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Weighted multiple majority games with unions: Generating functions and applications to the European Union

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ABSTRACT

An a priori system of unions or coalition structure is a partition of a finite set of players into disjoint coalitions which have made a prior commitment to cooperate in playing a game. This paper provides "ready-to-apply" procedures based on generating functions that are easily implementable to compute coalitional power indices in weighted multiple majority games. As an application of the proposed procedures, we calculate and compare coalitional power indices under the decision rules prescribed by the Treaty of Nice and the new rules proposed by the Council of the European Union.

1. Introduction

The classical power indices, derived from cooperative game theory, are the Banzhaf and Shapley–Shubik power indices (Banzhaf, 1965; Shapley and Shubik, 1954). These indices are unable to help us in measuring the distribution of power in situations where the players vote according to a coalition structure, because ignoring the existing information about the coalition structure may mislead the inference about the distribution of power. The main strength of the Banzhaf and Shapley–Shubik power indices is to provide a measure of the probability that some player might be pivotal during a vote, in the sense that he might be able to transform a losing coalition into a winning one. From a theoretical point of view, the a priori unions framework initiated by Owen (1977) turned out to be the relevant one for addressing the aforementioned problem. The main advantage of this model is to propose a new collection of power indices for games restricted by a system of unions: the so called coalitional power indices. These numerical measures of power take into account the modification of the probability space, while keeping track of their fundamental axiomatic properties with respect to both *within* and *between* unions allocation of rewards.

The aim of this paper is precisely to provide some " ready-to-apply" procedures based on generating functions that allow to compute the coalitional power indices in a particular case: the weighted multiple majority games. Roughly, a generating function is a polynomial that allows to enumerate the set of possible coalitions, while keeping track of their respective weights. One strength of these procedures is that they allow to get exact values of these indices, including games with a large number of players. This method, drawn from Cantor (see Lucas (1983)) and Brams and Affuso (1976) early work, provides algorithms to compute the Banzhaf and Shapley–Shubik power indices of weighted voting games.

Fernández et al. (2002) extended the generating functions method for computing the Myerson value in weighted voting games restricted by a communication graph. The same method is used by Bilbao et al. (2002) to compute classical power indices of weighted double and triple majority games. The main application of this work is to analyze the voting system in the Council of Ministers of the European Union under the Treaty of Nice. The next step was given by Algaba et al. (2003) who calculate generating functions for power indices of weighted multiple majority games and analyze its algorithmic complexity. The first application of generating functions to compute coalitional power indices was proposed by Alonso-Meijide and Bowles (2005). They use this method to study the distribution of power in the

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International Monetary Fund (IMF) modelled as a weighted majority game. Previously, the voting power in the IMF has been analyzed by Leech (2002) by using an algorithm that combines exact methods with an approximate procedure. Also, the voting power of members of a voting body with blocs has been studied by Leech and Leech (2006) by using the Penrose index (the absolute Banzhaf index). They have shown that the power of an individual bloc member can be modelled in terms of two contrasting components: the power of the bloc that increases with bloc size; and the power of the individual member over bloc decisions, which declines with bloc size. Moreover, Leech and Leech (2004) provide access to computer software for voting power analysis in internet. The programs calculate the Shapley and Shubik and Banzhaf indices based on different coalition models. They also provide the Penrose and Coleman (the power to initiate action and the power to prevent action) indices.

We provide the generating functions procedures and apply them to the distribution of power in the enlarged European Union, modelled as a weighted multiple majority game with a coalition structure. In Algaba et al. (2007), the distribution of voting power for these countries is analyzed under the voting rules prescribed by the Treaty of Nice in December 2000 and the new rules proposed by the Council of the European Union in June 2007. Since Algaba et al. (2007) analyzed weighted multiple majority games without coalition structure and Alonso-Meijide and Bowles (2005) studied weighted voting games under a coalition structure, our proposal is a generalization of both papers. Furthermore, our paper allows to illustrate the distinguishing features and behavior of the coalitional power indices and its dependence of the coalition structure.

The paper is organized as follows. In Section 2, we recall some preliminary definitions and the notions of generating function and power indices for weighted multiple majority games with a coalition structure. In Section 3, we introduce the procedures to compute these power indices by means of generating functions. In Section 4, we apply these procedures to the enlarged European Union in order to illustrate by a large game the differences in behavior of the three coalitional power indices considered in this paper depending of the coalition structure. Finally, we include an appendix with the algorithms to compute coalitional power indices which are written using the `Mathematica` Code.

2. Preliminaries

A finite cooperative game with transferable utility (TU game) is a pair (N, v) where $N = \{1, 2, \ldots, n\}$ is the set of players and v, the characteristic function, is a real valued function defined on the subsets of N (coalitions) such that $v(\emptyset) = 0$. A simple game is a TU game in which the function v only takes the values 0 and 1, it is not identically 0, and obeys the condition of monotonicity, i.e., $v(T) \leqslant v(S)$ whenever $T \subseteq S$. A coalition S is winning if $v(S) = 1$, and losing if $v(S) = 0$. The collection of all winning coalitions is denoted by $\mathscr{W}$. We denote by $SI(N)$ the set of simple games with player set N.

A weighted majority game, represented by $[q; w_1, \ldots, w_n]$, is a simple game where the quota q, and the weights $w_1, \ldots, w_n$ are positive integers, and such that $S \in \mathscr{W}$ if and only if $w(S) \geqslant q$, where $w(S) = \sum_{i \in S} w_i$. Given the simple games $(N, v_1), \ldots, (N, v_r)$, we consider the simple games $(N, v_1 \wedge \cdots \wedge v_r)$ and $(N, v_1 \vee \cdots \vee v_r)$ defined by

$$(v_1 \wedge \cdots \wedge v_r)(S) = \min\{v_f(S) : 1 \leqslant f \leqslant r\},$$
$$(v_1 \vee \cdots \vee v_r)(S) = \max\{v_f(S) : 1 \leqslant f \leqslant r\}.$$

In the game $(N, v_1 \wedge \cdots \wedge v_r)$, a coalition S is winning if and only if S is winning in v_f, for all $1 \leqslant f \leqslant r$. A weighted multiple majority game is the simple game $(N, v_1 \wedge \cdots \wedge v_r)$ where $v_f = [q^f; w_1^f, \ldots, w_n^f]$, $1 \leqslant f \leqslant r$ are weighted majority games. Then

$$(v_1 \wedge \cdots \wedge v_r)(S) = \begin{cases} 1 & \text{if } w^f(S) \geqslant q^f, \quad 1 \leqslant f \leqslant r, \\ 0 & \text{otherwise,} \end{cases}$$

where $w^f(S) = \sum_{i \in S} w_i^f$.

Definition 1. A power index is a function $g : SI(N) \to \mathbb{R}^n$ which assigns to a simple game (N, v) a vector $g(N, v)$, where the real number $g_i(N, v)$ is the power of player i in the game (N, v) according to g.

Let S be a finite set. As it became common practice, given $i \in S$ we will for simplicity write $S \setminus i$ instead of $S \setminus \{i\}$, and given $i \notin S$ we will write $S \cup i$ instead of $S \cup \{i\}$. We will denote by s the cardinality of S.

The most important power indices are the Banzhaf index and the Shapley–Shubik index (hereafter BZ and SH). These indices can be written as

$$g_i(N, v) = \sum_{S \subseteq N \setminus i} p_S^i (v(S \cup i) - v(S)), \quad \text{for all } i \in N,$$

where $p_S^i = 1/2^{n-1}$ for the BZ index and $p_S^i = s!(n-s-1)!/n!$ for the SH index. These two probability measures are, according to Felsenthal and Machover (1998), the two principal ones underlying the various existing power indices in the literature. Intuitively, the basic unit of analysis for the BZ index is the coalition, while the SH index focuses on the ranking of individual preferences over a given outcome. In both cases, each basic measurable event (coalition or ranking of preferences) is assumed to have an equal probability.

Taking into account that for a simple game $v(T) = 1$ if $T \in \mathscr{W}$ and $v(T) = 0$ otherwise, it holds that $v(S \cup i) - v(S) = 1$ if and only if S is losing and $S \cup i$ is winning. In this case, we say that the pair of coalitions $(S, S \cup i)$ is a swing for player i. The calculation of the SH and BZ power indices requires to know the number of swings for every player i.

For large games, the computation of the previous indices needs a great number of operations. The generating function is one of the procedures to compute them. The formal power series $f_a(t) = \sum_{k \geqslant 0} a_k t^k$ is called the generating function of the numbers $a = \{a_0, a_1, a_2, \ldots\}$. The variable t serves to identify a_k as the coefficient corresponding to t^k in $f_a(t)$.

Example 1. Let us take $\prod_{k=1}^{n}(1 + x_k t) = \sum_{k=0}^{n} a_k t^k$, where $a_o = 1$ and, for $k > 0$, a_k is given by

$$a_k = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} x_{i_1} x_{i_2} \ldots x_{i_k}.$$

Coefficients a_k are symmetric functions of $x_1, x_2, \ldots, x_n$. The cardinality of terms of coefficient a_k coincides with the number of combinations of k elements of a set formed by n elements. If all the values x_k are equal to 1 it holds that

$$(1+t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k.$$

Then, the function $f(t) = (1+t)^n$ is the generating function of the binomial coefficients $a_k = \binom{n}{k}$ for $k = 0, 1, \ldots, n$.

In some cases, we will use generating functions of several variables, for example

$$S(x,y,z) = \sum_{k\geqslant 0} \sum_{j\geqslant 0} \sum_{l\geqslant 0} c(k,j,l) x^k y^j z^l,$$

where $c(k,j,l)$ are real numbers that depend on k, j, and l.

