) *Null-Player:* $\psi_i(b) = 0$ whenever $i \in N$ is a null-player in $b \in \mathcal{BG}^N$, i.e. $b(S \cup i, T) - b(S, T \cup i) = 0$ for every $(S,T) \in 3^{N\setminus i}$.
(iii) *Efficiency:* $\sum_{i\in N}\psi(b) = b(N,\emptyset) - b(\emptyset,N)$ for every $b \in \mathcal{BG}^N$.
(iv) *Positivity:* $\psi_i(b) \geqslant 0$ whenever $b \in \mathcal{BG}^N$ is almost positive and $b(\emptyset, N\setminus i) - b(\emptyset,N) \geqslant 0$ for all $i \in N$.

**Theorem 8.** *Let $\psi : \mathcal{BG}^N \to \mathbb{R}^N$ be a value on $\mathcal{BG}^N$. The value $\psi$ is a sharing value if and only if $\psi$ satisfies Additivity, Efficiency, Positivity and Null-Player axioms.*

**Proof.** It is easy to check that a sharing value satisfies the desired axioms. Attention will now be restricted to the opposite direction of the characterization.

For every $(S,T) \in 3^N \setminus \{(\emptyset,N)\}$and $i \in N \setminus T$ define $q_i^{(S,T)} = \psi_i(\bar{u}_{(S,T)})$. Clearly, if $i \in T$, then $i$ is a null-player in game $\bar{u}_{(S,T)}$. By using Efficiency, Null-Player and Positivity axioms it follows that $q = (q^{(S,T)})_{(S,T)\in 3^N\setminus\{(\emptyset,N)\}}$ is a sharing system. In order to prove that $\psi = s^q$ it is, in view of Additivity of $\psi$ and $s^q$, sufficient to prove that for a superior unanimity game $\bar{u}_{(S,T)}$ and $\alpha \in \mathbb{R}$, it holds $\psi(\alpha\bar{u}_{(S,T)}) = s^q(\alpha\bar{u}_{(S,T)})$. Since $s^q(\alpha\bar{u}_{(S,T)}) = \alpha s^q(\bar{u}_{(S,T)}) = \alpha\psi(\bar{u}_{(S,T)})$ by definition of $q$ and $s^q$, it is sufficient to prove $\psi(\alpha\bar{u}_{(S,T)}) = \alpha\psi(\bar{u}_{(S,T)})$. The rest of the proof is analogous to that of cooperative case [10], changing the unanimity cooperative game $u_S$ by the superior unanimity bicooperative game $\bar{u}_{(S,T)}$. □

## 6. Concluding remarks

The elements of the Weber set of a bicooperative game are characterized in [4] as the values satisfying Linearity, Dummy, Monotonicity and Efficiency axioms. Furthermore, the Shapley value is the only one that satisfies Linearity, Dummy, Anonymity, Efficiency and Structural axioms (see [3]). It is possible to obtain a characterization of these values by using Additivity instead of Linearity and Null-Player instead of Dummy-Player. In relation with the characterization of the elements of the Weber set, Derks introduces in [11] an alternative proof for Weber's axiomatization of the compatible-order values. Derks et al. [12] analyze another way to link the Weber set and the selectope and provide two characterizations of the set of all sharing systems of the dividends whose associated Harsanyi payoff vectors are compatible-order values. Their approach can be adapted in the context of bicooperative games by using the Möbius function on the lattice $3^N$, given by

$$\mu((S,T),(S',T')) = \begin{cases} (-1)^{|S'\setminus S|+|T\setminus T'|} & \text{if } (S,T) \sqsubseteq (S',T') \quad \text{and} \quad T \cap S' = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the results in papers [11,12] can also be extended to the bicooperative case by using similar arguments as in our paper.

