The distribution of power in the European Cluster Game

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Abstract

Cooperative games under combinatorial restrictions are cooperative games in which the players have restricted communication possibilities, which are defined by a combinatorial structure. This paper provides a "ready-to-apply" procedure to compute the Shapley-Shubik index power of games restricted by combinatorial structures. These restrictions are determined by the positions of the players in a communication graph or in a permission structure. We study the *European Cluster Game* defined by a partition of the states of the European Union. We consider six blocks according to the distinction between large, medium and small members. Furthermore, we present the implementation of the algorithms in the computer system Mathematica.

1 Introduction

The weighted voting games are mathematical models which are used to analyze the distribution of the decision power of a nation in a supranational organization like the Council of Ministers of the European Union, the Security Council of the United Nations or the International Monetary Fund. In these institutions, each nation has associated a number of votes and a proposal is approved if a coalition of nations has enough votes to reach an established quota.

The power of a country in a supranational organization is a numerical measure of its capacity to decide the approval of a motion. This decisive character is measured calculating the number of times that the vote of a country converts to a coalition that does not reach the quota to take decisions in a winning coalition. The power indices are *a priori* measures of this power, the most useful are the Shapley-Shubik [11] and Banzhaf [3] indices.

Hagemann and De Clerck-Sachsse [8] obtained the next observation about the coalition formation in the European Union:

> "But it can be concluded that a consistent pattern can be observed in the distinction between large, medium and small mem-

bers; the following will reveal whether this differentiation also holds after the enlargement."

In this contribution, we study the *European Cluster Game* defined by the following six players:

Big1 = {Germany},
Big2 = {France,United Kingdom,Italy},
Big3 = {Spain,Poland},
Med1 = {Romania,Netherlands,Greece,Portugal,Belgium,Czech Rep.,Hungary},
Med2 = {Sweden,Austria,Bulgaria,Denmark,Slovakia,Finland,Ireland,Lithuania},
Small = {Latvia,Slovenia,Estonia,Cyprus,Luxembourg,Malta}.

The new voting rule proposed by the European Convention for the future European Treaty changes in a very remarkable way the power of the countries in the Council. The reason is that the weighted votes, that were approved in Nice are removed and a coalition only needs 15 votes, which at least sum up by 65% of the population to approve a decision with the new rule. Furthermore, the minimum number of countries to block a proposal is four and the abstentions are not counted.

Cooperative games under combinatorial restrictions are cooperative games in which the players have restricted communication possibilities, which are defined by a combinatorial structure. The first model in which the restrictions are defined by the connected subgraphs of a graph is introduced by Myerson [9]. Contributions on graph-restricted games include Owen [10], Borm, Owen, and Tijs [5]. In these models the possibilities of coalition formation are determined by the positions of the players in a *communication graph.*

Another type of combinatorial structure introduced by Gilles, Owen and van den Brink [7] and van den Brink [6] is equivalent to a subclass of antimatroids. This line of research focuses on the possibilities of coalition formation determined by the positions of the players in the so-called *permission structure.*

We will analyze the *European Cluster Game* by using the restricted cooperation model derived from a combinatorial structure called *augmenting system.* This structure is a generalization of the antimatroid structure and the system of connected subgraphs of a graph. Furthermore, this new set system includes the conjunctive and disjunctive systems derived from a permission structure (see Bilbao [4]). We will present a "ready-to-apply" procedure to compute the Shapley-Shubik index power of games restricted by combinatorial structures. Finally, we use the Mathematica computer system by Wolfram [12] to compute the distribution of power in the European Cluster Game.

2 Augmenting systems

Let N be a finite set. A *set system* over N is a pair $(N, \mathcal{F})$ where $\mathcal{F} \subseteq 2^N$ is a family of subsets. The sets belonging to $\mathcal{F}$ are called *feasible*. We will write $S \cup i$ and $S \setminus i$ instead of $S \cup \{i\}$ and $S \setminus \{i\}$ respectively.

Definition 1 *An augmenting system is a set system $(N, \mathcal{F})$ with the following properties:*

P1 $\emptyset \in \mathcal{F}$,

P2 for $S, T \in \mathcal{F}$ *with* $S \cap T \neq \emptyset$, *we have* $S \cup T \in \mathcal{F}$,

P3 for $S, T \in \mathcal{F}$ *with* $S \subset T$, *there exists* $i \in T \setminus S$ *such that* $S \cup i \in \mathcal{F}$.