Let us consider a finite set $N = \{1, \ldots, n\}$. A partition of N is a collection $P = \{P_1, \ldots, P_m\}$ of non-empty subsets of N such that $P_i \cap P_j = \emptyset$ if $i \neq j$, and $P_1 \cup \cdots \cup P_m = N$. We call P_i a block of P and we say that P has m blocks, denoted $|P| = m$. We will denote by $\mathbb{P}(N)$ the set of all partitions of N. A partition $P \in \mathbb{P}(N)$ is called a coalition structure for N. A simple game with a coalition structure is a triple (N, υ, P), where $(N, \upsilon) \in SI(N)$ and $P \in \mathbb{P}(N)$. The family of all simple games with player set N and a coalition structure $P \in \mathbb{P}(N)$ will be denoted by $SCS(N)$.

A weighted multiple majority game with a coalition structure is given by $(N, \upsilon_1 \wedge \cdots \wedge \upsilon_r, P)$ where N is the set of players, $\upsilon_1 \wedge \cdots \wedge \upsilon_r$ is a weighted multiple majority game, and $P \in \mathbb{P}(N)$. For simple games with a coalition structure, a power index is a function $g : SCS(N) \to \mathbb{R}^n$ which assigns to a simple game with a coalition structure (N, υ, P) a vector $g(N, \upsilon, P)$, where the real number $g_i(N, \upsilon, P)$ is the power of player i in the game (N, υ, P) according to g.

We consider three power indices for $SCS(N)$: the Banzhaf–Owen index (hereafter BO), the Symmetric Coalitional Banzhaf index (hereafter SCB), and the Owen index (hereafter OW). These indices can be written as

$$g_i(N, \upsilon, P) = \sum_{L \subseteq M\backslash k} \sum_{T \subseteq P_k \backslash i} p^i_{L,T} (\upsilon(Q \cup T \cup i) - \upsilon(Q \cup T)), \quad \text{for all } i \in N,$$

where $M = \{1, \ldots, m\}$, $P = \{P_1, \ldots, P_m\}$, $Q = \bigcup_{l \in L} P_l$, and $P_k \in P$ is the block such that $i \in P_k$. For the case of the BO index the coefficients are

$$p^i_{L,T} = \frac{1}{2^{m-1}} \frac{1}{2^{p_k - 1}},$$

for the SCB index

$$p^i_{L,T} = \frac{1}{2^{m-1}} \frac{t!(p_k - t - 1)!}{p_k!},$$

and finally, for the OW index

$$p^i_{L,T} = \frac{l!(m-l-1)!}{m!} \frac{t!(p_k - t - 1)!}{p_k!}$$

(see Owen, 1982; Alonso-Meijide and Fiestras-Janeiro, 2002; Owen, 1977, respectively). Other characterizations of the Owen value can be seen in Winter (1992), Amer and Carreras (1995), and Vázquez-Brage et al. (1997). Characterizations of the Banzhaf–Owen index appear in Albizuri (2001), Amer et al. (2002), and Alonso-Meijide et al. (2007).

According to the probability model underlying the choice of these probability distributions, one can see that all the three coalitional power indices can be derived from a two-level bargaining process: among coalitions and among players in the same coalition. First, applying the SH or BZ type of allocation in the game among coalitions, and secondly, using again the SH or the BZ type of allocation in a reduced game inside each coalition.

3. Coalitional power indices by using generating functions

This section is dedicated to the computation of the generating function of the coalitional power indices in weighted multiple majority games with a coalition structure, that will allow us to obtain them by the generating functions method. We recall two definitions presented in Alonso-Meijide and Bowles (2005).

Definition 2. Let $P = \{P_1, \ldots, P_m\}$ be a coalition structure for N. A coalition $S \subseteq N$ is compatible with a block $P_j \in P$ if $S = (\bigcup_{k \in L \subseteq M \backslash j} P_k) \cup T$, where $T \subseteq P_j$. We denote by $C(j, P)$ the set of compatible coalitions with the block $P_j \in P$.

Example 2. Let us consider the partition $P = \{\{1\}, \{2,4,5\}, \{3,6\}\}$. The sets $C(j, P)$ of compatible coalitions for $j \in \{1,2,3\}$ are

$$C(1,P) = \{\{\}, \{1\}, \{3,6\}, \{1,3,6\}, \{2,4,5\}, \{1,2,4,5\}, \{2,3,4,5,6\}, \{1,2,3,4,5,6\}\};$$

$$\begin{aligned} C(2,P) = \{&\{\}, \{1\}, \{2\}, \{4\}, \{5\}, \{1,2\}, \{1,4\}, \{1,5\}, \{2,4\}, \{2,5\}, \{3,6\}, \{4,5\}, \{1,2,4\}, \{1,2,5\}, \{1,3,6\}, \{1,4,5\}, \{2,3,6\}, \\ &\{2,4,5\}, \{3,4,6\}, \{3,5,6\}, \{1,2,3,6\}, \{1,2,4,5\}, \{1,3,4,6\}, \{1,3,5,6\}, \{2,3,4,6\}, \{2,3,5,6\}, \{3,4,5,6\}, \{1,2,3,4,6\}, \\ &\{1,2,3,5,6\}, \{1,3,4,5,6\}, \{2,3,4,5,6\}, \{1,2,3,4,5,6\}\}; \end{aligned}$$

$$\begin{aligned} C(3,P) = \{&\{\}, \{1\}, \{3\}, \{6\}, \{1,3\}, \{1,6\}, \{3,6\}, \{1,3,6\}, \{2,4,5\}, \{1,2,4,5\}, \{2,3,4,5\}, \{2,4,5,6\}, \{1,2,3,4,5\}, \{1,2,4,5,6\}, \\ &\{2,3,4,5,6\}, \{1,2,3,4,5,6\}\}. \end{aligned}$$

Definition 3. Let (N, v, P) be a simple game with a coalition structure and let P_j be a block of P. A compatible swing for a player $i \in P_j$, is a pair $(S, S \cup i)$ such that $S \in C(j, P)$, $v(S) = 0$, and $v(S \cup i) = 1$. We denote by $\eta_i(N, v, P)$ the number of compatible swings for a player $i \in N$.

Example 3. Let us consider a triple majority game $(N, v_1 \wedge v_2 \wedge v_3, P)$ with $N = \{1, 2, 3, 4, 5, 6\}$ and $P = \{P_1, P_2, P_3\}$, where

$$v_1 = [30; 28, 16, 5, 4, 3, 3], \quad v_2 = [51; 40, 20, 15, 10, 8, 7], \quad v_3 = [4; 1, 1, 1, 1, 1, 1],$$

and

$$P_1 = \{1\}, \quad P_2 = \{2, 4, 5\}, \quad P_3 = \{3, 6\}.$$

Although $v(\{2, 3, 4\}) = 0$ and $v(\{1, 2, 3, 4\}) = 1$, the pair $(\{2, 3, 4\}, \{1, 2, 3, 4\})$ is not a compatible swing for player 1 because $\{2, 3, 4\} \notin C(1, P)$. But the pair $(\{2, 4, 5\}, \{1, 2, 4, 5\})$ is a compatible swing for player 1.

3.1. The Banzhaf–Owen index

Owen (1982) proposed the BO index as a modification of the Banzhaf value for games with a coalition structure. It is based on the assumption that there are two bargaining levels: between blocks and between players within each block. For the BO index, the assessments in each step are given for the Banzhaf value. Alonso-Meijide and Fiestras-Janeiro (2002) established that the BO value fails to satisfy two interesting properties: symmetry in the quotient game and the quotient game property.

The computation of this value applied to simple games with a coalition structure is based on the number of compatible swings for every player, in a similar way to the Banzhaf index. Thus, for a weighted multiple majority game with a coalition structure $(N, v_1 \wedge \cdots \wedge v_r, P)$, where $v_f = [q^f; w_1^f, \ldots, w_n^f]$, $1 \leqslant f \leqslant r$ and $P \in \mathbb{P}(N)$, the BO index of a player $i \in P_j$, is given by

$$\Psi_i(N, v, P) = \frac{\eta_i(N, v_1 \wedge \cdots \wedge v_r, P)}{2^{m+p_j-2}}.$$

In the next result, we propose a method to compute $\eta_i(N, v_1 \wedge \cdots \wedge v_r, P)$ in a weighted multiple majority game.

Lemma 1. *Let $(N, v_1 \wedge \cdots \wedge v_r, P) \in SCS(N)$ where $v_f = [q^f; w_1^f, \ldots, w_n^f]$, $1 \leqslant f \leqslant r$ and $P = \{P_1, \ldots, P_m\}$. The number of compatible swings for a player $i \in P_j$ is*

$$\eta_i(N, v_1 \wedge \cdots \wedge v_r, P) = \sum_{\substack{k_f = q^f - w_i^f \\ 1 \leqslant f \leqslant r}}^{w^f(N \setminus i)} b^i_{k_1 \cdots k_r} - \sum_{\substack{k_f = q^f \\ 1 \leqslant f \leqslant r}}^{w^f(N \setminus i)} b^i_{k_1 \cdots k_r},$$

where $b^i_{k_1 \cdots k_r}$ is the number of coalitions $S \in C(j, P)$ such that $i \notin S$ with $w^f(S) = k_f$ for all $1 \leqslant f \leqslant r$.

Proof. First of all, we consider the set of all coalitions $S \in C(j, P)$ such that $i \notin S$ with $w^f(S) \geqslant q^f - w_i^f$ for all $1 \leqslant f \leqslant r$. Its cardinal is given by

$$s_i^1 = \sum_{k_1 = q^1 - w_i^1}^{w^1(N \setminus i)} \cdots \sum_{k_r = q^r - w_i^r}^{w^r(N \setminus i)} b^i_{k_1 \cdots k_r}.$$

As $w^f(S \cup i) \geqslant q^f$, $1 \leqslant f \leqslant r$, then s_1^i coincides with the number of compatible winning coalitions for players of the block P_j in which the player i participates.

