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Games on fuzzy communication structures with Choquet players

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ABSTRACT

Myerson (1977) used graph-theoretic ideas to analyze cooperation structures in games. In his model, he considered the players in a cooperative game as vertices of a graph, which undirected edges defined their communication possibilities. He modified the initial games taking into account the graph and he established a fair allocation rule based on applying the Shapley value to the modified game. Now, we consider a fuzzy graph to introduce leveled communications. In this paper players play in a particular cooperative way: they are always interested first in the biggest feasible coalition and second in the greatest level (Choquet players). We propose a modified game for this situation and a rule of the Myerson kind.

1. Preliminary

A *transferable utility cooperative game* is a pair (N, v) where N is a finite set and $v : 2^N \to \mathbb{R}$ is a function with $v(\emptyset) = 0$. Cooperative games was introduced by von Neumann and Morgensten (1944). The elements of $N = \{1, 2, \ldots, n\}$ are called *players*, the subsets $S \subseteq N$ *coalitions* and $v(S)$ is the *worth* (benefits) of S. A game (N, v) is *superadditive* iff for all $S, T \subset N$ with $S \cap T = \emptyset$ happen $v(S \cup T) \geqslant v(S) + v(T)$. In a superadditive game disjoint groups of players always prefer to join in a bigger coalition. We are only interested in superadditive games, so when we will mention a game we will want to say superadditive game. For a game v, assuming that the big coalition N will be formed, a *solution* concept will prescribe how distribute the profit $v(N)$ among the players. Hence, a solution is an efficient vector $x \in \mathbb{R}^N$ for (N, v), this is $\sum_{i \in N} x_i = v(N)$. There exist in the literature many solution concepts: some of them give a set of payoff vectors and others produce only one outcome. A *value* is a solution concept that assigns to each game just one payment for each player. One of the most important values is the *Shapley value*, introduced by Shapley (1953) as

$$\phi(v) = \sum_{\{S \subseteq N : i \in S\}} \frac{(n - |S|)!(|S| - 1)!}{n!} [v(S) - v(S \setminus \{i\})] \tag{1}$$

for each cooperative game (N, v). The Shapley value verifies the following properties: $\phi_i(a_1 v_1 + a_2 v_2) = a_1 \phi_i(v_1) + a_2 \phi_i(v_2)$ for all $a_1, a_2 \in \mathbb{R}$ and games v_1, v_2 (*Lineality*), if $i \in N$ satisfies $v(S) = v(S \setminus \{i\})$ for all S with $i \in S$ then $\phi_i(v) = 0$ (*Null player*), and $\sum_{i \in N} \phi_i(v) = v(N)$ (*Efficiency*). The reader can use Driessen (1988) to get more information about cooperative games.

Partial cooperation in games studies cases between universal cooperation of the players (all players will cooperate with each other) and no cooperation. In this area, Myerson (1977) established the communication possibilities of the players in a game by a graph. Players in the game are the vertices in the graph and a coalition of players is feasible if it is possible to connect all the vertices of the coalition in the graph. A (*crisp*) *graph* $g = (N, L)$ over the finite set N is defined by a set L of unordered pairs of different (without loops) members of N. The elements of N are named *vertices* and the elements of L are named *links* or edges. He identified the set of players in a cooperative game (N, v) as the set of vertices and therefore if $(i, j) \in L$ then he understood that there is a communication link between i and j. We denote by gN the complete graph, this is all the links among players of N are possible. The set of all the crisp graphs over N is denoted by G^N. Myerson identified G^N with the set of the (crisp) *communication structures* in N, and so gN represents the total cooperation. Let $g = (N, L) \in G^N$ be a graph. A *subgraph* $g' = (N', L')$ of g is other graph verifying that $N' \subseteq N$ and $L' \subseteq L$. A path in g is defined by a sequence of vertices $(i_k)_{k=1,\ldots,m}$ verifying that $(i_k, i_{k+1}) \in L$ is a different link in g for each $k = 1, \ldots, m-1$. Two players $i, j \in N$ are connected in g if it is possible to join both vertices by a path. When any pair of players are connected in g the graph g is called *connected*. The crisp communication structure g has a *cycle* iff there is

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a path $(i_k)_{k=1,\dots,m}$ with $m \geqslant 4$ verifying $i_1 = i_m$. If g does not have cycles then g is a *forest*. Particularly, a connected forest g is called *tree*. A *spanning tree* g' in a connected crisp communication structure g is a tree subgraph of g connecting all the vertices of g. The *connected components* of g are the maximal subgraphs of g which are connected. If $S \subseteq N$ is a coalition of players then we denote by g_S the subgraph of g using only the vertices in S and the links among them, and

$$S/g = \{T \subseteq S : g_T \text{ is a connected component of } g_S\}. \quad (2)$$

Coalitions in S/g are disjoints.

An *allocation rule* for the game (N, v) is any function $\Psi^v : G^N \to \mathbb{R}^N$ such that is efficient by components

$$\sum_{i\in S} \Psi_i^v(g) = v(S)$$

for all $g \in G^N$ and $S \in N/g$. Myerson (1977) indicated two desirable conditions for an allocation rule. The allocation rule Ψ^v is *stable* iff for all $g = (N,L)$ and for all $(i,j) \in L$ we have

$$\Psi_i^v(g) \geqslant \Psi_i^v(g_{-ij}) \text{ and } \Psi_j^v(g) \geqslant \Psi_j^v(g_{-ij}),$$

where g_{-ij} represents the subgraph $g_{-ij} = (N,L\backslash(i,j))$. The allocation rule Ψ^v is *fair* iff for all $g = (N,L)$ and for all $(i,j) \in L$ we have

$$\Psi_i^v(g) - \Psi_i^v(g_{-ij}) = \Psi_j^v(g) - \Psi_j^v(g_{-ij}).$$

Myerson defined a particular allocation rule modifying the superadditive game (N, v) for each crisp communication structure. For each graph $g \in G^N$ he takes a new game $(N, v/g)$ so that

$$v/g(S) = \sum_{T\in S/g} v(T) \quad (3)$$

for all $S \subseteq N$. The *Myerson value* satisfies

$$\mu^v(g) = \phi(v/g) \quad (4)$$

for every $g \in G^N$, and it is the unique allocation rule for v which is fair and stable. It is possible to extend this solution for the game v to graphs $g = (S,L)$ where $S \subset N$. In that case $\mu_i^v(g) = 0$ if $i \in N\backslash S$.

But, Myerson supposes that the communications are feasible or not feasible. In this paper we introduce a fuzzy graph to analyze the communication among the players. So, we can discuss which is the outcome of a game depending on the level of the communication among the players. The paper is organized as follows. In Section 2 the basic fuzzy graph structure background is briefly reviewed. We also introduce some new definitions which are necessary in the paper. In Section 3 we introduce fuzzy communication structures and the fuzzy Myerson value. Finally, in Section 4 we study two interesting cases: fuzzy graphs complete by links and forests.

2. Fuzzy graphs

We introduce now some concepts about fuzzy graphs for which the reader can see Mordeson and Nair (2000). In this paper we use the operators $\wedge$, $\vee$ as the minimum and the maximum respectively.

Let N be a finite set. A *fuzzy set* in N is an application τ: $N \to [0,1]$. We denote by $[0,1]^N$ the family of fuzzy sets in N. If $\tau \in [0,1]^N$ then a *fuzzy relation* over τ is $\rho \in [0,1]^{N\times N}$ such that $\rho(i,j) \leqslant \tau(i) \wedge \tau(j)$ for all $i, j \in N$. The fuzzy relation ρ over τ is *reflexive* if $\rho(i,i) = \tau(i)$ for each $i \in N$, and it is *symmetric* if $\rho(i,j) = \rho(j,i)$ for all $i, j \in N$. We denote by $[0,1]_0^{N\times N}$ the set of fuzzy relations which are reflexive and symmetric for some $\tau \in [0,1]^N$. If $\rho \in [0,1]_0^{N\times N}$ then $\rho(i,j) \leqslant \rho(i,i) \wedge \rho(j,j)$ for all $i, j \in N$. The set $[0,1]_0^{N\times N}$ has $\wedge$, $\vee$ as inner operators.