Example. The following collections of subsets of $N = \{1, \ldots, n\}$, given by $\mathcal{F} = 2^N$, $\mathcal{F} = \{\emptyset, \{i\}\}$ where $i \in N$, and $\mathcal{F} = \{\emptyset, \{1\}, \ldots, \{n\}\}$, are augmenting systems over N.

Example. In a communication graph $G = (N, E)$, the set system $(N, \mathcal{F})$ given by $\mathcal{F} = \{S \subseteq N : (S, E(S)) \text{ is a connected subgraph of } G\}$, is an augmenting system.

The next characterization of the augmenting systems derived from the connected subgraphs of a graph is proved by Algaba, Bilbao, and Slikker [2]

Theorem 2 *An augmenting system $(N, \mathcal{F})$ is the system of connected subgraphs of the graph $G = (N, E)$, where $E = \{S \in \mathcal{F} : |S| = 2\}$ if and only if $\{i\} \in \mathcal{F}$ for all $i \in N$.*

Example. Gilles et al. [7] showed that the feasible coalition system $(N, \mathcal{F})$ derived from the conjunctive or disjunctive approach contains the empty set, the ground set N, and that it is closed under union. Algaba et al. [1] showed that the coalition systems derived from the conjunctive and disjunctive approach were identified to *poset antimatroids* and *antimatroids with the path property* respectively. Thus, these coalition systems are augmenting systems.

Let $N = \{1, \ldots, n\}$ be a set of players with $n > 2$ and we consider a subset S of starting players. If $i \in S$ then the coalition $\{i\}$ is feasible. Each starting player i looks for a player $k \notin S$ to generate a new feasible coalition $\{i, k\}$. These coalitions with cardinality 2 searching for new players which agree to join one by one. If we assume that common elements of two feasible coalitions are intermediaries between the two coalitions in order to establish the feasibility of its union, we obtain an augmenting system $(N, \mathcal{F})$. Since the individual players $k \notin S$ are not feasible coalitions, the family $\mathcal{F}$ is not generated by the connected subgraphs of a graph. Moreover, if players $i, j \in S$ then $\{i\}, \{j\} \in S$ and $\{i, j\} \notin S$ and hence $(N, \mathcal{F})$ is not an antimatroid.

Example. Let $N = \{1, 2, 3, 4\}$ and we consider $S_1 = \{1, 2, 4\}$ and $S_2 = \{1, 4\}$. By using the above coalition formation model we can obtain the following augmenting systems.

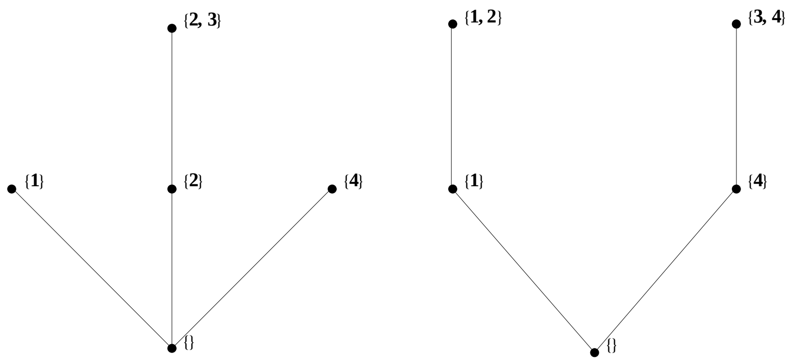

Figure 1

The sets of maximal feasible coalitions are partitions of the players into disjoint coalitions, that is, the coalition structures $CS_1 = \{\{1\}, \{4\}, \{2, 3\}\}$ and $CS_2 = \{\{1, 2\}, \{3, 4\}\}$.

Example. Let us consider $N = \{1, 2, 3, 4\}$ and

$$\mathcal{F} = \{\emptyset, \{1\}, \{4\}, \{1, 2\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, N\}.$$

Since $\{1, 2, 3\}$ and $\{2, 3, 4\}$ are feasible coalitions, property P2 implies that the grand coalition N is also feasible.