On the other hand, inside of the set of the compatible winning coalitions that contain player i, we consider the subset of those coalitions in which player i is not necessary to win. The cardinal of this subset coincides with the set of all coalitions $S \in C(j, P)$ such that $i \notin S$ with $w^f(S) \geqslant q^f$, $1 \leqslant f \leqslant r$, and it is given by

$$s_i^2 = \sum_{k_1 = q^1}^{w^1(N \setminus i)} \cdots \sum_{k_r = q^r}^{w^r(N \setminus i)} b^i_{k_1 \cdots k_r}.$$

Therefore, the number of compatible swings of player i in the previous game is $\eta_i(N, v_1 \wedge \cdots \wedge v_r, P) = s_i^1 - s_i^2$. □

Now, we present the generating function of the numbers $\{b^i_{k_1 \cdots k_r}\}_{k_1, \ldots, k_r \geqslant 0}$.

Theorem 1. *Let $(N, v_1 \wedge \cdots \wedge v_r, P) \in SCS(N)$ where $v_f = \left[q^f; w_1^f, \ldots, w_n^f\right]$, $1 \leqslant f \leqslant r$ and $P = \{P_1, \ldots, P_m\}$. For each player $i \in P_j$, the generating function of $\{b^i_{k_1 \cdots k_r}\}_{k_1, \ldots, k_r \geqslant 0}$ is given by*

$$B_i(x_1, \ldots, x_r) = \prod_{v=1, v \neq j}^{m} \left(1 + x_1^{w^1(P_v)} \cdots x_r^{w^r(P_v)}\right) \prod_{l=1, j_l \neq i}^{p_j} \left(1 + x_1^{w_{j_l}^1} \cdots x_r^{w_{j_l}^r}\right),$$

where $w^f(P_v) = \sum_{j \in P_v} w_j^f$, $1 \leqslant f \leqslant r$ and $P_j = \{j_1, j_2, \ldots, j_{p_j}\}$.

Proof. Expanding the function

$$B(x_1, \ldots, x_r) = \prod_{v=1, v \neq j}^{m} \left(1 + x_1^{w^1(P_v)} \cdots x_r^{w^r(P_v)}\right) \prod_{l=1}^{p_j} \left(1 + x_1^{w_{j_l}^1} \cdots x_r^{w_{j_l}^r}\right) = 1 + \sum_{S \in C(j,P)} \prod_{l \in S} x_1^{w_l^1} \cdots x_r^{w_l^r} = 1 + \sum_{S \in C(j,P)} x_1^{\sum_{l \in S} w_l^1} \cdots x_r^{\sum_{l \in S} w_l^r}$$

$$= 1 + \sum_{S \in C(j,P)} x_1^{w^1(S)} \cdots x_r^{w^r(S)}.$$

We take $k_f = w^f(S)$ for all $1 \leqslant f \leqslant r$, and hence the previous function is equal to

$$\sum_{\substack{k_f=0\\1\leqslant f\leqslant r}}^{w^f(N)} b_{k_1\cdots k_r} x_1^{k_1}\cdots x_r^{k_r},$$

where $b_{0\cdots 0}=1$, and $b_{k_1\cdots k_r}$ is the number of compatible coalitions $S\in C(j,P)$ such that $w^f(S)=k_f$, for all $1\leqslant f\leqslant r$, where $k_f>0$ for at least an element $f\in\{1,2,\ldots,r\}$. To obtain the numbers $\{b^i_{k_1\cdots k_r}\}_{k_1,\ldots,k_r\geqslant 0}$, it is sufficient to drop the factor $(1+x_1^{w_i^1}\cdots x_r^{w_i^r})$ in the polynomial $B(x_1,\ldots,x_r)$. $\square$

Example 4. Let us consider the triple majority game with 6 players and 3 a priori unions given in Example 3. We compute by hand the Banzhaf–Owen index of player 4. We take the generating function

$$\begin{aligned}B_4(x_1,x_2,x_3) &= (1+x_1^{28}x_2^{40}x_3)(1+x_1^{8}x_2^{22}x_3^2)(1+x_1^{16}x_2^{20}x_3)(1+x_1^{3}x_2^{8}x_3)\\ &= 1+x_1^{28}x_2^{40}x_3+x_1^{8}x_2^{22}x_3^2+x_1^{36}x_2^{62}x_3^3+x_1^{16}x_2^{20}x_3+x_1^{44}x_2^{60}x_3^2+x_1^{24}x_2^{42}x_3^3+x_1^{52}x_2^{82}x_3^4+x_1^{3}x_2^{8}x_3+x_1^{31}x_2^{48}x_3^2+x_1^{11}x_2^{30}x_3^3+x_1^{39}x_2^{70}x_3^4\\ &\quad +x_1^{19}x_2^{28}x_3^2+x_1^{47}x_2^{68}x_3^3+x_1^{27}x_2^{50}x_3^4+x_1^{55}x_2^{90}x_3^5.\end{aligned}$$

We choose the coefficients of terms $x_1^{k_1}x_2^{k_2}x_3^{k_3}$ such that k_1 takes values between 26 and 55, k_2 takes values between 41 and 90, and k_3 takes values between 3 and 5. In this case, there are 6 terms:

$$x_1^{36}x_2^{62}x_3^3+x_1^{52}x_2^{82}x_3^4+x_1^{39}x_2^{70}x_3^4+x_1^{47}x_2^{68}x_3^3+x_1^{27}x_2^{50}x_3^4+x_1^{55}x_2^{90}x_3^5.$$

Then we obtain $s_4^1=6$. We must eliminate those coalitions that remain winning without player 4. That is, those corresponding with terms $x_1^{k_1}x_2^{k_2}x_3^{k_3}$ such that k_1 takes values between 30 and 55, k_2 takes values between 51 and 90, and k_3 takes values between 4 and 5. In this case, there are 3 terms:

$$x_1^{52}x_2^{82}x_3^4+x_1^{39}x_2^{70}x_3^4+x_1^{55}x_2^{90}x_3^5.$$

Then we obtain $s_4^2=3$ and $s_4^1-s_4^2=\eta_4(N,\upsilon_1\wedge\upsilon_2\wedge\upsilon_3,P)=3$. The Banzhaf–Owen index of player 4 is

$$\Psi_4(N,\upsilon_1\wedge\upsilon_2\wedge\upsilon_3,P)=\frac{\eta_4(N,\upsilon_1\wedge\upsilon_2\wedge\upsilon_3,P)}{2^{m+p_2-2}}=\frac{3}{2^{3+3-2}}=\frac{3}{16}.$$

3.2. The symmetric coalitional Banzhaf index

Following a similar argument to that used by Owen to define the BO value and the OW value, Alonso-Meijide and Fiestras-Janeiro (2002) defined the Symmetric Coalitional Banzhaf value. They modified the two-step allocation scheme and use the Banzhaf value for sharing in the game played among blocks and the Shapley value within blocks. This value satisfies the properties of symmetry in the quotient game and quotient game. For the particular case of simple games (similar to the SH index) the computation of this value for a player $i\in P_j$ depends on the size of the swings, in particular, of the numbers of players of the coalition P_j in the swings. Given a weighted multiple majority game with a coalition structure $(N,\upsilon_1\wedge\cdots\wedge\upsilon_r,P)$, where $\upsilon_f=[q^f;w_1^f,\ldots,w_n^f]$, $1\leqslant f\leqslant r$ and $P\in\mathbb{P}(N)$, the SCB index of a player $i\in P_j$, denoted by $\Lambda_i(N,\upsilon_1\wedge\cdots\wedge\upsilon_r,P)$ is equal to

$$\sum_{\{S\in C(i,P):S\notin W,S\cup i\in W\}}\frac{1}{2^{m-1}}\frac{|S\cap P_j|!(p_j-|S\cap P_j|-1)!}{p_j!}.$$

Then it should be noted that

$$\Lambda_i(N,\upsilon_1\wedge\cdots\wedge\upsilon_r,P)=\sum_{l=0}^{p_j-1}\frac{1}{2^{m-1}}\frac{l!(p_j-l-1)!}{p_j!}d_l^i,$$

where d_l^i is the number of compatible swings $(S,S\cup i)$, for player i in the game $(N,\upsilon_1\wedge\cdots\wedge\upsilon_r,P)$, such that $|S\cap P_j|=l$. For any value of l between 0 and p_j-1, we have

$$d_l^i=\sum_{\substack{k_f=q^f-w_i^f\\1\leqslant f\leqslant r}}^{w^f(N\setminus i)}a^i_{k_1\cdots k_r l}-\sum_{\substack{k_f=q^f\\1\leqslant f\leqslant r}}^{w^f(N\setminus i)}a^i_{k_1\cdots k_r l},$$

where $a^i_{k_1\cdots k_r l}$ is the number of compatible swings $(S,S\cup i)$, for player i in the game $(N,\upsilon_1\wedge\cdots\wedge\upsilon_r,P)$ with $w^f(S)=k_f$, $1\leqslant f\leqslant r$ and $|S\cap P_j|=l$.

Theorem 2. *Let* $(N,\upsilon_1\wedge\cdots\wedge\upsilon_r,P)\in SCS(N)$ *where* $\upsilon_f=\left[q^f;w_1^f,\ldots,w_n^f\right]$, $1\leqslant f\leqslant r$ *and* $P=\{P_1,\ldots,P_m\}$. *For each player* $i\in P_j$, *the generating function of* $\{a^i_{k_1\cdots k_r l}\}_{k_1,\ldots,k_r\geqslant 0,l\geqslant 0}$ *is given by*

$$S_i(x_1,\ldots,x_r,z)=\prod_{\upsilon=1,\upsilon\neq j}^{m}\left(1+x_1^{w^1(P_\upsilon)}\cdots x_r^{w^r(P_\upsilon)}\right)\prod_{l=1,j_l\neq i}^{p_j}\left(1+x_1^{w^1_{j_l}}\cdots x_r^{w^r_{j_l}}z\right),$$

where $w^f(P_\upsilon)=\sum_{j\in P_\upsilon}w_j^f$, $1\leqslant f\leqslant r$ *and* $P_j=\{j_1,\ldots,j_{p_j}\}$.