A (*undirect*) *fuzzy graph* is a terna (N,τ,ρ) with N a finite set, $\tau \in [0,1]^N$ and ρ a symmetric fuzzy relation over τ. We consider only fuzzy graphs where ρ is reflexive, and we can identify the family of these fuzzy graphs with $[0,1]_0^{N\times N}$ taking $\tau(i) = \rho(i,i)$. We denote by $\rho = 0$ the null fuzzy graph where $\rho(i,j) = 0$ for all $i, j \in N$.

Let $\rho \in [0,1]_0^{N\times N}$ be a fuzzy graph. Another fuzzy graph ρ' verifies that $\rho' \leqslant \rho$ iff $\rho'(i,j) \leqslant \rho(i,j)$ for all $i, j \in N$. If $S \subseteq N$ then $\rho_S \in [0,1]_0^{N\times N}$ is defined as $\rho_S(i,j) = \rho(i,j)$ if $i,j \in S$ and $\rho_S(i,j) = 0$ otherwise. The *support* of ρ is the set

$$\text{supp}(\rho) = \{(i,j) \in N \times N : \rho(i,j) \neq 0\}.$$

Particularly, the *set of vertices* of ρ is vert $(\rho) = \{i \in N: (i,i) \in \text{supp}(\rho)\}$ and the *set of links* is link $(\rho) = \{(i,j) \in \text{supp}(\rho): i \neq j\}$. The *crisp version* of ρ is the crisp graph $g^\rho = (\text{vert}(\rho), \text{link}(\rho))$. For each $t \in [0,1]$ the *crisp t-version* of ρ is a crisp graph $g_t^\rho = (N_t^\rho, L_t^\rho)$ satisfying

$$N_t^\rho = \{i \in N : \rho(i,i) \geqslant t\}, \quad L_t^\rho = \{(i,j) \in N \times N : i \neq j,\ \rho(i,j) \geqslant t\}.$$

Example 1. We consider the fuzzy graph over $N = \{1,2,3,4\}$ given by $\rho(i,i) = 1$ for all $i \in N$, $\rho(1,2) = 0.3$, $\rho(1,3) = 0.2$, $\rho(1,4) = 0.3$, $\rho(2,3) = 0.4$, $\rho(2,4) = 0.5$ and $\rho(i,j) = 0$ otherwise. Fig. 1 represents this fuzzy graph.

We put in increasing order the different non-null images of ρ, that is $0.2 < 0.3 < 0.4 < 0.5 < 1$, and so we make the different crisp versions in Fig. 2. For instance, if $t \in (0.3, 0.4)$ then $g_t^\rho = g_{0.4}^\rho$.

Every crisp graph $g = (S,L)$ is a fuzzy graph. The graph g is identified with $g \in [0,1]_0^{N\times N}$ such that $g(i,j) = 1$ iff $(i,j) \in L$ or $i = j \in S$, and $g(i,j) = 0$ otherwise.

A fuzzy graph $\rho \in [0,1]_0^{N\times N}$ is *connected* iff its crisp version g^ρ is connected. The *connection level* of ρ is the maximum $t \in [0,1]$ such that g_t^ρ is connected. If ρ is not connected then its connection level is zero. The fuzzy graph ρ is a *forest (tree)* iff g^ρ is a forest (tree). A *spanning tree* of a connected fuzzy graph ρ is a tree $\rho' \leqslant \rho$ such that $g^{\rho'}$ is a spanning tree of g^ρ and $\rho' = \rho$ in supp (ρ'). A connected fuzzy graph ρ' is a

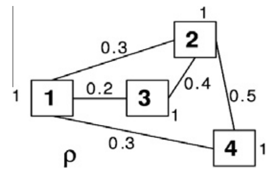

Fig. 1. Fuzzy graph (Example 1).

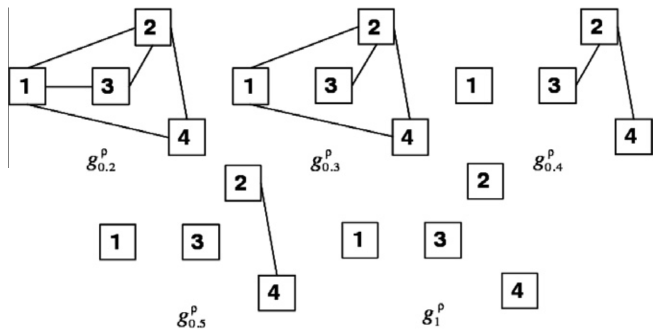

Fig. 2. Crisp versions.

connected component of ρ iff $\rho' \leqslant \rho$ and there is not any $\rho' < \rho'' \leqslant \rho$ such that ρ'' is connected. This is, the connected components are the maximum connected fuzzy graphs into ρ.

We need to define three operations among fuzzy graphs.

Definition 1. Let $\rho, \rho' \in [0,1]_0^{N\times N}$ be two fuzzy graphs over N.

(1) If $\rho(i,i) + \rho'(i,i) \leqslant 1$ for all $i \in N$ then the sum is a new fuzzy graph with $(\rho + \rho')(i,j) = \rho(i,j) + \rho'(i,j)$ for all $i, j \in N$.
(2) If $\rho' \leqslant \rho$ then the subtraction is a new fuzzy graph where for all i, $j \in N$, $(\rho - \rho')(i,j) = [\rho(i,j) - \rho'(i,j)] \wedge [\rho(i,i) - \rho'(i,i)] \wedge [\rho(j,j) - \rho'(j,j)]$.
(3) If $t \in [0,1]$ then there is a new fuzzy graph defined by $(t\rho)(i,j) = t\rho(i,j)$ for all $i, j \in N$.

The reader can observe that sum and subtraction of fuzzy graphs are not opposite operations because the special definition of the subtraction, but we obtain the next result.

Proposition 1. *If $\rho, \rho', \rho'' \in [0,1]_0^{N\times N}$ are three fuzzy graphs over N such that $\rho'' \leqslant \rho - \rho'$ and $\rho' \leqslant \rho$ then*

(1) $(\rho - \rho') - \rho'' = \rho - (\rho' + \rho'')$ and
(2) $(\rho - \rho') - \rho'' = (\rho - \rho'') - \rho'$.

Proof

(1) We take $i, j \in N$. Since $\rho'' \leqslant \rho - \rho'$ and $\rho' \leqslant \rho$ we get

$$\rho''(i,j) \leqslant (\rho - \rho')(i,j) \leqslant \rho(i,j) - \rho'(i,j),$$

thus $\rho'(i,j) + \rho''(i,j) \leqslant \rho(i,j)$. Therefore both members of the equality are possible. Now we have

$$\begin{aligned}[(\rho - \rho') - \rho''](i,j) &= [(\rho - \rho')(i,j) - \rho''(i,j)] \wedge [(\rho - \rho')(i,i) - \rho''(i,i)] \wedge [(\rho - \rho')(j,j) - \rho''(j,j)] \\ &= [\rho(i,j) - \rho'(i,j) - \rho''(i,j)] \wedge [\rho(i,i) - \rho'(i,i) - \rho''(i,j)] \wedge [\rho(j,j) - \rho'(j,j) - \rho''(i,j)] \wedge [\rho(i,i) - \rho'(i,i) - \rho''(i,i)] \\ &\quad \wedge [\rho(j,j) - \rho'(j,j) - \rho''(j,j)] \\ &= [\rho(i,j) - \rho'(i,j) - \rho''(i,j)] \wedge [\rho(i,i) - \rho'(i,i) - \rho''(i,i)] \wedge [\rho(j,j) - \rho'(j,j) - \rho''(j,j)] \\ &= [\rho(i,j) - (\rho'(i,j) + \rho''(i,j))] \wedge [\rho(i,i) - (\rho'(i,i) + \rho''(i,i))] \wedge [\rho(j,j) - (\rho'(j,j) + \rho''(j,j))] = [\rho - (\rho' + \rho'')](i,j),\end{aligned}$$

because if $\rho''(i,j) \leqslant \rho''(i,i)$ then $\rho(i,i) - \rho'(i,i) - \rho''(i,i) \leqslant \rho(i,i) - \rho'(i,i) - \rho''(i,j)$.
(2) We only have to see that $\rho' \leqslant \rho - \rho''$ because in that case by (1)

$$(\rho - \rho') - \rho'' = \rho - (\rho' + \rho'') = (\rho - \rho'') - \rho'.$$

We take $i, j \in N$ again. As $\rho''(i,j) \leqslant \rho(i,j) - \rho'(i,j)$ we obtain $\rho'(i,j) \leqslant \rho(i,j) - \rho''(i,j)$. If $\rho'(i,j) > \rho(i,i) - \rho''(i,i)$ then $\rho'(i,j) > \rho'(i,i)$ because $-\rho''(i,i) \geqslant -\rho(i,i) + \rho'(i,i)$. Thus we get $\rho'(i,j) \leqslant \rho(i,i) - \rho''(i,i)$. □