On the other hand, in the cooperative case the sharing values corresponding to consistent sharing systems are weighted Shapley values (see [24,26]). In Ref. [26] it is shown that weighted Shapley values cover the core and in Ref. [10] is showed that these values are the only ones that satisfy Additivity, Null-Player, Efficiency, Positivity and Partnership axioms. For the bicooperative case, a consistent sharing system $q$ can be defined as follows. The sharing system $q$ is consistent if, for all coalitions $(A,B),(S,T) \in 3^N \setminus \{(\emptyset,N)\}$, it holds

$$\text{If } (A,B) \subseteq (S,T) \quad \text{with } i \in N \setminus B \quad \text{and} \quad \sum_{j\in N\setminus B} q_j^{(S,T)} > 0, \text{ then}$$

$$q_i^{(A,B)} = \frac{q_i^{(S,T)}}{\sum_{j\in N\setminus B} q_j^{(S,T)}}.$$

The solution concepts for bicooperative games given in Section 3 have a straightforward analogy with well-known concepts of cooperative games. Its advantages lies in the automatic fashion in which the properties of bicooperative games can be obtained from them. In other words, the results obtained in this paper and in Refs. [2–4] imply a narrow closeness between cooperative and bicooperative models with respect to the selectope, the core and the Weber set. For this reason, we state now, without further elaboration, that weighted Shapley values for a bicooperative game can be also characterized by a set of similar axioms to the cooperative case and that these values cover the core of a bicooperative game.

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## Appendix A

**Proof** (Theorem 3). Let us consider the superior unanimity game $\bar{u}_{(S,T)} \in \mathscr{BG}^N$, with $(S,T) \neq (\emptyset,\emptyset)$. Clearly $d_{\bar{u}_{(S,T)}}(S,T) = 1$, and $d_{\bar{u}_{(S,T)}}(A,B) = 0$ for all $(A,B) \neq (S,T)$.

*Sufficient condition.* Let $\alpha \in \mathscr{A}(3^N)$ be a selector such that there exists a maximal chain $\theta \in \Theta(3^N)$ satisfying $p^\alpha(b) = a^\theta(b)$, for every $b \in \mathscr{BG}^N$ and let $(S,T) \in 3^N \setminus \{(\emptyset,N)\}$ with $\alpha(S,T) = i$. As $p_i^\alpha(\bar{u}_{(S,T)}) = 1$ and by hypothesis $p_i^\alpha(\bar{u}_{(S,T)}) = a_i^\theta(\bar{u}_{(S,T)})$, then

$$\bar{u}_{(S,T)}(\Lambda^{-1}[\theta(i)]) - \bar{u}_{(S,T)}(\Lambda^{-1}[\theta(i)\setminus i]) + \bar{u}_{(S,T)}(\Lambda^{-1}[\theta(-i)]) - \bar{u}_{(S,T)}(\Lambda^{-1}[\theta(-i)\setminus -i]) = 1.$$

Since $\Lambda^{-1}[\theta(-i)\setminus i] \sqsubseteq \Lambda^{-1}[\theta(-i)] \sqsubseteq \Lambda^{-1}[\theta(i)\setminus i] \sqsubseteq \Lambda^{-1}[\theta(i)]$, there is two possibilities:

*Case 1:* $(S,T) \sqsubseteq \Lambda^{-1}[\theta(i)]$ and $(S,T)$ is not included in $\Lambda^{-1}[\theta(i)\setminus i]$ and hence $i \in S$. If $(A,B) \sqsubseteq (S,T)$and $\alpha(S,T) = i \in A$, then

$$a_i^\theta(\bar{u}_{(A,B)}) = M_i^\theta(\bar{u}_{(A,B)}) + m_i^\theta(\bar{u}_{(A,B)}) = 1.$$

By hypothesis $p_i^\alpha(\bar{u}_{(A,B)}) = a_i^\theta(\bar{u}_{(A,B)})$, which implies $p_i^\alpha(\bar{u}_{(A,B)}) = 1$ and for this, it must be verified that $i = \alpha(A,B)$.