Proof. Let us take a game $(N, v_1 \wedge \cdots \wedge v_r, P) \in SCS(N)$ where $v_f = \left[q^f; w_1^f, \ldots, w_n^f\right]$, $1 \leqslant f \leqslant r$ and $i \in P_j$. We expand the function

$$S(x_1, \ldots, x_r, z) = \prod_{v=1, v \neq j}^{m} \left(1 + x_1^{w^1(P_v)} \cdots x_r^{w^r(P_v)}\right) \prod_{l=1}^{p_j} \left(1 + x_1^{w^1_{j_l}} \cdots x_r^{w^r_{j_l}} z\right) = 1 + \sum_{S \in C(j,P)} \prod_{l \in S \setminus P_j} x_1^{w_l^1} \cdots x_r^{w_l^r} \prod_{l \in S \cap P_j} x_1^{w_l^1} \cdots x_r^{w_l^r} z$$
$$= 1 + \sum_{S \in C(j,P)} x_1^{w^1(S)} \cdots x_r^{w^r(S)} z^{|S \cap P_j|}.$$

We take $k_f = w^f(S)$ for all $1 \leqslant f \leqslant r$, and hence the previous function is equal to

$$\sum_{\substack{k_f=0 \\ 1 \leqslant f \leqslant r}}^{w^f(N)} \sum_{l=0}^{p_j} a_{k_1 \cdots k_r l} x_1^{k_1} \cdots x_r^{k_r} z^l,$$

where $a_{0 \cdots 0} = 1$, and $a_{k_1 \cdots k_r l}$ is the number of compatible coalitions $S \in C(j, P)$ such that $w^f(S) = k_f$, for all $1 \leqslant f \leqslant r$, and $|S \cap P_j| = l$, where at least one k_f with $f \in \{1, 2, \ldots, r\}$ or l is greater than 0. In order to compute $\{a^i_{k_1 \cdots k_r l}\}_{k_1, \ldots, k_r \geqslant 0, l \geqslant 0}$, it is sufficient to drop the factor $\left(1 + x_1^{w_i^1} \cdots x_r^{w_i^r} z\right)$ in the polynomial $S(x_1, \ldots, x_r, z)$. $\square$

The previous result gives a method to compute $\{a^i_{k_1 \cdots k_r l}\}_{k_1, \ldots, k_r \geqslant 0, l \geqslant 0}$. Now, it is necessary to obtain the values d_l^i. These values can be identified by the coefficients of

$$g_i(z) = \sum_{l=0}^{p_j - 1} d_l^i z^l,$$

and, taking into account that

$$d_l^i = \sum_{\substack{k_f = q^f - w_i^f \\ 1 \leqslant f \leqslant r}}^{w^f(N \setminus i)} a^i_{k_1 \cdots k_r l} - \sum_{\substack{k_f = q^f \\ 1 \leqslant f \leqslant r}}^{w^f(N \setminus i)} a^i_{k_1 \cdots k_r l},$$

it holds that

$$g_i(z) = \sum_{l=0}^{p_j - 1} d_l^i z^l = \sum_{l=0}^{p_j - 1} \left[\sum_{\substack{k_f = q^f - w_i^f \\ 1 \leqslant f \leqslant r}}^{w^f(N \setminus i)} a^i_{k_1 \cdots k_r l} - \sum_{\substack{k_f = q^f \\ 1 \leqslant f \leqslant r}}^{w^f(N \setminus i)} a^i_{k_1 \cdots k_r l} \right] z^l.$$

Hence, we obtain that

$$g_i(z) = \sum_{\substack{k_f = q^f - w_i^f \\ 1 \leqslant f \leqslant r}}^{w^f(N \setminus i)} \left[\sum_{l=0}^{p_j - 1} a^i_{k_1 \cdots k_r l} z^l \right] - \sum_{\substack{k_f = q^f \\ 1 \leqslant f \leqslant r}}^{w^f(N \setminus i)} \left[\sum_{l=0}^{p_j - 1} a^i_{k_1 \cdots k_r l} z^l \right].$$

Finally, we have that:

$$S_i(x_1, x_2, \ldots, x_r, z) = \sum_{\substack{k_f = q^f - w_i^f \\ 1 \leqslant f \leqslant r}}^{w^f(N \setminus i)} \left[\sum_{l=0}^{p_j - 1} a^i_{k_1 \cdots k_r l} z^l \right] x_1^{k_1} \ldots x_r^{k_r}.$$

We determine a polynomial in z whose coefficients represent the number of winning coalitions that contain player i

$$s_i^1(z) = \sum_{\substack{k_f = q^f - w_i^f \\ 1 \leqslant f \leqslant r}}^{w^f(N \setminus i)} \sum_{l=0}^{p_j - 1} a^i_{k_1 \cdots k_r l} z^l.$$

On the other hand, we consider a polynomial in z whose coefficients represent the number of winning coalitions that contain player i but his presence is not necessary to win

$$s_i^2(z) = \sum_{\substack{k_f = q^f \\ 1 \leqslant f \leqslant r}}^{w^f(N \setminus i)} \sum_{l=0}^{p_j - 1} a^i_{k_1 \cdots k_r l} z^l.$$

The coefficients of $s_i^1(z)$ can be computed, selecting for any exponent of variable z in $S_i(x_1, \ldots, x_r, z)$, the coefficients of those terms $x_1^{k_1} \cdots x_r^{k_r} z^l$ such that k_f takes values greater or equal than $q^f - w_i^f$, for $1 \leqslant f \leqslant r$. The coefficients of $s_i^2(z)$ can be computed, selecting for any exponent of variable z in $S_i(x_1, \ldots, x_r, z)$, the coefficients of those terms $x_1^{k_1} \cdots x_r^{k_r} z^l$ such that k_f takes values greater or equal than q^f, for $1 \leqslant f \leqslant r$. Finally

$$g_i(z) = \sum_{l=0}^{p_j-1} d_l^i z^l = s_i^1(z) - s_i^2(z).$$

Example 5. Let us consider the triple majority game of Example 3 and we compute the Symmetric Coalitional Banzhaf index of player 4. We take the function:

$$\begin{aligned} S_4(x_1,x_2,x_3,z) &= \left(1+x_1^{28}x_2^{40}x_3\right)\left(1+x_1^{8}x_2^{22}x_3^{2}\right)\left(1+x_1^{16}x_2^{20}x_3z\right)\left(1+x_1^{3}x_2^{8}x_3z\right) \\ &= 1+x_1^{28}x_2^{40}x_3+x_1^{8}x_2^{22}x_3^{2}+x_1^{36}x_2^{62}x_3^{3}+x_1^{16}x_2^{20}x_3z+x_1^{44}x_2^{60}x_3^{2}z+x_1^{24}x_2^{42}x_3^{3}z+x_1^{52}x_2^{82}x_3^{4}z+x_1^{3}x_2^{8}x_3z+x_1^{31}x_2^{48}x_3^{2}z+x_1^{11}x_2^{30}x_3^{3}z \\ &\quad +x_1^{39}x_2^{70}x_3^{4}z+x_1^{19}x_2^{28}x_3^{2}z^2+x_1^{47}x_2^{68}x_3^{3}z^2+x_1^{27}x_2^{50}x_3^{4}z^2+x_1^{55}x_2^{90}x_3^{5}z^2. \end{aligned}$$

We choose the coefficients of terms $x_1^{k_1}x_2^{k_2}x_3^{k_3}z^l$ such that k_1 takes values between 26 and 55, k_2 takes values between 41 and 90, and k_3 takes values between 3 and 5. In this case, there are 6 terms:

$$x_1^{36}x_2^{62}x_3^{3}+x_1^{52}x_2^{82}x_3^{4}z+x_1^{39}x_2^{70}x_3^{4}z+x_1^{47}x_2^{68}x_3^{3}z^2+x_1^{27}x_2^{50}x_3^{4}z^2+x_1^{55}x_2^{90}x_3^{5}z^2.$$

Then we obtain that $s_4^1(z) = 1+2z+3z^2$. We must eliminate those coalitions that remain winning without player 4. That is, those corresponding with terms $x_1^{k_1}x_2^{k_2}x_3^{k_3}z^l$ such that k_1 takes values between 30 and 55, k_2 takes values between 51 and 90, and k_3 takes values between 4 and 5. In this case, there are 3 terms:

$$x_1^{52}x_2^{82}x_3^{4}z+x_1^{39}x_2^{70}x_3^{4}z+x_1^{55}x_2^{90}x_3^{5}z^2.$$

Then we obtain that $s_4^2(z) = 2z+z^2$ and $g_4(z) = s_4^1(z) - s_4^2(z) = 1+2z^2$. The Symmetric Coalitional Banzhaf index of player 4 is

$$\Lambda_4(N, \upsilon_1 \wedge \upsilon_2 \wedge \upsilon_3, P) = \sum_{l=0}^{p_j-1} \frac{1}{2^{m-1}} \frac{l!(p_j-l-1)!}{p_j!} d_l^i = \frac{1}{2^2}\frac{0!2!}{3!} + \frac{1}{2^2}\frac{2!0!}{3!}2 = \frac{1}{4}.$$

Note that the coefficients d_l^i give rise of the expression

$$g_i(z) = \sum_{l=0}^{p_j-1} d_l^i z^l.$$

3.3. The Owen index

The Owen value was proposed by Owen (1977) as a modification of the Shapley value for games with a coalition structure. The OW value uses the Shapley value in the two step process. Similar to the SCB value, the OW value satisfies the properties of symmetry in the quotient game and quotient game. For the particular case of simple games, the computation of this value for a player $i \in P_j$ depends on the size of the swings, in this case, of the numbers of players of the coalition P_j in the swings and of the number of coalitions different of P_j in the swings. Given a weighted multiple majority game with a coalition structure $(N, \upsilon_1 \wedge \cdots \wedge \upsilon_r, P)$, where $\upsilon_f = [q^f; w_1^f, \ldots, w_n^f]$, $1 \leqslant f \leqslant r$ and $P \in \mathbb{P}(N)$, the OW index of a player $i \in P_j$, denoted by $\Phi_i(N, \upsilon_1 \wedge \cdots \wedge \upsilon_r, P)$ is equal to

$$\sum_{\{S \in C(i,P): S \notin W, S \cup i \in W\}} \frac{|m_j(S)|!(m-|m_j(S)|-1)!}{m!} \frac{|S \cap P_j|!(p_j - |S \cap P_j| - 1)!}{p_j!}.$$