3. Fuzzy communication structures

Let (N, v) be a superadditive cooperative game. We consider a fuzzy graph $\rho \in [0,1]_0^{N\times N}$ which changes the initial situation. The number $\rho(i,i)$ is interpreted as the real level of involvement of player $i \in N$ in the game and $\rho(i,j)$ represents the maximum level which the link (i,j) can be used. Therefore, we understand a *fuzzy communication structure* over N as a fuzzy graph without loops over N. For each fuzzy communication structure ρ over N we would take the different possibilities of communication among the players. We suppose that players will construct crisp coalitions as the initial game establishes, but into a coalition they will allocate their involvement levels in the best way for the coalition as the fuzzy communication structure allows. Considering the big coalition they would choose a communication by a tree linking a coalition of players with a feasible level in the structure.

Definition 2. Let $\rho \in [0,1]_0^{N\times N}$ be a fuzzy communication structure and $t \in (0,1]$. Another fuzzy graph $\rho' \in [0,1]_0^{N\times N}$ is named a t-tree in ρ if $\rho' \leqslant \rho$ is a tree and $\rho'(i,j) = t$ for all $(i,j) \in \text{supp}\,(\rho')$. The t-tree ρ' in ρ is maximal iff there is a connected component of ρ with the same set of vertices than ρ' and connection level t.

Players would select a t-tree in the fuzzy communication structure, but as players can keep looking for cooperation then they choose t-trees until using up their involvements. Given ρ a fuzzy graph over N we define a *ρ-partition* as a finite sequence $P = \{(S_k, s_k)\}_{k=1}^{k=m}$ such that for all $k = 1,\ldots,m$ there exists $\rho_k \in [0,1]_0^{N\times N}$ verifying:

(1) vert $(\rho_k) = S_k$,
(2) ρ_k is a s_k-tree in $\rho - \sum_{l=1}^{k-1}\rho_l$ and
(3) $\rho - \sum_{k=1}^{m}\rho_k = 0$.

In the set of ρ-partitions we can do equivalence classes in the sense that $P = Q$ with $P = \{(S_k, s_k)\}_{k=1}^{k=m}$ and $Q = \{(T_k, t_k)\}_{k=1}^{k=p}$ iff $\sum_{\{k:S_k=S\}} s_k = \sum_{\{k:T_k=S\}} t_k$ for every $S \subseteq N$. Furthermore, $Q \leqslant P$ iff we can find a partition for each S_k, this is $(R_l^k)_{l=1}^{l=m_k}$ disjoint sets and covering S_k, such that $P' = \left\{\left\{\left(R_l^k, s_k\right)\right\}_{l=1}^{l=m_k}\right\}_{k=1}^{k=m}$ is other ρ-partition with $P' = Q$.

Next proposition will prove that a ρ-partition is not an ordered set actually.

Proposition 2. *Let ρ be a fuzzy graph. If $P = \{(S_k, s_k)\}_{k=1}^{k=m}$ is a ρ-partition then any order of the elements of P obtains the same ρ-partition.*

Proof. Let π be a permutation of the set $\{1,\ldots,m\}$. We denote $(\rho_k)_{k=1}^{k=m}$ the set of t-trees that we have used to construct P. We will prove that $Q = \{(S_{\pi(k)}, s_{\pi(k)})\}_{k=1}^{k=m}$ is a ρ-partition with t-trees $(\rho_{\pi(k)})_{k=1}^{k=m}$. In that case it is obvious the equality $Q = P$. We get that

$$\rho - \sum_{k=1}^{m}\rho_{\pi(k)} = \rho - \sum_{k=1}^{m}\rho_k = 0.$$

Therefore, we only need to see that for each $k = 0,\ldots,m-1$ the claim

$$\rho_{\pi(k+1)} \leqslant \rho - \sum_{l=1}^{k}\rho_{\pi(l)}.$$

If $k = 0$ then $\rho_{\pi(1)} \leqslant \rho$. We suppose $k < m-1$ such that $\rho_{\pi(k)} \leqslant \rho - \sum_{l=1}^{k-1}\rho_{\pi(l)}$ and we take $k+1$. We have

$$\rho_{\pi(k+1)} \leqslant \rho - \sum_{\{l\leqslant k:\pi(k+1)<\pi(l)\}}\rho_l$$

and we consider q such that $\pi(q) = \vee_{\{l\leqslant k:\ \pi(k+1)>\pi(l)\}}\pi(l)$. But this element q verifies

$$\rho_{\pi(q)} \leqslant \rho - \sum_{\{l\leqslant k:\pi(k+1)<\pi(l)\}}\rho_l - \rho_{\pi(k+1)}$$

and hence by Proposition 1(2)

$$\rho_{\pi(k+1)} \leqslant \rho - \sum_{\{l\leqslant k:\pi(k+1)<\pi(l)\}}\rho_l - \rho_{\pi(q)}.$$

We can repeat the same idea with the set $\{l \leqslant k\colon \pi(l) < \pi(q)\}$ and going on we obtain the claim. □

If $P = \{(S_k, s_k)\}_{k=1}^{k=m}$ is a ρ-partition then we denote

$$v(P) = \sum_{k=1}^{m} s_k v(S_k). \tag{5}$$

Obviously, if $Q = P$ then $v(Q) = v(P)$, but we also have the following result.

Proposition 3. *Let P, Q be ρ-partitions with ρ a fuzzy communication structure. If $Q \leqslant P$ and v is a superadditive game then $v(Q) \leqslant v(P)$.*

Proof. We take $P = \{(S_k, s_k)\}_k$. As $Q \leqslant P$ then there is a partition (classical) of each S_k, $\{T_k^p\}_p$, such that $P' = \bigcup_k\{(T_k^p, s_k)\}_p$ is a ρ-partition with $P' = Q$. Therefore $v(P') = v(Q)$. But then we obtain

$$v(Q) = \sum_k\sum_p s_k v(T_{s_k}^p) \leqslant \sum_k s_k v(S_k) = v(P),$$

because v is superadditive. □

So, among the ρ-partitions we would look for the best profits for the big coalition, but the set of ρ-partitions is infinite.

Cooperative games with fuzzy coalitions (fuzzy games) were introduced by Aubin (1981) as a pair (N, v) where $v : [0,1]^N \to \mathbb{R}$ is the fuzzy characteristic function with $v(0) = 0$. Tsurumi et al. (2001) defined an extension to a fuzzy game from a crisp game v. The *Choquet extension* of v is the fuzzy game $(N, ch(v))$ where for any $\tau \in [0,1]^N$

$$ch(v)(\tau) = \sum_{k=1}^{m} [h_k - h_{k-1}] v(\{i \in N : \tau_i \geqslant h_k\}) \tag{6}$$

with $h_1 < \cdots < h_m$ the different values of τ and $h_0 = 0$. We can observate that the Choquet extension supposes players in a fuzzy coalition τ first look for the biggest feasible coalition in τ and second look for the greatest feasible level, thus they go on cooperating in this way. Therefore we understand as "Choquet action" of players when they want to find in this order, a maximal feasible coalition and the maximal level for this coalition. It is possible to interpret this kind of action as the inertia produced from the original game.

Definition 3. Let ρ be a fuzzy communication structure over N. A Choquet ρ-partition is a finite sequence $P = \{(S_k, s_k)\}_{k=1}^{k=m}$ such that for all $k = 1, \ldots, m$ there exists $\rho_k \in [0, 1]_0^{N \times N}$ verifying:

(1) vert $(\rho_k) = S_k$,
(2) ρ_k is a maximal s_k-tree in $\rho - \sum_{l=1}^{k-1} \rho_l$ and
(3) $\rho - \sum_{k=1}^{m} \rho_k = 0$.