*Case 2:* $(S,T) \sqsubseteq \Lambda^{-1}[\theta(-i)]$ and $(S,T)$ is not included in $\Lambda^{-1}[\theta(-i)\setminus -i]$ and hence $i \notin S \cup T$. Note also that for $(A,B) \sqsubseteq (S,T)$ and $\alpha(S,T) = i \notin A \cup B$, it holds

$$a_i^\theta(\bar{u}_{(A,B)}) = M_i^\theta(\bar{u}_{(A,B)}) + m_i^\theta(\bar{u}_{(A,B)}) = 1.$$

Since $p_i^\alpha(\bar{u}_{(A,B)}) = a_i^\theta(\bar{u}_{(A,B)})$ by hypothesis, then $p_i^\alpha(\bar{u}_{(A,B)}) = 1$ and or this, it must be verified that $i = \alpha(A,B)$. Therefore, $\alpha$ is consistent.

*Necessary condition.* Let $\alpha \in \mathscr{A}(3^N)$ be a consistent selector. First of all, we prove that there exists a maximal chain $\theta \in \Theta(3^N)$ for which $a^\theta(b) = p^\alpha(b)$ for every $b \in \mathscr{BG}^N$. Consider the maximal chain $\theta \in \Theta(3^N)$

$$(\emptyset,N) \sqsubset (S_1,T_1) \sqsubset \cdots \sqsubset (S_j,T_j) \sqsubset \cdots \sqsubset (S_{2n-1},T_{2n-1}) \sqsubset (N,\emptyset),$$

associated to the ordering $\theta = (i_1,\ldots,i_{2n})$ on $\overline{N}$ in such a way that $\Lambda(S_j,T_j) = \theta(i_j)$ for all $j = 1,\ldots,2n$, where

$$i_{2n} = \alpha(N,\emptyset) = \alpha(\Lambda^{-1}[\theta(i_{2n})]),$$

$$i_k = \begin{cases} \alpha(\Lambda^{-1}[\theta(i_{2n})\setminus\{i_{2n-1},\ldots,i_{k+1}\}]) & \text{if } i_k \in S_k, \\ -\alpha(\Lambda^{-1}[\theta(i_{2n})\setminus\{i_{2n-1},\ldots,i_{k+1}\}]) & \text{if } -i_k \in N\setminus(S_k\cup T_k), \end{cases}$$

for $1 \leqslant k < 2n$. We prove that the selector $\alpha$ coincides with $\beta \in \mathscr{A}(3^N)$ defined in Proposition 2 for this chain, i.e.

$$\beta(S,T) = \begin{cases} i & \text{if } i \in S \quad \text{and} \quad (S,T) \sqsubseteq \Lambda^{-1}[\theta(i)], \\ j & \text{if } j \in N \setminus (S \cup T) \quad \text{and} \quad (S,T) \sqsubseteq \Lambda^{-1}[\theta(-j)]. \end{cases}$$

Indeed, it is obvious that $\alpha(S,T) = \beta(S,T)$ for every pair $(S,T)$ in the chain $\theta$. If $(S,T)$ is not in the chain $\theta$, then $\beta(S,T) \in N \setminus T$ and we consider two possibilities:

*Case 1:* $i \in S$ and $(S,T) \sqsubseteq \Lambda^{-1}[\theta(i)]$. Then $\alpha(\Lambda^{-1}[\theta(i)]) = i$ and the selector $\alpha$ is consistent. It follows that $\alpha(S,T) = i$.
*Case 2:* $j \in N \setminus (S \cup T)$ and $(S,T) \sqsubseteq \Lambda^{-1}[\theta(-j)]$. Then $\alpha(\Lambda^{-1}[\theta(-j)]) = j$ and the selector $\alpha$ is consistent. It follows that $\alpha(S,T) = j$.

Therefore, $\beta(S,T) = \alpha(S,T)$ for every pair $(S,T) \in 3^N, (S,T) \neq (\emptyset, N)$. Applying Proposition 2 to the selector $\alpha$, we have that $a^\theta(b) = p^\alpha(b)$ for every $b \in \mathscr{BG}^N$.