Then it should be noted that

$$\Phi_i(N, \upsilon_1 \wedge \cdots \wedge \upsilon_r, P) = \sum_{g=0}^{m-1} \sum_{l=0}^{p_j-1} \frac{g!(m-g-1)!}{m!} \frac{l!(p_j-l-1)!}{p_j!} d_{gl}^i,$$

where $m_j(S) = \{k \in M \setminus j : P_k \subseteq S\}$ and d_{gl}^i is the number of compatible swings $(S, S \cup i)$, for a player $i \in P_j$ in $(N, \upsilon_1 \wedge \cdots \wedge \upsilon_r, P)$, such that $|m_j(S)| = g$ and $|S \cap P_j| = l$. Then, for any value of g between 0 and $m-1$ and for any value of l between 0 and $p_j - 1$, we obtain

$$d_{gl}^i = \sum_{\substack{k_f = q^f - w_i^f \\ 1 \leqslant f \leqslant r}}^{w^f(N \setminus i)} a_{k_1 \cdots k_r g l}^i - \sum_{\substack{k_f = q^f \\ 1 \leqslant f \leqslant r}}^{w^f(N \setminus i)} a_{k_1 \cdots k_r g l}^i,$$

where $a_{k_1 \cdots k_r g l}^i$ is the number of compatible swings $(S, S \cup i)$, for player i in the game $(N, \upsilon_1 \wedge \cdots \wedge \upsilon_r, P)$ with $w^f(S) = k_f$, $1 \leqslant f \leqslant r$ and $|S \cap P_j| = l$ and $|m_j(S)| = g$.

Theorem 3. *Let $(N, \upsilon_1 \wedge \cdots \wedge \upsilon_r, P) \in SCS(N)$ where $\upsilon_f = [q^f; w_1^f, \ldots, w_n^f]$, $1 \leqslant f \leqslant r$ and $P = \{P_1, \ldots, P_m\}$. For each player $i \in P_j$, the generating function of $\{a_{k_1 \cdots k_r g l}^i\}_{k_1, \ldots, k_r \geqslant 0, g \geqslant 0, l \geqslant 0}$ is given by*

$$S_i(x_1, \ldots, x_r, t, z) = \prod_{\upsilon=1, \upsilon \neq j}^{m} \left(1 + x_1^{w^1(P_\upsilon)} \cdots x_r^{w^r(P_\upsilon)} t\right) \prod_{l=1, j_l \neq i}^{p_j} \left(1 + x_1^{w_{j_l}^1} \cdots x_r^{w_{j_l}^r} z\right),$$

where $w^f(P_\upsilon) = \sum_{j \in P_\upsilon} w_j^f$, $1 \leqslant f \leqslant r$ and $P_j = \{j_1, \ldots, j_{p_j}\}$.

Proof. Let us take a game $(N, v_1 \wedge \cdots \wedge v_r, P) \in SCS(N)$ where $v_f = [q^f; w_1^f, \ldots, w_n^f]$, $1 \leqslant f \leqslant r$ and $i \in P_j$, where $P_j = \{i_1, \ldots, i_{p_j}\}$. We expand the function

$$S(x_1,\ldots,x_r,t,z) = \prod_{v=1,v\neq j}^{m}(1+x_1^{w^1(P_v)}\cdots x_r^{w^r(P_v)}t)\prod_{l=1}^{p_j}(1+x_1^{w_{j_l}^1}\cdots x_r^{w_{j_l}^r}z) = 1+\sum_{S\in C(j,P)}\prod_{g\in m_j(S)}(x_1^{w^1(P_g)}\cdots x_r^{w^r(P_g)}t)\prod_{l\in S\cap P_j}x_1^{w_l^1}\cdots x_r^{w_l^r}z$$

$$= 1+\sum_{S\in C(j,P)}x_1^{\sum_{l\in S}w_l^1}\cdots x_r^{\sum_{l\in S}w_l^r}t^{|m_j(S)|}z^{|S\cap P_j|} = 1+\sum_{S\in C(j,P)}x_1^{w^1(S)}\cdots x_r^{w^r(S)}t^{|m_j(S)|}z^{|S\cap P_j|}.$$

We take $k_f = w^f(S)$ for all $1 \leqslant f \leqslant r$, then the previous function is equal to

$$\sum_{\substack{k_f=0\\1\leqslant f\leqslant r}}^{w^f(N)}\sum_{g=0}^{m-1}\sum_{l=0}^{p_j}a_{k_1\cdots k_r gl}x_1^{k_1}\cdots x_r^{k_r}t^g z^l,$$

where $a_{0\cdots0}=1$, and $a_{k_1\cdots k_r gl}$ is the number of compatible coalitions $S \in C(j,P)$ such that $w^f(S) = k_f$, for all $1 \leqslant f \leqslant r$, $m_j(S) = g$, and $|S \cap P_j| = l$, where at least one k_f with $f \in \{1,2,\ldots,r\}$, g or l is greater than 0. To obtain $\{a^i_{k_1\cdots k_r gl}\}_{k_1,\ldots,k_r\geqslant0,g\geqslant0,l\geqslant0}$, it is sufficient to drop the factor $(1+x_1^{w_i^1}\cdots x_r^{w_i^r}z)$ in the polynomial $S(x_1,\ldots,x_r,t,z)$. □

The previous result gives a method to compute $\{a^i_{k_1\cdots k_r gl}\}_{k_1,\ldots,k_r\geqslant0,g\geqslant0,l\geqslant0}$. Now, it is necessary to obtain the values d^i_{gl}. These values can be identified by the coefficients of

$$g_i(t,z) = \sum_{g=0}^{m-1}\sum_{l=0}^{p_j-1}d^i_{gl}t^g z^l,$$

and, taking into account that

$$d^i_{gl} = \sum_{\substack{k_f=q^f-w_i^f\\1\leqslant f\leqslant r}}^{w^f(N\setminus i)}a^i_{k_1\cdots k_r gl} - \sum_{\substack{k_f=q^f\\1\leqslant f\leqslant r}}^{w^f(N\setminus i)}a^i_{k_1\cdots k_r gl},$$

it holds that

$$g_i(t,z) = \sum_{g=0}^{m-1}\sum_{l=0}^{p_j-1}d^i_{gl}t^g z^l = \sum_{g=0}^{m-1}\sum_{l=0}^{p_j-1}\left[\sum_{\substack{k_f=q^f-w_i^f\\1\leqslant f\leqslant r}}^{w^f(N\setminus i)}a^i_{k_1\cdots k_r l} - \sum_{\substack{k_f=q^f\\1\leqslant f\leqslant r}}^{w^f(N\setminus i)}a^i_{k_1\cdots k_r l}\right]t^g z^l.$$

By using Theorem 3 we have that

$$S_i(x_1,\ldots,x_r,t,z) = \sum_{\substack{k_f=q^f-w_i^f\\1\leqslant f\leqslant r}}^{w^f(N\setminus i)}\left[\sum_{g=0}^{m-1}\sum_{l=0}^{p_j-1}a^i_{k_1\cdots k_r l}t^g z^l\right]x_1^{k_1}\cdots x_r^{k_r}.$$

We determine a polynomial in z and t whose coefficients represent the number of winning coalitions that contain player i

$$s_i^1(t,z) = \sum_{\substack{k_f=q^f-w_i^f\\1\leqslant f\leqslant r}}^{w^f(N\setminus i)}\sum_{g=0}^{m-1}\sum_{l=0}^{p_j-1}a^i_{k_1\cdots k_r l}t^g z^l.$$

On the other hand, we consider a polynomial in z and t whose coefficients represent the number of winning coalitions that contain player i but his/her presence is not necessary to win

$$s_i^2(t,z) = \sum_{\substack{k_f=q^f\\1\leqslant f\leqslant r}}^{w^f(N\setminus i)}\sum_{g=0}^{m-1}\sum_{l=0}^{p_j-1}a^i_{k_1\cdots k_r l}t^g z^l.$$

The coefficients of $s_i^1(t,z)$ can be computed, selecting for any exponent of variables z and t in $S_i(x_1,\ldots,x_r,t,z)$, the coefficients of those terms $x_1^{k_1}\cdots x_r^{k_r}t^g z^l$ such that k_f takes values greater or equal than $q^f - w_i^f$, for $1 \leqslant f \leqslant r$. The coefficients of $s_i^2(t,z)$ can be computed, selecting for any exponent of variables z and t in $S_i(x_1,\ldots,x_r,t,z)$, the coefficients of those terms $x_1^{k_1}\cdots x_r^{k_r}t^g z^l$ such that k_f takes values greater or equal than q^f, for $1 \leqslant f \leqslant r$. Finally

$$g_i(t,z) = \sum_{g=0}^{m-1}\sum_{l=0}^{p_j-1}d^i_{gl}t^g z^l = s_i^1(t,z) - s_i^2(t,z).$$

Example 6. Let us consider again the triple majority game given in Example 3 and we compute the Owen index of player 4. We take the function

$$S_4(x_1,x_2,x_3,t,z) = (1+x_1^{28}x_2^{40}x_3t)(1+x_1^{8}x_2^{22}x_3^{2}t)(1+x_1^{16}x_2^{20}x_3z)(1+x_1^{3}x_2^{8}x_3z)$$
$$= 1+x_1^{28}x_2^{40}x_3t+x_1^{8}x_2^{22}x_3^{2}t+x_1^{36}x_2^{62}x_3^{3}t^2+x_1^{16}x_2^{20}x_3z+x_1^{44}x_2^{60}x_3^{2}zt+x_1^{24}x_2^{42}x_3^{3}zt+x_1^{52}x_2^{82}x_3^{4}zt^2+x_1^{3}x_2^{8}x_3z+x_1^{31}x_2^{48}x_3^{2}zt$$
$$+x_1^{11}x_2^{30}x_3^{3}zt+x_1^{39}x_2^{70}x_3^{4}zt^2+x_1^{19}x_2^{28}x_3^{2}z^2+x_1^{47}x_2^{68}x_3^{3}z^2t+x_1^{27}x_2^{50}x_3^{4}z^2t+x_1^{55}x_2^{90}x_3^{5}z^2t^2.$$