The set of all the Choquet ρ-partitions is denoted by $\mathcal{F}_\rho$.

In this paper we are going to suppose that players' behavior is Choquet, they select communicated coalitions step by step and they look in for the biggest coalition with the greatest level in this order for every step. Therefore players choose a Choquet partition for each fuzzy communication structure.

Example 2. We consider Example 1. In this case ρ is connected with connection level $t = 0.3$ (see Fig. 2). Looking at $g_{0.3}^{\rho}$ in Fig. 2 we test that there are three different maximal 0.3-trees. We choose one of them ρ_1 and we have a new connected fuzzy communication structure $\rho - \rho_1$. It is in Fig. 3 with its crisp versions.

The connection level of $\rho - \rho_1$ is $t = 0.2$. In this case there is only one maximal 0.2-tree, ρ_2. We repeat the process with the new fuzzy communication situation $\rho - \rho_1 - \rho_2 = \rho - (\rho_1 + \rho_2)$. Its crisp versions are painted in Fig. 4.

Now, $\rho - \rho_1 - \rho_2$ has two connected components and both of them have the same connection level, $t = 0.1$. We get the unique maximal 0.1-tree in each connected component, ρ_3 and ρ_4 (see Fig. 4).

Finally, we obtain the fuzzy graph $\rho - \rho_1 - \rho_2 - \rho_3 - \rho_4$ which has only individual connected components. So, we consider four maximal 0.4-trees $(\rho_5, \rho_6, \rho_7, \rho_8)$ with only one vertex, and now $\rho - \sum_{k=1}^{8} \rho_k = 0$. We obtain the Choquet ρ-partition P

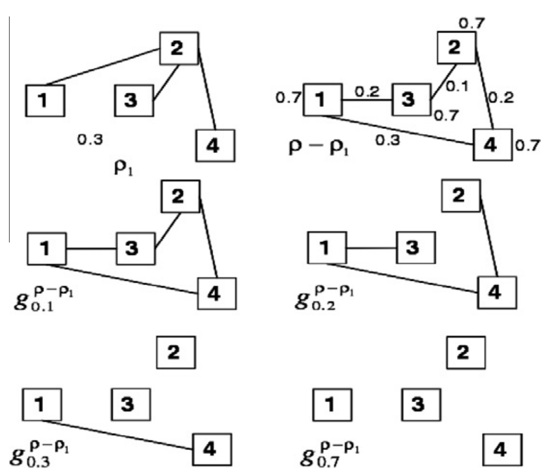

Fig. 3. Choquet partition. First step.

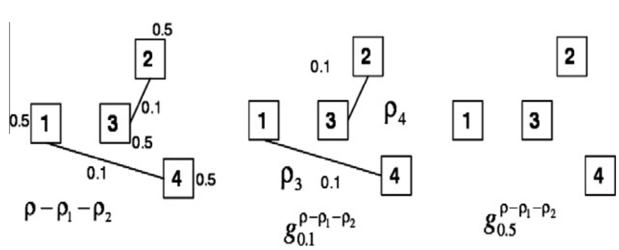

Fig. 4. Choquet partition. Other steps.

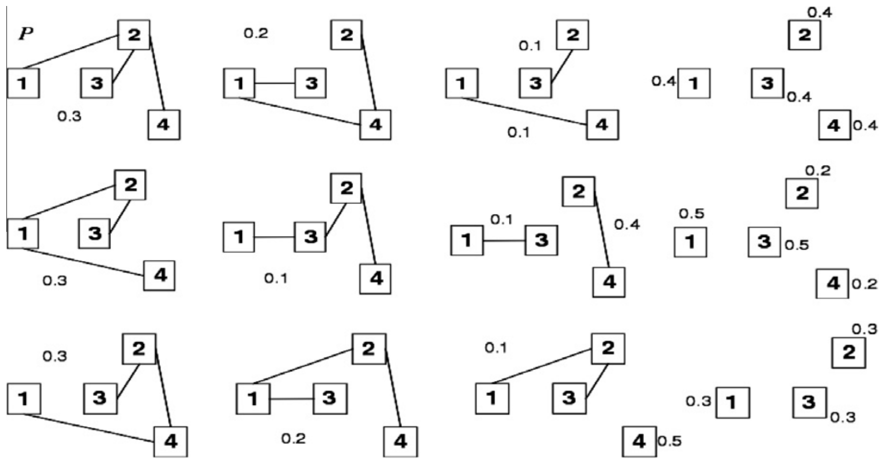

Fig. 5. Choquet partitions.

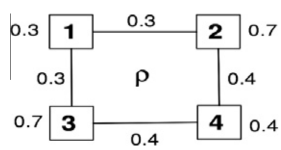

Fig. 6. Fuzzy graph (Example 3).

$$P = \{(N,0.3),(N,0.2),(\{1,4\},0.1),(\{2,3\},0.1),(\{1\},0.4),(\{2\},0.4),(\{3\},0.4),(\{4\},0.4)\}.$$

In Fig. 5 we can see the three feasible Choquet ρ-partitions.

Unfortunately, the best profit is not always obtained in a Choquet partition as we can test in the next example.

Example 3. We consider the following fuzzy graph ρ with four players in Fig. 6.

The ρ-partition $P = \{(\{1,2,3\},0.3),(\{2,3,4\},0.4)\}$ is not Choquet and there is not a Choquet ρ-partition Q with $P < Q$. The only Choquet ρ-partition is $Q = \{(N,0.3),(\{2,3,4\},0.1),(\{2\},0.2),(\{3\},0.2)\}$. If we take the game $\upsilon(S) = |S|$ for every coalition S then $\upsilon(Q) = 1.9$ and $\upsilon(P) = 2.1$.

Next lemma about Choquet partitions will be used in the paper.

Lemma 4. *Let $\rho \in [0,1]_0^{N\times N}$ be a fuzzy graph and $\rho' \leqslant \rho$ such that $\rho'(i,i) = \rho(i,i)$ for every $i \in N$. If $Q \in \mathcal{F}_{\rho'}$ then Q is a ρ-partition and there exists $P \in \mathcal{F}_\rho$ with $Q \leqslant P$.*

Proof. We take $Q = \{(S_k, s_k)\}_k \in \mathcal{F}_{\rho'}$ with s_k-trees associated ρ'_k. Obviously, $\rho'_k \leqslant \rho$ for each k. The partition Q is a ρ-partition because the s_k-trees of Q use up players' levels in ρ, this is $\rho - \sum_k \rho'_k = 0$, by the special election of ρ'.

We are going to construct a Choquet ρ-partition P such that $Q \leqslant P$. By Proposition 2 we can arrange the elements of Q according to the connected components of ρ. This is, we take any connected component ρ^* of ρ with connection level t_1 and we put joint every S_k with vertices in this component. Now, we put these elements in order not to repeat vertices until to cover all the vertices of ρ^* once and so we repeat the process until use all the elements. This is possible because Q is Choquet and ρ^* is union of connected components of ρ'. Let m be the number of elements in the first group of the component ρ^* until repeating players. As partition Q is Choquet then each one of these elements represents the vertex set of a connected component in ρ'. The fuzzy graph $\rho'' = \sum_{k=1}^m \rho'_k$ is a forest subgraph of ρ^*. Since ρ^* is connected there is at least a spanning tree of ρ^* greater than ρ''. To obtain this spanning tree we increase in level each link in the forest until its level in ρ, and we only add new edges from link $(\rho)\backslash$link (ρ') because if not some ρ_k is not maximal. We are going to select a maximal t_1-tree ρ_1 in the following way. If we can find the above spanning tree with connection level t then we take the only maximal t_1-tree in it as ρ_1. Otherwise, we have an spanning tree of ρ^* with connection level less than t_1. We consider an edge (i,j) in the spanning tree with $\rho(i,j) < t_1$. If this edge is not in ρ'' then we can always change this link in ρ_1 by another one with level greater or equal than t_1 since ρ^* has connection level t_1. Therefore, we suppose $(i,j) \in$ link (ρ_k) for some k. Another time, as the connection level of ρ^* is t_1 we can change this edge in ρ_1 by a sequence of links in link $(\rho)\backslash$link (ρ') joining i with j, with levels greater or equal than t_1. So, we obtain always a maximal t_1-tree ρ_1 in ρ^* which contains ρ'' or a subgraph of its and edges out of ρ'. Hence, $\rho - \rho_1 \geqslant \rho' - t_1 g^{\rho''}$ and

$$\bigcup_{k=1}^m S_k = T_1 = \mathrm{vert}(\rho_1).$$

We have a partition of T_1 and enough levels in $\rho - \rho_1$ to cover $\rho' - t_1 g^{\rho''}$. If we follow this idea with $\rho - \rho_1$ and so on we obtain $P \geqslant Q$ with $P \in \mathcal{F}_\rho$. $\square$