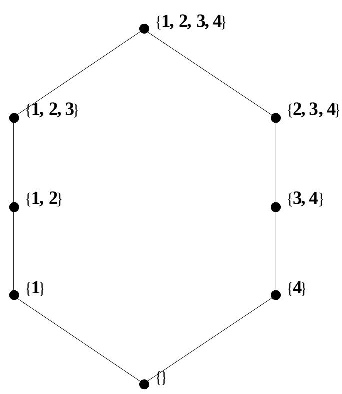

Figure 2

Example. The set system given by $N = \{1, 2, 3, 4\}$ and

$$\begin{aligned} \mathcal{F} &= \{\emptyset, \{1\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}, \\ &\quad \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, N\}. \end{aligned}$$

is an augmenting system. Since $\{1, 4\} \notin \mathcal{F}$ the system $(N, \mathcal{F})$ is not an antimatroid. Moreover, $\{1, 2\} \cap \{2, 4\} = \{2\} \notin \mathcal{F}$ and hence $(N, \mathcal{F})$ is not a convex geometry.

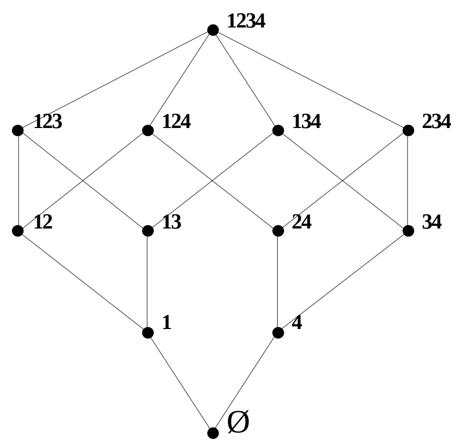

Figure 3

Definition 3 *Let $(N, \mathcal{F})$ be an augmenting system. For a feasible coalition $S \in \mathcal{F}$, we define the set $S^* = \{i \in N \setminus S : S \cup i \in \mathcal{F}\}$ of augmentations of S and the set $S^+ = S \cup S^* = \{i \in N : S \cup i \in \mathcal{F}\}$.*

Let $(N, \mathcal{F})$ be a set system and let $S \subseteq N$ be a subset. The maximal non-empty feasible subsets of S are called *components* of S. We denote by $C_{\mathcal{F}}(S)$ the set of the components of a subset $S \subseteq N$. Observe that the set $C_{\mathcal{F}}(S)$ may be the empty set. This set will play a role in the concept of a game restricted by an augmenting system.

Proposition 4 *A set system $(N, \mathcal{F})$ satisfies property P2 if and only if for any $S \subseteq N$ with $C_{\mathcal{F}}(S) \neq \emptyset$, the components of S form a partition of a subset of S.*

3 Games restricted by augmenting systems

Definition 5 *Let* $v : 2^N \to \mathbb{R}$ *be a cooperative game and let* $(N, \mathcal{F})$ *be an augmenting system. The restricted game* $v^{\mathcal{F}} : 2^N \to \mathbb{R}$, *is defined by*

$$v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T).$$

If $(N, \mathcal{F})$ is the augmenting system given by the connected subgraphs of a graph $G = (N, E)$, then the game $\left(N, v^{\mathcal{F}}\right)$ is a graph-restricted game which is studied by Myerson [9] and Owen [10]. Note that if $S \in \mathcal{F}$ then $v^{\mathcal{F}}(S) = v(S)$.

Let (N, v) be a game and let $(N, \mathcal{F})$ be an augmenting system. The *Shapley value* for player i in the restricted game $v^{\mathcal{F}}$ is given by

$$\Phi_i\left(N, v^{\mathcal{F}}\right) = \sum_{\{S \subseteq N \,:\, i \in S\}} \frac{(s-1)!(n-s)!}{n!}\left[v^{\mathcal{F}}(S) - v^{\mathcal{F}}(S \setminus i)\right],$$

where $n = |N|$ and $s = |S|$. This value is an average of the *marginal contributions* $v^{\mathcal{F}}(S) - v^{\mathcal{F}}(S \setminus i)$ of a player i to all coalitions $S \in 2^N \setminus \{\emptyset\}$. In this value, the sets S of different size get different weight. The *Banzhaf value* for player i in the restricted game $v^{\mathcal{F}}$ is given by

$$\beta_i'\left(N, v^{\mathcal{F}}\right) = \sum_{\{S \subseteq N \,:\, i \in S\}} \frac{1}{2^{n-1}}\left[v^{\mathcal{F}}(S) - v^{\mathcal{F}}(S \setminus i)\right],$$

for all $i \in N$. If the number of players is n, then the function that measures the worst case running time for computing these indices is in $O\left(n2^n\right)$ (see Deng and Papadimitriou [**?**]) Moreover, to obtain the restricted game $v^{\mathcal{F}}$ we need to compute the set of the components $C_{\mathcal{F}}(S)$ of every subset $S \subseteq N$. Then it is necessary to consider all the feasible subsets of S and hence the time complexity is $O(t)$ where

$$t = \sum_{s=0}^{n} \binom{n}{s} 2^s = 3^n.$$

Bilbao [4] obtain the following *explicit formula,* in terms of v, for the Shapley value of the players in the restricted game $v^{\mathcal{F}}$. The time complexity of the formula is polynomial in the cardinality $|\mathcal{F}|$.