We choose the coefficients of terms $x_1^{k_1}x_2^{k_2}x_3^{k_3}t^gz^l$ such that k_1 takes values between 26 and 55, k_2 takes values between 41 and 90, and k_3 takes values between 3 and 5. In this case, there are 6 terms:

$$x_1^{36}x_2^{62}x_3^{3}t^2+x_1^{52}x_2^{82}x_3^{4}zt^2+x_1^{39}x_2^{70}x_3^{4}zt^2+x_1^{47}x_2^{68}x_3^{3}z^2t+x_1^{27}x_2^{50}x_3^{4}z^2t+x_1^{55}x_2^{90}x_3^{5}z^2t^2.$$

Then we obtain that $s_4^1(t,z)=t^2+2zt^2+2z^2t+z^2t^2$. We must eliminate those coalitions that remain winning without player 4. That is, those corresponding with terms $x_1^{k_1}x_2^{k_2}x_3^{k_3}t^gz^l$ such that k_1 takes values between 30 and 55, k_2 takes values between 51 and 90, and k_3 takes values between 4 and 5. In this case, there are 3 terms:

$$x_1^{52}x_2^{82}x_3^{4}zt^2+x_1^{39}x_2^{70}x_3^{4}zt^2+x_1^{55}x_2^{90}x_3^{5}z^2t^2.$$

Then we obtain that $s_4^2(t,z)=2zt^2+z^2t^2$ and $g_4(t,z)=s_4^1(t,z)-s_4^2(t,z)=t^2+2z^2t$. The Owen index of player 4 is

$$\Phi_4(N,\upsilon_1\wedge\upsilon_2\wedge\upsilon_3,P)=\sum_{r=0}^{m-1}\sum_{l=0}^{p_j-1}\frac{r!(m-r-1)!}{m!}\frac{l!(p_j-l-1)!}{p_j!}d_{rl}^i=\frac{2!0!}{3!}\frac{0!2!}{3!}+\frac{1!1!}{3!}\frac{2!0!}{3!}2=\frac{2}{9}.$$

In this case, the coefficients d_{rl}^i give rise of the expression

$$g_i(t,z)=\sum_{r=0}^{m-1}\sum_{l=0}^{p_j-1}d_{rl}^it^rz^l.$$

4. An application to the European Union

The Council of Ministers of the EU represents the national governments of the member states. The Council uses a voting system of qualified majority to pass new legislation. The Nice European Council in December 2000 established the decision rule for the EU enlarged to 27 countries. This rule is contained in the *Declaration on the enlargement of the European Union* and the *Declaration on the qualified majority threshold and the number of votes for a blocking minority in an enlarged Union* (Official Journal of the European Communities 10.3.2001, C 80/80-85).

Felsenthal and Machover (2001, 2004) analyzed in terms of a *priori* measures of power the decision rules for the Council of Ministers of the European Union, under the hypothesis that the behavior of the representatives in the Council is independent and there are not commitments to cooperate. In this section we apply the procedures presented in previous sections to the particular case of the distribution of power in the European Union according to a coalition structure. Each member state represented in the European Council is considered an individual player.

The players in the European Council and the population weights are showed in Table 1. These weights are the rate per ten thousands of the population of each country with respect to the total population of the EU. The data of population used to calculate the mentioned weights are those provided by the Office of the Census of EUROSTAT corresponding to January 2008.

The voting rule prescribed by the Treaty of Nice is the weighted triple majority game $\upsilon_1\wedge\upsilon_2\wedge\upsilon_3$, where the three weighted voting games corresponding to votes, countries and population, are the following:

$$\upsilon_1=[255;29,29,29,29,27,27,14,13,12,12,12,12,12,10,10,10,7,7,7,7,7,4,4,4,4,4,3],$$

$$\upsilon_2=[14;1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1],$$

$$\upsilon_3=[6200;1652,1281,1231,1197,910,766,433,330,225,214,214,209,202,185,167,154,110,109,106,89,68,46,41,27,16,10,8].$$

The game υ_3 is defined assigning to every country, a number of votes equal to the rate per ten thousand of its population over the total population and the quota represents the 62% of the total population. So, a voting will be favorable if it counts on the support of 14 countries with at least 255 votes, and with at least the 62% of the population.

The new voting rule proposed by the European Council in June 2007 changes in a very remarkable way the power of the countries in the Council. The reason is that the weighted voting game υ_3, that were approved in Nice is removed and a coalition only needs 15 votes, which at least sum up by 65% of the population to approve a decision with the new rule. Furthermore, the minimum number of countries to block a proposal is four and the abstentions are not counted. Therefore, the new voting rule proposed by the Council of the European Union is the game $(\upsilon_2'\wedge\upsilon_3')\vee bc$, where

$$\upsilon_2'=[15;1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1],$$

$$\upsilon_3'=[6500;1652,1281,1231,1197,910,766,433,330,225,214,214,209,202,185,167,154,110,109,106,89,68,46,41,27,16,10,8],$$

Table 1
Population in January 2008 and population weights for the 27 EU members

Countries	Population	Weights
Germany	82,221,808	1652
France	63,753,140	1281
United Kingdom	61,270,283	1231
Italy	59,578,359	1197
Spain	45,283,259	910
Poland	38,115,641	766
Romania	21,528,627	433
Netherlands	16,404,282	330
Greece	11,214,992	225
Belgium	10,660,770	214
Portugal	10,633,006	214
Czech Republic	10,381,130	209
Hungary	10,045,000	202
Sweden	9,182,927	185
Austria	8,331,930	167
Bulgaria	7,640,238	154
Denmark	5,475,791	110
Slovakia	5,400,998	109
Finland	5,296,826	106
Ireland	4,419,859	89
Lithuania	3,366,357	68
Latvia	2,270,894	46
Slovenia	2,025,866	41
Estonia	1,340,935	27
Cyprus	794,580	16
Luxembourg	483,800	10
Malta	410,584	8

$$bc = [24; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1].$$

In the next lemma, we compute the number of compatible swings for player $i \in P_j$ in the Council of the European Union game with a coalition structure $(N, (v'_2 \wedge v'_3) \vee bc, P)$. We omit the proof which may be obtained by using similar arguments to those found in Algaba et al. (2007).

Lemma 2. *Let $(N, (v'_2 \wedge v'_3) \vee bc, P)$ be a weighted double majority game with blocking with a coalition structure, given by $v'_2 = [p; 1, \ldots, 1]$, $v'_3 = [q; w_1, \ldots, w_n]$, $bc = [|N| - b; 1, \ldots, 1]$ with $p < |N| - b$ and $P = \{P_1, \ldots, P_m\}$. The number of compatible swings for a player $i \in P_j$ is*

$$\eta_i(N, (v'_2 \wedge v'_3) \vee bc, P) = \sum_{k=q-w_i}^{q-1} \sum_{r=p}^{|N|-b-2} b^i_{kr} + \sum_{k=q-w_i}^{w(N\setminus i)} b^i_{k,p-1} + \sum_{k=1}^{q-1} b^i_{k,|N|-b-1},$$

where b^i_{kr} is the number of coalitions $S \in C(j, P)$ such that $i \notin S$ with $w(S) = k$ and $|S| = r$.

We will apply the procedures presented in the previous section to two different coalition structures, the first one, based on the population size of the countries, and the second one, based on cultural and religious cleavages. As a referee has pointed us, in the second case, different structures to that we elect, could have sense, too. With the mentioned tools we present in this paper, the interested readers could calculate the coalitional power indices with others coalition structures. Hagemann and De Clerck-Sachsse (2004) obtained with an empirical analysis, the next observation about the coalition formation in the European Union:

" But it can be concluded that a consistent pattern can be observed in the distinction between large, medium and small members; the following will reveal whether this differentiation also holds after the enlargement".

Then we consider the following coalition structure based in the population size of the countries:

P_1={Germany},
P_2={France, United Kingdom, Italy},
P_3={Spain, Poland},
P_4={Romania, Netherlands, Greece, Belgium, Portugal, Czech R., Hungary},
P_5={Sweden, Austria, Bulgaria, Denmark, Slovakia, Finland, Ireland, Lithuania},
P_6={Latvia, Slovenia, Estonia, Cyprus, Luxembourg, Malta}.

In terms of the enumeration of the countries, the above coalition structure is given by

$$P = \{\{1\}, \{2, 3, 4\}, \{5, 6\}, \{7, 8, 9, 10, 11, 12, 13\}, \{14, 15, 16, 17, 18, 19, 20, 21\}, \{22, 23, 24, 25, 26, 27\}\}.$$

Huntington (1996) presented the main arguments of his book *The Clash of Civilizations and the Remaking of World Order* as follows:

" The argument in my book on the clash of civilization was well reflected in that short quote saying that the relations between countries in the coming decade are most likely to reflect their cultural commitments, their cultural ties and antagonism with other countries... So the question really is what will be the central focus of global politics in the coming decades and my argument is that cultural identities and cultural antagonisms and affiliations will play not the only role but a major role. Countries will cooperate with each other, and are more likely to cooperate with each other when they share a common culture, as is most dramatically illustrated in the European Union".

According to the Huntington's model (see Chapter 7, section 'Bounding the West'), we also consider another coalition structure based on cultural and religious cleavages:

P'_1={Germany, France},
P'_2={United Kingdom},
P'_3={Netherlands, Belgium, Luxembourg},
P'_4={Italy, Spain, Portugal, Ireland, Slovenia, Malta},
P'_5={Sweden, Denmark, Finland, Lithuania, Latvia, Estonia},
P'_6={Poland, Czech R., Hungary, Austria, Slovakia},
P'_7={Romania, Greece, Bulgaria, Cyprus}.