Hence Choquet players in a superadditive game choose a Choquet ρ-partition for each fuzzy graph ρ. We consider that players look for an optimal partition for v and then we denote:

$$v(\rho) = \vee_{P \in \mathcal{F}_\rho} v(P). \tag{7}$$

Definition 4. A fuzzy allocation rule for a game v is an application $\xi^v : [0,1]_0^{N\times N} \to \mathbb{R}^N$ such that for every $\rho \in [0,1]_0^{N\times N}$ we have

$$\sum_{i\in S} \xi_i^v(\rho) = v(\rho_S)$$

for all $S \in N/g^\rho$, and $\xi_i^v(\rho) = 0$ if $i \notin$ vert (ρ).

Particularly, the fuzzy allocation rule ξ^v is stable iff for all $\rho \in [0,1]_0^{N\times N}$, $(i,j) \in$ link (ρ) and $t \in (0, \rho(i,j)]$ we have

$$\xi_i^v(\rho) \geqslant \xi_i^v(\rho - te_{ij}) \quad \text{and} \qquad \xi_j^v(\rho) \geqslant \xi_j^v(\rho - te_{ij}),$$

where $e_{ij}(i,j) = 1$ and zero otherwise. The fuzzy allocation rule ξ^v is fair iff

$$\xi_i^v(\rho) - \xi_i^v(\rho - te_{ij}) = \xi_j^v(\rho) - \xi_j^v(\rho - te_{ij}).$$

Following Myerson we define a new game taking in account the fuzzy communication structure.

Definition 5. Let ρ be a fuzzy communication structure and v a superadditive game. The vertex fuzzy game of v is a new cooperative game defined for all $S \subseteq N$ by $v^\rho(S) = v(\rho_S)$.

These new crisp games are superadditive for all the fuzzy communication structures as we will prove in the next proposition.

Proposition 5. *Let ρ be a fuzzy communication structure and v a game. The vertex fuzzy game of v is a superadditive game.*

Proof. We take $S, T \subseteq N$ such that $S \cap T = \emptyset$. We have $\rho_S + \rho_T \leqslant \rho_{S\cup T}$ and if $i \in N$ then $\rho_S(i,i) + \rho_T(i,i) = \rho_{S\cup T}(i,i)$.

There exists $Q_S \in \mathcal{F}_{\rho_S}$ such that $v^\rho(S) = v(Q_S)$. There is also $Q_T \in \mathcal{F}_{\rho_T}$ such that $v^\rho(T) = v(Q_T)$. Since $S \cap T = \emptyset$ then $Q_S \cup Q_T \in \mathcal{F}_{\rho_S+\rho_T}$. By Lemma 4 there is $P \in \mathcal{F}_{\rho_{S\cup T}}$ with $Q_S \cup Q_T \leqslant P$. Using Proposition 3 we get

$$v^\rho(S) + v^\rho(T) = v(Q_S) + v(Q_T) = v(Q_S \cup Q_T) \leqslant v(P) \leqslant v^\rho(S\cup T).$$

So, the vertex fuzzy game is superadditive. □

Finally, we define a fuzzy allocation rule.

Definition 6. The fuzzy Myerson value of a game v is defined for each $\rho \in [0,1]_0^{N\times N}$ as $\eta^v(\rho) = \phi(v^\rho)$, where ϕ is the Shapley value.

Theorem 6. *The fuzzy Myerson value is the only fair fuzzy allocation rule and it is stable.*

Proof. First we will prove that there is only one fair fuzzy allocation rule. Let $\xi^1 \neq \xi^2$ be fair allocation rules for the game v. If link $(\rho) = \emptyset$ then each player $i \in$ vert (ρ) satisfies $\{i\} \in N/g^\rho$ and

$$\xi_i^1(\rho) = \xi_i^2(\rho) = \rho(i,i)v(\{i\}).$$

We take $\rho > 0$ a minimal in links fuzzy communication structure with $\xi^1(\rho) \neq \xi^2(\rho)$, this is for all $\rho' \leqslant \rho$ with $|\text{link}(\rho')| \leqslant |\text{link}(\rho)|$ happens $\xi^1(\rho') = \xi^2(\rho')$. By definition of ρ we have for each i, j with $\rho(i,j) > 0$.

$$\xi^1(\rho - \rho(i,j)e_{ij}) = \xi^2(\rho - \rho(i,j)e_{ij}).$$

Using the fair condition,

$$\xi_i^1(\rho) - \xi_j^1(\rho) = \xi_i^1(\rho - \rho(i,j)e_{ij}) - \xi_j^1(\rho - \rho(i,j)e_{ij}) = \xi_i^2(\rho - \rho(i,j)e_{ij}) - \xi_j^2(\rho - \rho(i,j)e_{ij}) = \xi_i^2(\rho) - \xi_j^2(\rho),$$

and we obtain that $\xi_i^1(\rho) - \xi_i^2(\rho) = \xi_j^1(\rho) - \xi_j^2(\rho)$. Now, let $S \in N/g^\rho$ be. When $S = \{i\}$ it happens $\xi_i^1(\rho) = \xi_i^2(\rho)$ because both of them are fuzzy allocation rules. If $|S| > 1$ then there is a constant K such that $\xi_i^1(\rho) - \xi_i^2(\rho) = K$ for all $i \in S$. As both applications are fuzzy allocation rules these sums are equal, $\sum_{i\in S}\xi_i^1(\rho) = \sum_{i\in S}\xi_i^2(\rho)$. Therefore,

$$|S|K = \sum_{i\in S} \xi_i^1(\rho) - \xi_i^2(\rho) = \sum_{i\in S} \xi_i^1(\rho) - \sum_{i\in S} \xi_i^2(\rho) = 0,$$

and this fact implies $K = 0$. We obtain $\xi^1(\rho) = \xi^2(\rho)$.

Second, we are going to test that the fuzzy Myerson value is a allocation rule. Let $\rho \in [0,1]_0^{N\times N}$ be. For any $R \subseteq N$ there is $P \in \mathcal{F}_{\rho_R}$ such that $v(\rho_R) = v(P)$ by (7). Obviously, we can write $P = \bigcup_{T\in N/g^\rho} Q_T$ with $Q_T \in \mathcal{F}_{\rho_{R\cap T}}$ for each $T \in N/g^\rho$ and $Q_T = \emptyset$ if $R \cap T = \emptyset$. Moreover, $v(\rho_{R\cap T}) = v(Q_T)$ for every T. As these partitions are disjoints then

$$v(\rho_R) = \sum_{T\in N/g^\rho} v(\rho_{R\cap T}),$$

and by Definition 6

$$v^{\rho} = \sum_{T \in N/g^{\rho}} v^{\rho_T}.$$

We take a fixed $S \in N/g^{\rho}$. If $T \in N/g^{\rho}$ with $T \neq S$ then any $i \in S$ is a null player for v^{ρ_T}. Hence, we obtain using lineality, null player and efficiency properties of the Shapley value (1),

$$\sum_{i \in S} \eta_i^v(\rho) = \sum_{i \in S} \phi_i(v^{\rho}) = \sum_{i \in S} \sum_{T \in N/g^{\rho}} \phi_i(v^{\rho_T}) = \sum_{T \in N/g^{\rho}} \sum_{i \in S} \phi_i(v^{\rho_T}) = \sum_{i \in S} \phi_i(v^{\rho_S}) = v^{\rho_S}(S) = v(\rho_S).$$