Theorem 6 *Let* $(N, v, \mathcal{F})$ *be an augmenting structure. Then*

$$\Phi_i\left(N, v^{\mathcal{F}}\right) = \sum_{\{T \in \mathcal{F} \,:\, i \in T\}} \frac{(t-1)!\, t^*!}{t^+!} v(T) - \sum_{\{T \in \mathcal{F} \,:\, i \in T^*\}} \frac{t!\, (t^*-1)!}{t^+!} v(T),$$

where $i \in N$, $t = |T|$, $t^* = |T^*|$ *and* $t^+ = |T^+|$.

Remark. Notice that if $\mathcal{F} = 2^N$, then $T^* = N \setminus T$ and $T^+ = N$, for every $T \in \mathcal{F}$. Thus, the formula obtained in the above theorem is equal to the classical Shapley value [11] for the game v.

The algorithm showed in Theorem 6 computes the Shapley value $\Phi\left(N, v^{\mathcal{F}}\right)$ and written in the Mathematica computer system (Wolfram [12]) it is the following:

```
<<DiscreteMath`Combinatorica`

Feasible[i_,F_List]:=Feasible[i,F]=Select[F,(MemberQ[#,i])&]

SupInt[S_List,F_List]:=Select[T,(MemberQ[F,Union[S,{#}]])&]

Augmentation[i_,F_List]:=Augmentation[i,F]=Select[F,
(DeleteCases[#,i]==#)&&(MemberQ[F,Union[#,{i}]])& ]

co1[S_List]:=co1[S]=(Length[S]-1)!*
(Length[SupInt[S,F]]-Length[S])!/Length[SupInt[S,F]]!;
co2[S_List]:=co2[S]=co1[S]*Length[S]/
(Length[SupInt[S,F]]-Length[S]);

RestrictedShapleyValue[game_:Null]:=Module[{value},
value=Table[Apply[Plus,If[#=={},0,
co1[#] (v[#])]& /@ Feasible[i,F]]-
Apply[Plus,If[#=={},0,co2[#] v[#]]& /@ Augmentation[i,F]],
{i,Length[T]}];Return[value]];
```

4 The European Cluster Game

The *European Cluster Game* is defined by the following six players:

Big1 = {Germany (171)},
Big2 = {France (123), United Kingdom (123), Italy (118)},
Big3 = {Spain (86), Poland (79)},
Med1 = {Romania (45), Netherlands (33), Greece (23), Portugal (22), Belgium (21), Czech Rep. (21), Hungary (21)},
Med2 = {Sweden (18), Austria (17), Bulgaria (16), Denmark (11), Slovakia (11), Finland (11), Ireland (8), Lithuania (7)},
Small = {Latvia (5), Slovenia (4), Estonia (3), Cyprus (1), Luxembourg (1), Malta (1)}.

The voting method approved in the summit of Brussels on 18th June, 2004, for its incorporation to the European Constitution, is based on a double voting system and a blocking clause. To approve a proposal in the Council of Ministers of the 27 members of the European Union, it is needed at least 15 countries that sum up more or equal than 65% of the population. Moreover, the minimum number of countries to block a proposal is four and the abstentions are not counted. This game `Cluster` is defined as follows:

```
Cluster:= (T = Range[6]; Clear[w, z, p, q, v];
w[1]:= 171; w[2]:= 364; w[3]:= 165;w[4] := 186;
w[5]:= 99; w[6]:= 15 ;
p[S_List] := Plus @@ w /@ S;
z[1]:= 1; z[2]:= 3; z[3]:= 2; z[4]:= 7; z[5]:= 8; z[6]:= 6;
q[S_List] := Plus @@ z /@ S; v[{}] := 0;
v[S_ /; (p[S] >= 650 && q[S] >= 15) || q[S] >= 24] := 1;
v[S_ /; (p[S] < 650 && q[S] < 24) || q[S] < 15] := 0;)
```

We can also define the game `Cluster1` given only by the double majority game withour the blocking clause.

```
(v[{}] := 0; v[S_ /; (p[S] >= 650 && q[S] >= 15)] := 1;
 v[S_ /; (p[S] < 650 || q[S] < 15)] := 0;)
```

The classical Shapley values of these games are:

```
ShaCluster =  {0.10,  0.25, 0.10,  0.25, 0.15, 0.15},
ShaCluster1 = {0.067, 0.42, 0.067, 0.22, 0.12, 0.12}.
```

Let us consider the following star and wheel graphs.