In terms of the enumeration of the countries, the above coalition structure is given by

$$P' = \{\{1,2\},\{3\},\{8,10,26\},\{4,5,11,20,23,27\},\{14,17,19,21,22,24\},\{6,12,13,15,18\},\{7,9,16,25\}\}.$$

We apply Theorems 1–3 and Lemma 2, to compute coalitional power indices for the games $(N, v_1 \wedge v_2 \wedge v_3, P)$ and $(N, v_1 \wedge v_2 \wedge v_3, P')$ based on the Nice rule, and also for the games $(N, (v'_2 \wedge v'_3) \vee bc, P)$ and $(N, (v'_2 \wedge v'_3) \vee bc, P')$, based on the European Council rule, where P and P' are the two coalition structures considered. Next, we present three tables which contain the Banzhaf–Owen, Symmetric Coalitional Banzhaf and Owen indices of the 27 countries of the European Union. Since all the weights are integer numbers the power indices given by *Mathematica* are exact rational numbers. With the objective to compare the collected data, we will approximate these rational numbers by *rounding* to present four correct digits to the right of the decimal point.

Note that the Banzhaf–Owen indices are the swing probabilities of the countries and the sum by columns, in general, is not equal to 1. If we need a comparison between the voting rules and partitions then we can use their normalized values rather their magnitudes. With this purpose, we normalize the columns Council P and Council P' to add up to 1. The normalized data are presented in columns NC P and NC P', respectively.

The first conclusion that is obtained from these data is the monotonicity of the power indices presented in the first column. The Nice rule with the coalition structure P, based in the size of the population, provides an increasing power in relation to the population. This property is not true with the coalition structure P', since the United Kingdom becomes the most powerful country, Portugal loses a lot of power and Luxembourg becomes a dummy player.

The Council rule is characterized to use weights given by the percentage of the population that have a very high dispersion. Then the oscillations of the power when the coalition structures are introduced have a very high value. For example, the data in columns NC P and NC P' imply that Germany increases its power in the Huntington's model with a percentage greater than the double, the United Kingdom and Italy lose power, whereas Poland increases it and Spain remains stable (see Table 2).

Since the Symmetric Coalitional Banzhaf indices are very similar to the Banzhaf–Owen indices we omit its normalization values in Table 3. As a result of this similarity, all the commentaries made on the indices of Banzhaf–Owen can be applied to these indices (see Table 4).

Since the Owen index enjoys the efficiency property, the sum of all the columns is equal to 1. Then it is possible to have a comparison between the classical Shapley–Shubik power indices and the coalitional Owen indices with the coalition structures P and P', for the two games given by the Nice and the Council voting rules respectively. In the columns called Nice and Council are included the Shapley–Shubik

Table 2
The Banzhaf–Owen indices for the 27 EU members

Countries	Nice P	Nice P'	Council P	NC P	Council P'	NCP'
Germany	0.1250	0.0859	0.0625	0.0530	0.1953	0.1332
France	0.0859	0.0859	0.1250	0.1060	0.1484	0.1012
United Kingdom	0.0859	0.1094	0.1250	0.1060	0.1250	0.0852
Italy	0.0859	0.0835	0.1250	0.1060	0.1206	0.0822
Spain	0.0625	0.0796	0.0781	0.0663	0.0981	0.0669
Poland	0.0625	0.0791	0.0469	0.0398	0.0791	0.0539
Romania	0.0522	0.0430	0.0415	0.0352	0.0625	0.0426
Netherlands	0.0415	0.0391	0.0376	0.0319	0.0469	0.0320
Greece	0.0317	0.0312	0.0366	0.0311	0.0391	0.0266
Belgium	0.0317	0.0391	0.0356	0.0302	0.0469	0.0320
Portugal	0.0317	0.0190	0.0356	0.0302	0.0366	0.0250
Czech Republic	0.0317	0.0361	0.0347	0.0294	0.0381	0.0260
Hungary	0.0317	0.0361	0.0337	0.0286	0.0381	0.0260
Sweden	0.0242	0.0273	0.0234	0.0199	0.0327	0.0223
Austria	0.0242	0.0283	0.0229	0.0195	0.0361	0.0246
Bulgaria	0.0242	0.0312	0.0229	0.0195	0.0391	0.0266
Denmark	0.0173	0.0215	0.0229	0.0195	0.0259	0.0176
Slovakia	0.0173	0.0146	0.0229	0.0195	0.0322	0.0220
Finland	0.0173	0.0215	0.0229	0.0195	0.0259	0.0176
Ireland	0.0173	0.0161	0.0225	0.0191	0.0298	0.0203
Lithuania	0.0173	0.0215	0.0225	0.0191	0.0239	0.0163
Latvia	0.0127	0.0117	0.0312	0.0265	0.0220	0.0150
Slovenia	0.0127	0.0103	0.0312	0.0265	0.0269	0.0183
Estonia	0.0127	0.0117	0.0312	0.0265	0.0210	0.0143
Cyprus	0.0127	0.0078	0.0293	0.0249	0.0234	0.0160
Luxembourg	0.0127	0.0000	0.0273	0.0232	0.0312	0.0213
Malta	0.0010	0.0044	0.0273	0.0232	0.0220	0.0150
Sum	0.9835	0.9949	1.1782	1.0001	1.4668	1.0000

indices of these games. The columns labeled with P and P', corresponding to Owen indices for the mentioned games with the respective coalition structure.

With respect to coalition structure P, the countries in the blocks P_2, P_4, and P_6 win power by the new Council rule with the exception of Romania. On the other hand, Germany, Poland and all the countries belonging to P_5 lose power with respect to the Nice rule. Moreover, the coalitional power indices are very sensitive with respect to the Council and Nice voting rules. For example, Germany loses half of its power in all the indices considered in the Council voting game with coalition structure P. This sensitivity is greater when the player belongs to a coalition of size 1.

The Huntington's model of coalition structure P' implies several consequences. The most important is that Germany and France gain power compared to Nice and the block P'_1 becomes the central coalition in the European Union. Otherwise, the isolated position of the

Table 3
The symmetric coalitional Banzhaf indices for the 27 EU members

Countries	Nice P	Nice P'	Council P	Council P'
Germany	0.1250	0.0859	0.0625	0.1953
France	0.0833	0.0859	0.1250	0.1484
United Kingdom	0.0833	0.1094	0.1250	0.1250
Italy	0.0833	0.0820	0.1250	0.1211
Spain	0.0625	0.0789	0.0781	0.1008
Poland	0.0625	0.0815	0.0469	0.0750
Romania	0.0528	0.0391	0.0473	0.0677
Netherlands	0.0372	0.0391	0.0452	0.0469
Greece	0.0320	0.0312	0.0447	0.0443
Belgium	0.0320	0.0391	0.0442	0.0469
Portugal	0.0320	0.0174	0.0442	0.0388
Czech Republic	0.0320	0.0372	0.0437	0.0372
Hungary	0.0320	0.0372	0.0432	0.0372
Sweden	0.0363	0.0271	0.0353	0.0310
Austria	0.0363	0.0294	0.0309	0.0359
Bulgaria	0.0363	0.0312	0.0309	0.0443
Denmark	0.0282	0.0208	0.0309	0.0266
Slovakia	0.0282	0.0177	0.0309	0.0333
Finland	0.0282	0.0208	0.0309	0.0266
Ireland	0.0282	0.0128	0.0301	0.0318
Lithuania	0.0282	0.0208	0.0301	0.0250
Latvia	0.0240	0.0099	0.0349	0.0240
Slovenia	0.0240	0.0091	0.0349	0.0297
Estonia	0.0240	0.0099	0.0349	0.0232
Cyprus	0.0240	0.0078	0.0286	0.0312
Luxembourg	0.0240	0.0000	0.0271	0.0312
Malta	0.0052	0.0029	0.0271	0.0216
Sum	1.1250	0.9841	1.3125	1.5000

Table 4
The Owen indices for the 27 EU members

Countries	Nice	Nice P	Nice P'	Council	Council P	Council P'
Germany	0.0874	0.1167	0.0881	0.1560	0.0500	0.1333
France	0.0872	0.0722	0.0881	0.1138	0.0944	0.1048
United Kingdom	0.0870	0.0722	0.1095	0.1087	0.0944	0.0881
Italy	0.0869	0.0722	0.0871	0.1053	0.0944	0.0861
Spain	0.0802	0.0583	0.0831	0.0780	0.0583	0.0747
Poland	0.0799	0.0583	0.0787	0.0663	0.0417	0.0594
Romania	0.0398	0.0460	0.0393	0.0412	0.0348	0.0480
Netherlands	0.0367	0.0321	0.0381	0.0325	0.0337	0.0270
Greece	0.0340	0.0277	0.0321	0.0241	0.0334	0.0278
Belgium	0.0340	0.0277	0.0381	0.0233	0.0331	0.0270
Portugal	0.0340	0.0277	0.0152	0.0233	0.0331	0.0279
Czech Republic	0.0340	0.0277	0.0387	0.0229	0.0328	0.0249
Hungary	0.0340	0.0277	0.0387	0.0223	0.0325	0.0249
Sweden	0.0281	0.0302	0.0294	0.0210	0.0251	0.0172
Austria	0.0281	0.0302	0.0327	0.0196	0.0227	0.0241
Bulgaria	0.0281	0.0302	0.0321	0.0186	0.0227	0.0278
Denmark	0.0195	0.0252	0.0208	0.0152	0.0227	0.0150
Slovakia	0.0195	0.0252	0.0208	0.0151	0.0227	0.0215
Finland	0.0195	0.0252	0.0208	0.0149	0.0227	0.0150
Ireland	0.0195	0.0252	0.0123	0.0136	0.0223	0.0189
Lithuania	0.0195	0.0252	0.0208	0.0120	0.0223	0.0141
Latvia	0.0110	0.0222	0.0089	0.0103	0.0269	0.0137
Slovenia	0.0110	0.0222	0.0084	0.0100	0.0269	0.0177
Estonia	0.0110	0.0222	0.0089	0.0089	0.0269	0.0132
Cyprus	0.0110	0.0222	0.0060	0.0081	0.0236	0.0179
Luxembourg	0.0110	0.0222	0.0000	0.0077	0.0228	0.0175
Malta	0.0082	0.0056	0.0034	0.0075	0.0228	0.0128

United Kingdom loses power under the Council rule. The dummy situation of Luxembourg with respect to the Banzhaf–Owen, Symmetric Coalitional Banzhaf and Owen indices should disappear with the new voting rule. The power of Spain and Poland decreases and this fact explains their opposition to the Council rule. Also decreases the power of the medium-sized countries with the exceptions of Romania, Portugal, Slovakia, and Ireland. Finally, all the small countries gain power under the Council rule.