Finally, we see that our allocation rule is fair and stable. We define for a fixed fuzzy graph ρ, any $(i,j) \in$ link (ρ) and $t \in (0, \rho(i,j)]$ the game $w = v^{\rho} - v^{\rho - te_{ij}}$. By construction, $w(S) \neq 0$ iff $i,j \in S$, and in that case $\phi_i(w) = \phi_j(w)$. By lineality of the Shapley value we have

$$\eta_i^v(\rho) - \eta_i^v(\rho - te_{ij}) = \eta_j^v(\rho) - \eta_j^v(\rho - te_{ij}).$$

Thus, η^v is a fair rule. Now, we take a coalition S such that $i, j \in S$. We can observe that $(\rho - te_{ij})_S = \rho_S - te_{ij} \leqslant \rho_S$ and $(\rho_S - te_{ij})(k,k) = \rho_S(k,k)$ for all $k \in N$. By Lemma 4, if $Q \in \mathcal{F}_{\rho_S - te_{ij}}$ such that $v(\rho - te_{ij}) = v(Q_S)$ then there is $P \in \mathcal{F}_{\rho_S}$ with $Q_S \leqslant P$. We get $v(\rho - te_{ij}) \leqslant v(\rho_S)$ by (7) and Proposition 3, this is $v^{\rho - te_{ij}}(S) \leqslant v^{\rho}(S)$. This fact implies that $w(S) \geqslant 0$. But then the marginal contributions of players i, j for game w are non negative, and $\phi_i(w)$, $\phi_j(w) \geqslant 0$. The fuzzy Myerson value is also stable. □

4. Particular cases

Returning to the Choquet extension (4) introduced by Tsurumi et al. (2001), we are going to establish the relationship between this extension and the fuzzy Myerson value.

Definition 7. A fuzzy graph $\rho \in [0,1]_0^{N \times N}$ is complete by links iff it satisfies that $\rho(i,j) = \rho(i,i) \wedge \rho(j,j)$ for all $i, j \in N$.

Each link in ρ complete by links can be used at maximum permitted from its vertices. Hence, we can identify the complete by links fuzzy communication structures with the fuzzy coalitions in N by the relation $\rho(i,i) = \tau(i)$ for each $i \in N$ when $\tau \in [0,1]^N$. The Myerson value (4), coincides with the Shapley value (1) when we take the complete graph. Now, we will prove that the fuzzy Myerson value coincides with the fuzzy Shapley function defined by Tsurumi et al. (2001) when we take a fuzzy graph complete by links. For the game v and the coalition S we denote by v_S the restriction of v to S, this is $v_S(T) = v(S \cap T)$ for all $T \subseteq N$. Tsurumi define the *fuzzy Shapley function* for the Choquet extension (6) of v for each $\tau \in [0,1]^N$ as:

$$\psi(ch(v))(\tau) = \sum_{k=1}^{m} [h_k - h_{k-1}] \phi(v_{H_k}), \tag{8}$$

where $h_1 < \cdots < h_m$ are the different non-null values of τ, $h_0 = 0$ and $H_k = \{i \in N: \tau(i) \geqslant h_k\}$ for each $k = 1, \ldots, m$.

Lemma 7. *If ρ is a complete by links fuzzy graph then there is a unique Choquet ρ-partition. Moreover, if $h_1 < \cdots < h_m$ are the different non-null values of ρ, $h_0 = 0$ and $H_k = \{i \in N: \rho(i,i) \geqslant h_k\}$ for each $k = 1, \ldots, m$, then the only Choquet ρ-partition is $\{(H_k, h_k - h_{k-1})\}_{k=1}^{k=m}$.*

Proof. Let $i_1 \in N$ be such that

$$h_1 = \rho(i_1, i_1) = \bigwedge_{i \in \mathrm{vert}(\rho)} \rho(i,i).$$

Therefore, for all $(i,j) \in$ supp (ρ) we have $\rho(i,j) \geqslant h_1$. For each spanning tree g of g^{ρ} (ρ is connected) we obtain $h_1 g$ maximal h_1-tree in ρ. Furthermore, $\rho - h_1 g$ does not contain any link incident with vertex i_1. But we are going to see that the fuzzy graph $\rho - h_1 g$ is always the same for every spanning tree g. We suppose two different spanning trees g, g' in g^{ρ}. If $g \neq g'$ then there is $(i,j) \in$ link $(h_1 g)$ such that $(i,j) \notin$ link $(h_1 g')$ and we take $\rho(i,i) \leqslant \rho(j,j)$. So $\rho(i,j) = \rho(i,i)$ and then

$$(\rho - h_1 g)(i,j) = [\rho(i,j) - h_1] \wedge [\rho(i,i) - h_1] \wedge [\rho(j,j) - h_1] = \rho(i,i) - h_1$$

but also

$$(\rho - h_1 g')(i,j) = [\rho(i,j)] \wedge [\rho(i,i) - h_1] \wedge [\rho(j,j) - h_1] = \rho(i,i) - h_1.$$

We can repeat the process with $\rho - h_1 g$ because it is a new complete by links fuzzy graph. For all $i, j \in N$ with $\rho(i,i) \leqslant \rho(j,j)$ we have

$$(\rho - h_1 g)(i,j) = \rho(i,i) - h_1 = (\rho - h_1 g)(i,i) \wedge (\rho - h_1 g)(j,j).$$

Hence, we have proved that there is a unique Choquet ρ-partition.

Now we consider $h_1 < \cdots < h_m$ are the different non-null values of ρ. Following the process, in the first step we have used $h_1 g$ to get the pair (H_1, h_1), where $H_1 = N = \{i \in N: \rho(i,i) \geqslant h_1\}$. In the second step the minimal level is $h_2 - h_1$ and then we obtain $(H_2, h_2 - h_1)$, where $H_2 = N = \{i \in N: \rho(i,i) \geqslant h_2\}$. We conclude that the only Choquet ρ-partition is $\{(H_k, h_k - h_{k-1})\}_{k=1}^{k=m}$. □

Theorem 8. *If ρ is a complete by links fuzzy communication and τ its fuzzy coalition associated then $\eta^v(\rho) = \psi(ch(v))(\tau)$.*

Proof. The different levels in ρ, $h_1, \ldots, h_m$, are the different values of τ because the fuzzy graph is complete by links. By the above lemma we have that the only Choquet ρ-partition is $\{(H_k, h_k - h_{k-1})\}_{k=1}^{k=m}$ with $H_k = \{i \in N: \rho(i,i) \geqslant h_k\}$ for each k. Therefore, for each $S \subseteq N$ by (7)

$$v^{\rho}(S) = v(\rho_S) = \sum_{k=1}^{m} [h_k - h_{k-1}]\, v(H_k \cap S)$$

because ρ_S is also complete by links and $\{(H_k \cap S, h_k - h_{k-1})\}_{k=1}^{k=m}$ is its only Choquet partition. This fact means

$$v^{\rho} = \sum_{k=1}^{m} [h_k - h_{k-1}]\, v_{H_k}.$$

Finally the lineality of the Shapley value implies

$$\eta^{v}(\rho) = \phi(v^{\rho}) = \sum_{k=1}^{m} [h_k - h_{k-1}] \phi(v_{H_k}) = \psi(ch(v))(\tau). \qquad \square$$

To finish the paper we study what happens when we take a forest. We denote by $[0,1]_1^{N\times N}$ the set of the forest fuzzy communication structures. In that case we can also prove that there is only one Choquet partition.

Lemma 9. *If $\rho \in [0,1]_1^{N\times N}$ then there exists a unique Choquet ρ-partition.*

Proof. Each connected component of ρ is a tree. Therefore there is a unique t-tree for component and we take ρ' the sum of these t-trees. The new fuzzy graph $\rho - \rho'$ is also a forest and we repeat the process. So, we obtain a unique Choquet ρ-partition. $\square$

The fuzzy Mayerson value for a forest can be calculated by Myerson values of its crisp versions.

Theorem 10. *Let $\rho \in [0,1]_1^{N\times N}$ be a forest fuzzy communication structure and v a game. The fuzzy Myerson value satisfies that*

$$\eta^{v}(\rho) = \sum_{k=1}^{m} [h_k - h_{k-1}] \mu^{v}\left(g_{h_k}^{\rho}\right),$$

where $h_1 < \cdots < h_m$ are the different ordered values of ρ and $h_0 = 0$.