Finally, we prove that this chain is the unique chain for which the equality $a^\theta(b) = p^\alpha(b)$ is satisfied for every $b \in \mathscr{BG}^N$. Consider the superior unanimity game $\bar{u}_{(N,\emptyset)} \in \mathscr{BG}^N$. For each $j \in N$, we have that

$$p_j^\alpha(\bar{u}_{(N,\emptyset)}) = \begin{cases} 1 & \text{if } \alpha(N,\emptyset) = j, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$a_j^\theta(\bar{u}_{(N,\emptyset)}) = \begin{cases} 1 & \text{if } \Lambda^{-1}[\theta(j)] = (N,\emptyset), \\ 0 & \text{otherwise.} \end{cases}$$

Since $\alpha(N,\emptyset) = i_{2n}$, both vectors coincide if and only if $\Lambda^{-1}[\theta(i_{2n})] = (N,\emptyset)$. We now consider $\bar{u}_{\Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]} \in \mathscr{BG}^N$. For every $j \in N$, we have that

$$p_j^\alpha(\bar{u}_{\Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]}) = \begin{cases} 1 & \text{if } \alpha(\Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]) = j, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} a_j^\theta\left(\bar{u}_{\Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]}\right) &= \bar{u}_{\Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]}(\Lambda^{-1}[\theta(j)]) - \bar{u}_{\Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]}(\Lambda^{-1}[\theta(j) \setminus j]) + \bar{u}_{\Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]}(\Lambda^{-1}[\theta(-j)]) - \bar{u}_{\Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]}\left(\Lambda^{-1}[\theta(-j) \setminus -j]\right) \\ &= \begin{cases} 1 & \text{if } \Lambda^{-1}[\theta(j)] \sqsupseteq \Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}] \quad \text{and} \quad \Lambda^{-1}[\theta(j) \setminus j] \not\sqsupseteq \Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}] \\ & \text{or} \\ & \Lambda^{-1}[\theta(-j)] \sqsupseteq \Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}] \quad \text{and} \quad \Lambda^{-1}[\theta(-j) \setminus -j] \not\sqsupseteq \Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}] \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We distinguish two cases:

(1) If $\Lambda^{-1}[\theta(j)] \sqsupseteq \Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]$ and $\Lambda^{-1}[\theta(j) \setminus j] \not\sqsupseteq \Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]$, then $\Lambda^{-1}[\theta(j)]$ is a pair $(A,B)$ such that $|A| = n, |B| = 0$ or $|A| = n-1, |B| = 0$, but in every maximal chain there is only one coalition with $|A| = k_1$ and $|B| = k_2$ elements, for all $0 \leqslant k_1, k_2 \leqslant n$, and the pair $(N,\emptyset)$ is excluded. Therefore,

$$a_j^\theta(\bar{u}_{\Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]}) = \begin{cases} 1 & \text{if } \Lambda^{-1}[\theta(j)] = \Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}], \\ 0 & \text{otherwise.} \end{cases}$$

Since $\alpha(\Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]) = i_{2n-1}$, it holds that

$$a^\theta(\bar{u}_{\Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]}) = p^\alpha(\bar{u}_{\Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]}) \Longleftrightarrow \Lambda^{-1}[\theta(i_{2n-1})] = \Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}].$$

(2) If $\Lambda^{-1}[\theta(-j)] \sqsupseteq \Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]$ and $\Lambda^{-1}[\theta(-j) \setminus -j] \not\sqsupseteq \Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]$, the above argument may be repeated and then

$$a_j^\theta(\bar{u}_{\Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]}) = \begin{cases} 1 & \text{if } \Lambda^{-1}[\theta(-j)] = \Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}], \\ 0 & \text{otherwise,} \end{cases}$$

Since $\alpha(\Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]) = -i_{2n-1}$, it holds that

$$a^\theta(\bar{u}_{\Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]}) = p^\alpha(\bar{u}_{\Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}]}) \Longleftrightarrow \Lambda^{-1}[\theta(-i_{2n-1})] = \Lambda^{-1}[\theta(i_{2n}) \setminus i_{2n}].$$

By repeating this argument, the maximal chain $\theta$ is obtained. □

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