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Appendix. Mathematica Code

The main reason for using *Mathematica* as programming language is the larger data sets of polynomial coefficients that we use in the generating functions method. *Mathematica* collects these data sets in lists and implements a large set of effective commands for manipulating lists. In general, list operations and data storage are useful and fast in *Mathematica* when dealing with large amounts of data. The Code is written for *Mathematica 5.2* and can be implemented under later versions. The functions BO3Power, SCB3Power and OW3Power allows to get the corresponding coalitional power indices for weighted triple majority games.

The Mathematica Code for the Banzhaf–Owen index is given by the following procedure. First, we obtain the function BO3FG which is the generating function of the numbers $\{b^i_{k_1\cdots k_r}\}_{k_1,\ldots,k_r\geqslant 0}$ for each player *i*. Next, we compute the Banzhaf–Owen index of player *i* in a weighted triple majority game by using the function BO3Index. Finally, BO3Power presents these indices in table form. Note that the input data requires to enter three sets of integer weights, a partition, and three integer quotas.

```
BO3FG[weightsA_List, weightsB_List, weightsC_List, P_List, i_Integer]:=
Module[{m,Pj,PartC,PlayerPjC,weightsAPartC,weightsAPlayerPjC,
weightsBPartC,weightsBPlayerPjC,weightsCPartC,weightsCPlayerPjC},
m = Length[weightsA]; Pj = Select[P, MemberQ[#1, i] &];
PartC = Complement[P, Pj];
weightsAPartC = (Plus@@ #1 &)/@ (PartC/.Table[j-> weightsA[[j]],{j,m}]);
weightsBPartC = (Plus@@ #1 &)/@ (PartC/.Table[j-> weightsB[[j]],{j,m}]);
weightsCPartC = (Plus@@ #1 &)/@ (PartC/.Table[j-> weightsC[[j]],{j,m}]);
PlayerPjC = Complement[Flatten[Pj, 1], {i}];
weightsAPlayerPjC = PlayerPjC/. Table[j -> weightsA[[j]], {j, m}];
weightsBPlayerPjC = PlayerPjC/. Table[j -> weightsB[[j]], {j, m}];
weightsCPlayerPjC = PlayerPjC/. Table[j -> weightsC[[j]], {j, m}];
Times@@(1 + x̂weightsAPartC*ŷweightsBPartC*z^weightsCPartC)*
Times@@(1 + x̂weightsAPlayerPjC*ŷ weightsBPlayerPjC*
ẑweightsCPlayerPjC)]

BO3Index[weightsA_List, weightsB_List, weightsC_List, P_List, i_Integer,
qA_Integer, qB_Integer, qC_Integer]:=
Module[{poly, coefi, swA, swB, swC, s1, s2},
poly = BO3FG[weightsA, weightsB, weightsC, P, i];
coefi = CoefficientList[poly, {x, y, z}]/. Table -> {};
swA = Length[coefi]; swB = Length[coefi[[1]]];
swC = Length[coefi[[1]][[1]]];
s1 = Plus@@ Flatten[coefi[[Range[Max[1,qA-weightsA[[i]] + 1],swA],
Range[Max[1,qB-weightsB[[i]] + 1],swB],
Range[Max[1,qC-weightsC[[i]] + 1],swC]]]];
s2 = If[qA + 1 > swA || qB + 1 > swB || qC + 1 > swC, 0,
Plus@@ Flatten[coefi[[Range[qA + 1, swA],
Range[qB + 1,swB],Range[qC + 1,swC]]]]];
s1 − s2]

BO3Power[weightsA_List,weightsB_List,weightsC_List,P_List,
qA_,qB_,qC_]:=
Table[BO3Index[weightsA,weightsB,weightsC,P,i,qA,qB,qC]/
2̂(Length[P] + Length[Flatten[Select[P, MemberQ[#1,i]&]]]-2),
{i, Length[weightsA]}]
```

The Mathematica Code for the Symmetric Coalitional Banzhaf index is

```
SCB3FG[weightsA_List,weightsB_List,weightsC_List,P_List,i _Integer]:=
Module[{m, Pj,PartC,PlayerPjC,weightsAPartC,weightsAPlayerPjC,
```

```
weightsBPartC,weightsBPlayerPjC,weightsCPartC,weightsCPlayerPjC},
m = Length[weightsA]; Pj = Select[P,MemberQ[#1,i]&];
PartC = Complement[P,Pj];
weightsAPartC = (Plus@@ #1 &)/@(PartC/.Table[j-> weightsA[[j]],{j,m}]);
weightsBPartC = (Plus@@ #1 &)/@(PartC/.Table[j-> weightsB[[j]],{j, m}]);
weightsCPartC = (Plus@@ #1 &)/@(PartC/.Table[j-> weightsC[[j]],{j, m}]);
PlayerPjC = Complement[Flatten[Pj,1],{i}];
weightsAPlayerPjC = PlayerPjC/.Table[j-> weightsA[[j]],{j,m}];
weightsBPlayerPjC = PlayerPjC/.Table[j-> weightsB[[j]],{j,m}];
weightsCPlayerPjC = PlayerPjC/.Table[j-> weightsC[[j]],{j, m}];
Times@@(1 + x̂weightsAPartC*ŷweightsBPartC*tˆweightsCPartC)*
Times@@(1 + x̂weightsAPlayerPjC*ŷ weightsBPlayerPjC*
t̂weightsCPlayerPjC*z)]

SCB3Power[weightsA_List,weightsB_List,weightsC_List,P_List,
q_Integer,p_Integer,m_Integer]:=
Module[{n = Length[weightsA],poly,sw,sp,sm,s1,s2,g,coefi},
Table[poly = SCB3FG[weightsA,weightsB,weightsC,P,i];
coefi = CoefficientList[poly,{x,y,t}]/.Table->{};
sw = Length[coefi]; sp = Length[coefi[[1]]]; sm = Length[coefi[[1]][[1]]];
s1 = Plus@@Flatten[coefi[[Range[Max[1,q-weightsA[[i]] + 1],sw],
Range[Max[1,p-weightsB[[i]] + 1],sp],
Range[Max[1,m-weightsC[[i]] + 1],sm]]]];
s2 = If[q + 1 > sw || p + 1 > sp || m + 1 > sm, 0,
Plus@@Flatten[coefi[[Range[q + 1,sw],Range[p + 1,sp],Range[m + 1,sm]]]]];
g = s1-s2; pj = Length[Flatten[Select[P, MemberQ[#1,i]&]]];
Sum[Coefficient[g,z,s]*((s!*(pj-s-1)!)/(pj!*2̂ (Length[P]-1))),
{s,0,pj-1}], {i,n}]]
```

The Mathematica Code for the Owen index is

```
OW3FG[weightsA_List,weightsB_List,weightsC_List,P_List,i _Integer]:=
Module[{m,Pj,PartC,PlayerPjC,weightsAPartC,weightsAPlayerPjC,
weightsBPartC,weightsBPlayerPjC,weightsCPartC,weightsCPlayerPjC},
m = Length[weightsA]; Pj = Select[P,MemberQ[#1,i]&];
PartC = Complement[P,Pj];
weightsAPartC = (Plus @@ #1&)/@ (PartC/.Table[j-> weightsA[[j]],{j,m}]);
weightsBPartC = (Plus @@ #1&)/@ (PartC/.Table[j-> weightsB[[j]],{j,m}]);
weightsCPartC = (Plus @@ #1&)/@ (PartC/.Table[j-> weightsC[[j]],{j,m}]);
PlayerPjC = Complement[Flatten[Pj, 1], {i}];
weightsAPlayerPjC = PlayerPjC/.Table[j-> weightsA[[j]],{j,m}];
weightsBPlayerPjC = PlayerPjC/.Table[j-> weightsB[[j]],{j,m}];
weightsCPlayerPjC = PlayerPjC/.Table[j-> weightsC[[j]],{j,m}];
Times @@(1 + x1̂weightsAPartC*x2̂weightsBPartC*x3ˆ weightsCPartC*t)*
Times @@(1 + x1̂weightsAPlayerPjC*x2̂ weightsBPlayerPjC*
x3̂weightsCPlayerPjC*z)]

OW3Power[weightsA_List,weightsB_List,weightsC_List,
P_List,qA_Integer,qB_Integer,qC_Integer]:=
Module[{n = Length[weightsA],m = Length[P],poly,sw,sp,sm,s1,s2,g,coefi},
Table[poly = OwenTripleFG[weightsA, weightsB, weightsC, P, i];
coefi = CoefficientList[poly, {x1,x2,x3}]/. Table -> {};
sw = Length[coefi]; sp = Length[coefi[[1]]]; sm = Length[coefi[[1]][[1]]];
s1 = Plus @@ Flatten[coefi[[Range[Max[1, qA - weightsA[[i]] + 1], sw],
Range[Max[1,qB-weightsB[[i]] + 1],sp],Range[Max[1,qC-weightsC[[i]] + 1], sm]]]];
s2 = If[qA + 1 > sw || qB + 1 > sp || qC + 1 > sm, 0,
Plus @@ Flatten[coefi[[Range[qA + 1,sw],Range[qB + 1,sp],Range[qC + 1,sm]]]]];
g = s1-s2; pj = Length[Flatten[Select[P, MemberQ[#1, i] &]]];
Sum[Coefficient[Sum[Coefficient[g,z,s]*((s!*(pj-s-1)!)/pj!), {s,0,pj-1}],t,r]*
(r!*((m-r-1)!/m!)),{r,0,m-1}],{i,n}]]
```

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