Proof. Let $\rho \in [0,1]_1^{N\times N}$ be a forest. We will prove that if $h_1 < \cdots < h_m$ are the different ordered values of ρ then the fuzzy vertex game of v verifies

$$v^{\rho} = \sum_{k=1}^{m} [h_k - h_{k-1}] \left(v/g_{h_k}^{\rho}\right), \tag{9}$$

where $h_0 = 0$.

We consider a coalition $S \subseteq N$. Since Lemma 9 there is only one Choquet ρ_S-partition P because ρ_S is a forest. The element $(T, t_T) \in P$ if we have subtracted maximal t-trees to ρ_S until obtaining a subgraph ρ' where $T \in S/g^{\rho'}$. In that case we have had to delete all the links between players in T and players in $S\backslash T$. Just to obtain ρ' we had another subgraph ρ_1 of ρ_S and, at that moment, we deleted the last links between T and $S\backslash T$. These links must have the same level in ρ,

$$\bigvee_{\{(j_1,j_2)\in \mathrm{link}(\rho): j_1 \in T, j_2 \in S\backslash T\}} \rho(j_1,j_2).$$

Obviously, we get $\rho'(i_1,i_2) = \rho_1(i_1,i_2) - \rho_1(j_1,j_2)$ for all $i_1, i_2 \in T$ with $(i_1,i_2) \in \mathrm{supp}\,(\rho)$ and where $(j_1,j_2) \in \mathrm{link}\,(\rho_1)$ with $j_1 \in T$ and $j_2 \in S\backslash T$. But (i_1,i_2) and (j_1,j_2) were in the same connected component of ρ_1, thus there exists a constant K such that $\rho_1(i_1,i_2) = \rho(i_1,i_2) - K$ and $\rho_1(j_1,j_2) = \rho(j_1,j_2) - K$. This fact implies

$$\rho'(i_1,i_2) = \rho(i_1,i_2) - \bigvee_{\{(j_1,j_2)\in \mathrm{link}(\rho): j_1 \in T, j_2 \in S\backslash T\}} \rho(j_1,j_2).$$

As ρ'_T is a tree then

$$t_T = \bigwedge_{\{(i_1,i_2)\in \mathrm{supp}(\rho): i_1, i_2 \in T\}} \rho'(i_1,i_2) = \bigwedge_{\{(i_1,i_2)\in \mathrm{supp}(\rho): i_1, i_2 \in T\}} \rho(i_1,i_2) - \bigvee_{\{(j_1,j_2)\in \mathrm{link}(\rho): j_1 \in T, j_2 \in S\backslash T\}} \rho(j_1,j_2),$$

taking in account that we have not lost in ρ' any link between vertices in T. There exist $p_T, q_T \in \{1,\ldots,m\}$, $p_T < q_T$, such that

$$h_{p_T} = \bigvee_{\{(j_1,j_2)\in \mathrm{link}(\rho): j_1 \in T, j_2 \in S\backslash T\}} \rho(j_1,j_2) \quad \text{and} \quad h_{q_T} = \bigwedge_{\{(i_1,i_2)\in \mathrm{supp}(\rho): i_1, i_2 \in T\}} \rho(i_1,i_2).$$

Therefore $(T, t_T) \in P$ iff $T \in S/g_{h_k}^{\rho}$ for all $k \in \{p_T,\ldots,q_T\}$. We get then

$$t_T = h_{q_T} - h_{p_T} = \sum_{k=p_T+1}^{q_T} [h_k - h_{k-1}].$$

Now, using the expressions (7), (5) and (3),

$$v^{\rho}(S) = \sum_{(T,t_T)\in P} t_T v(T) = \sum_{(T,t_T)\in P} \sum_{k=p_T+1}^{q_T} [h_k - h_{k-1}]\, v(T) = \sum_{k=1}^{m} [h_k - h_{k-1}] \sum_{T \in S/g_{h_k}^{\rho}} v(T) = \sum_{k=1}^{m} [h_k - h_{k-1}] \left(v/g_{h_k}^{\rho}\right)(S).$$

By the lineality property of the Shapley value (1) and the efficiency by components of the Myerson value (4) we obtain the result of the theorem. $\square$

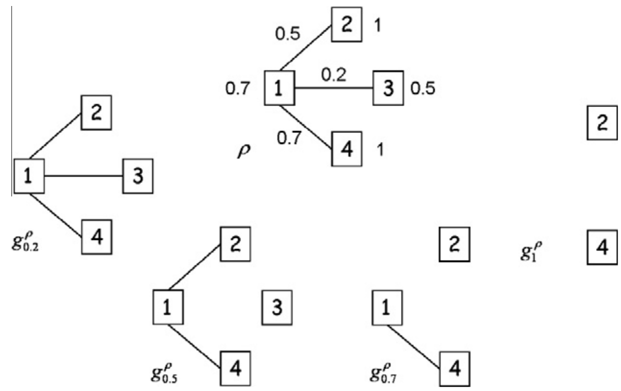

Fig. 7. Fuzzy graph and crisp versions (Example 4).

Example 4. We take the forest in Fig. 7 with two trees and the game $v(S) = |S|^2$. In the same figure we can see its crisp versions.

The Myerson values of the crisp versions are: $\mu^v(g^{\rho}_{0.2}) = (6, 3.33, 3.33, 3.33)$, $\mu^v(g^{\rho}_{0.5}) = (3.67, 2.67, 1, 2.67)$, $\mu^v(g^{\rho}_{0.7}) = (2, 1, 0, 2)$ and $\mu^v(g^{\rho}_{0.2}) = (0, 1, 0, 1)$. Now, we obtain the fuzzy Myerson value by the formula in Theorem 10,

$$\eta^v(\rho) = 0.2\mu^v(g^{\rho}_{0.2}) + 0.3\mu^v(g^{\rho}_{0.5}) + 0.2\mu^v(g^{\rho}_{0.7}) + 0.3\mu^v(g^{\rho}_{1}) = (2.7, 1.967, 0.967, 2.167).$$

The above formula is not useful in other cases because we do not always obtain a fuzzy allocation rule by this way. The reader can see this fact in the following example.

Example 5. We consider the connected fuzzy graph in Fig. 1 which crisps versions are in Fig. 2. We can apply the above formula to get an allocation rule but it is not in this case a fuzzy allocation rule because it is not an allocation of $v(\rho)$ for any superadditive game v. The allocation rule obtained over ρ would be

$$\xi^v(\rho) = 0.2\mu^v(g^{\rho}_{0.2}) + 0.1\mu^v(g^{\rho}_{0.3}) + 0.1\mu^v(g^{\rho}_{0.4}) + 0.1\mu^v(g^{\rho}_{0.5}) + 0.5\mu^v(g^{\rho}_{1}).$$

Therefore, as $N/g^{\rho} = \{N\}$ and the efficiency by components

$$\begin{aligned}\sum_{i\in N}\xi^v_i(\rho) &= 0.2\sum_{i\in N}\mu^v_i(g^{\rho}_{0.2}) + 0.1\sum_{i\in N}\mu^v_i(g^{\rho}_{0.3}) + 0.1\sum_{i\in N}\mu^v_i(g^{\rho}_{0.4}) + 0.1\sum_{i\in N}\mu^v_i(g^{\rho}_{0.5}) + 0.5\sum_{i\in N}\mu^v_i(g^{\rho}_{1})\\ &= 0.2v(N) + 0.1v(N) + 0.1[v(\{1\}) + v(\{2,3,4\})] + 0.1[v(\{1\}) + v(\{3\}) + v(\{2,4\})] + 0.5[v(\{1\}) + v(\{2\}) + v(\{3\}) + v(\{4\})]\\ &= 0.3v(N) + 0.1v(\{2,3,4\}) + 0.1v(\{2,4\}) + 0.7v(\{1\}) + 0.5v(\{2\}) + 0.6v(\{3\}) + 0.5v(\{4\}).\end{aligned}$$

Now, we take P the Choquet ρ-partition obtained in Example 2. As v is a superadditive game then by (5),

$$\sum_{i\in N}\xi^v_i(\rho) \leqslant v(P) = 0.5v(N) + 0.1v(\{1,4\}) + 0.1v(\{2,3\}) + 0.4v(\{1\}) + 0.4v(\{2\}) + 0.4v(\{3\}) + 0.4v(\{4\}).$$

If we choose for instance the game $v(S) = |S| - 1$ then

$$\sum_{i\in N}\xi^v_i(\rho) = 1.2 < v(P) = 1.7 \leqslant v(\rho).$$

The fuzzy Myerson value is continuous with regard to the forest when the game is non-negative.