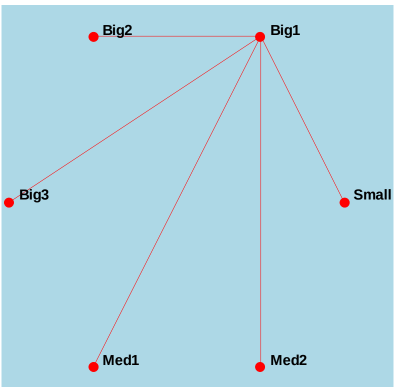

Figure 4

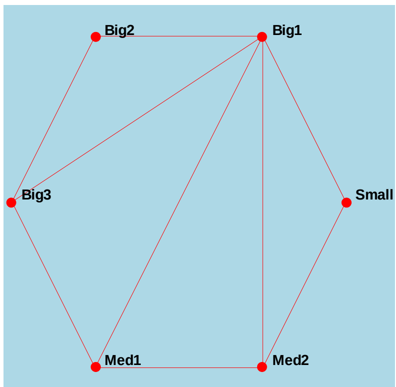

Figure 5

The augmenting system given by a graph G is the collection of all the connected subgraphs of G. For the above graphs, we obtain:

F_Star = {{}, {1}, {2}, {3}, {4}, {5}, {6}, {1, 2}, {1, 3}, {1, 4}, {1, 5}, {1, 6}, {1, 2, 3}, {1, 2, 4},
{1, 2, 5}, {1, 2, 6}, {1, 3, 4}, {1, 3, 5}, {1, 3, 6}, {1, 4, 5}, {1, 4, 6}, {1, 5, 6}, {1, 2, 3, 4},
{1, 2, 3, 5}, {1, 2, 3, 6}, {1, 2, 4, 5}, {1, 2, 4, 6}, {1, 2, 5, 6}, {1, 3, 4, 5}, {1, 3, 4, 6}, {1, 3, 5, 6},
{1, 4, 5, 6}, {1, 2, 3, 4, 5}, {1, 2, 3, 4, 6}, {1, 2, 3, 5, 6}, {1, 2, 4, 5, 6}, {1, 3, 4, 5, 6}, {1, 2, 3, 4, 5, 6}}

F_Wheel = {{}, {1}, {2}, {3}, {4}, {5}, {6}, {1, 2}, {1, 3}, {1, 4}, {1, 5}, {1, 6}, {2, 3}, {3, 4},
{4, 5}, {5, 6}, {1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {1, 2, 6}, {1, 3, 4}, {1, 3, 5}, {1, 3, 6}, {1, 4, 5}, {1, 4, 6},
{1, 5, 6}, {2, 3, 4}, {3, 4, 5}, {4, 5, 6}, {1, 2, 3, 4}, {1, 2, 3, 5}, {1, 2, 3, 6}, {1, 2, 4, 5}, {1, 2, 4, 6},
{1, 2, 5, 6}, {1, 3, 4, 5}, {1, 3, 4, 6}, {1, 3, 5, 6}, {1, 4, 5, 6}, {2, 3, 4, 5}, {3, 4, 5, 6}, {1, 2, 3, 4, 5},
{1, 2, 3, 4, 6}, {1, 2, 3, 5, 6}, {1, 2, 4, 5, 6}, {1, 3, 4, 5, 6}, {2, 3, 4, 5, 6}, {1, 2, 3, 4, 5, 6}}

The Shapley value for the Cluster and Cluster1 games restricted for these graphs are respectively:

```
ShaStarCluster =  {0.37, 0.17, 0.067, 0.17, 0.12,  0.12},
ShaStarCluster1 = {0.33, 0.33, 0.033, 0.13, 0.083, 0.083},


ShaWheelCluster =  {0.17, 0.22, 0.12,  0.22, 0.17, 0.12},
ShaWheelCluster1 = {0.13, 0.38, 0.083, 0.18, 0.13, 0.083}.
```


Acknowledgments

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