Lemma 11. *If v is a superadditive game with $v(S) \geqslant 0$ for every $S \subseteq N$ and g a graph with vertices in N then the Myerson value verifies that $0 \leqslant \mu^v_i(g) \leqslant v(N)$ for all player $i \in N$.*

Proof. A game v is monotone iff $v(T) \leqslant v(S)$ when $T \subseteq S$. It is known that if $v \geqslant 0$ is superadditive then v is monotone. We will prove that v/g is also monotone for all $g \in G^N$. Let $T \subseteq S$ be and $g \in G^N$. Each element $R \in T/g$ is contained in a coalition $H \in S/g$ but it can be into the same one. If $R_1, \ldots, R_m$ are the elements in T/g contained in a particular $H \in S/g$ then

$$\sum_{k=1}^{m} v(R_k) \leqslant v\left(\bigcup_{k=1}^{m} R_k\right) \leqslant v(H)$$

because v is superadditive and monotone. This fact implies that $v/g(T) \leqslant v/g(S)$ and then v/g is monotone.

Hence, the Myerson value of g satisfies $\mu^v_i(g) = \phi_i(v/g) \geqslant 0$. Finally, for each player $i \in N$ we have $\mu^v_i(g) \leqslant v(S_i)$, using efficiency by components, where S_i is the only coalition in N/g such that $i \in S_i$. We obtain $\mu^v_i(g) \leqslant v(N)$ because v is monotone. □

Corollary 12. *The fuzzy Myerson value is a continuous fuzzy allocation rule on forests regarding to the fuzzy communication structures when the game in non-negative.*

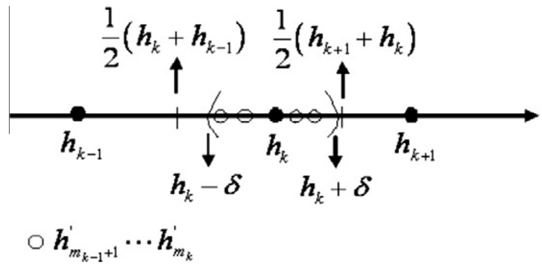

Fig. 8. Positions of the levels.

Proof. We define the following distance between forests, if $\rho, \rho' \in [0,1]_1^{N\times N}$ then

$$d(\rho,\rho') = \bigvee_{(i,j)\in N\times N} |\rho(i,j) - \rho'(i,j)|.$$

We are going to prove that $\eta_{i_0}^{v}$ is continuous for each $i_0 \in N$ in $\rho \in [0,1]_1^{N\times N}$. We consider the different ordered levels of ρ as $h_1 < \cdots < h_m$ and the set of pairs with a particular level as $A_k = \{(i,j) \in N \times N \colon \rho(i,j) = h_k\}$ for all $k = 1,\ldots,m$.

For any $\epsilon > 0$ we choose

$$\delta < \left[\frac{1}{2}\bigwedge_{k=1}^{m}(h_k - h_{k-1})\right] \wedge \frac{\epsilon}{4mv(N)}.$$

Let $\rho' \in [0,1]_1^{N\times N}$ be another forest such that $d(\rho,\rho') < \delta$. The different ordered levels of ρ' are denoted by $h'_1 < \cdots < h'_q$. By the election of δ we have for each $k \in \{1,\ldots,m\}$ that $\rho'(i_k,j_k) < \rho'(i_{k+1},j_{k+1})$ for all $(i_k,j_k) \in A_k$ and $(i_{k+1},j_{k+1}) \in A_{k+1}$. Therefore, we take $h'_{m_{k-1}+1} < \cdots < h'_{m_k}$ the different levels of ρ' for A_k with $k \in \{1,\ldots,m\}$.

We can get the following claim: level $h'_{m_{k-1}+1}$ verifies $L_{h_k}^{\rho} = L_{h'_{m_{k-1}+1}}^{\rho'}$ for every $k \in \{1,\ldots,m\}$, this is $g_{h_k}^{\rho} = g_{h'_{m_{k-1}+1}}^{\rho'}$. If $(i,j) \in L_{h_k}^{\rho}$ then $\rho(i,j) \geqslant h_k$ and $\rho'(i,j) \geqslant h_k - \delta$ by election of ρ'. Looking at Fig. 8 we have $\rho'(i,j) \geqslant h'_{m_{k-1}+1}$ and $(i,j) \in L_{h'_{m_{k-1}+1}}^{\rho'}$. On the other hand if $(i,j) \in L_{h'_{m_{k-1}+1}}^{\rho'}$ then $\rho'(i,j) \geqslant h'_{m_{k-1}+1}$ and $\rho(i,j) \geqslant h'_{m_{k-1}+1} - \delta$ by election of ρ'. Another time looking at Fig. 8 we have $\rho(i,j) \geqslant h_k$ and $(i,j) \in L_{h_k}^{\rho}$. We will also use that

$$|(h_k - h_{k-1}) - \left(h'_{m_{k-1}+1} - h'_{m_{k-1}}\right)| \leqslant 2\delta$$

and

$$\sum_{p=m_{k-1}+1}^{m_k}\left(h'_p - h'_{p-1}\right) \leqslant 2\delta.$$

Now, we get using the above lemma

$$\begin{aligned}
|\eta_{i_0}^{v}(\rho) - \eta_{i_0}^{v}(\rho')| &= \left|\sum_{k=1}^{m}(h_k - h_{k-1})\mu_{i_0}^{v}\left(g_{h_k}^{\rho}\right) - \sum_{p=1}^{q}\left(h'_p - h'_{p-1}\right)\mu_{i_0}^{v}\left(g_{h'_p}^{\rho'}\right)\right| \leqslant \sum_{k=1}^{m}\left|(h_k - h_{k-1})\mu_{i_0}^{v}\left(g_{h_k}^{\rho}\right) - \sum_{p=m_{k-1}+1}^{m_k}\left(h'_p - h'_{p-1}\right)\mu_{i_0}^{v}\left(g_{h'_p}^{\rho'}\right)\right| \\
&= \sum_{k=1}^{m}\left|(h_k - h_{k-1})\mu_{i_0}^{v}\left(g_{h_k}^{\rho}\right) - \left(h'_{m_{k-1}+1} - h'_{m_{k-1}}\right)\mu_{i_0}^{v}\left(g_{h'_{m_{k-1}+1}}^{\rho'}\right) - \sum_{p=m_{k-1}+2}^{m_k}\left(h'_p - h'_{p-1}\right)\mu_{i_0}^{v}\left(g_{h'_p}^{\rho'}\right)\right| \\
&\leqslant \sum_{k=1}^{m}\left[2\delta\mu_{i_0}^{v}\left(g_{h_k}^{\rho}\right) + \sum_{p=m_{k-1}+2}^{m_k}\left(h'_p - h'_{p-1}\right)\mu_{i_0}^{v}\left(g_{h'_p}^{\rho'}\right)\right] \leqslant mv(N)4\delta \leqslant \epsilon.
\end{aligned}$$

And this fact finishes the proof. □

Finally, we find a relationship between the vertex fuzzy game (Definition 5) and the Choquet integral when the fuzzy graph is a forest. Let $\rho \in [0,1]_1^{N\times N}$ be a forest. We denote by $2_0^{N\times N}$ the family of all the set of unordered pairs of players, this is the set of coalitions of vertices and links in the forest. For each coalition $S \subseteq N$ we define the capacity $w_S : 2_0^{N\times N} \to \mathbb{R}$ such that

$$w_S(\{(i,j) \in \mathrm{supp}(\rho_S) : \rho(i,j) \geqslant t\}) = v/g_t^{\rho}(S).$$

We can explain the vertex fuzzy game as a Choquet integral of the function ρ by the capacity w_S using (8)

$$v^{\rho}(S) = \int \rho\, dw_S.$$

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