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The core and the Weber set of games on augmenting systems

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ABSTRACT

This paper deals with cooperative games in which only certain coalitions are allowed to form. There have been previous models developed to confront the problem of unallowable coalitions. Games restricted by a communication graph were introduced by Myerson and Owen. In their model, the feasible coalitions are those that induce connected subgraphs. Another type of model is introduced in Gilles, Owen and van den Brink. In their model, the possibilities of coalition formation are determined by the positions of the players in a so-called permission structure. Faigle proposed a general model for cooperative games defined on lattice structures. In this paper, the restrictions to the cooperation are given by a combinatorial structure called augmenting system which generalizes antimatroid structure and the system of connected subgraphs of a graph. In this framework, the core and the Weber set of games on augmenting systems are introduced and it is proved that monotone convex games have a non-empty core. Moreover, we obtain a characterization of the convexity of these games in terms of the core of the game and the Weber set of the extended game.

1. Introduction

Cooperative games under combinatorial restrictions are cooperative games in which the players have restricted communication possibilities, which are defined by a combinatorial structure. The first model in which the restrictions are defined by the connected subgraphs of a graph is introduced by Myerson [10]. Since then, many other situations where players have communication restrictions have been studied in cooperative game theory. Contributions on graph-restricted games include Owen [11], Borm, Owen, and Tijs [4] and Hamiache [8]. In these models the possibilities of coalition formation are determined by the positions of the players in a *communication graph*. Another type of combinatorial structure introduced by Gilles, Owen and van den Brink [7] and van den Brink [5] is equivalent to a subclass of antimatroids. This line of research focuses on the possibilities of coalition formation determined by the positions of the players in the so-called *permission structure*. Faigle [6] adopts a different point of view. He considers a non-empty collection of feasible coalitions and a game defined on this collection and extends this game.

In the present paper, we use a restricted cooperation model derived from a combinatorial structure called *augmenting system*, introduced by Bilbao [3]. In Section 2, we recall preliminaries on this combinatorial structure which is a generalization of the antimatroid structure and the system of connected subgraphs of a graph. The aim of Section 3 is the introduction of the *non-negative core* and the *Weber set* for a game on an augmenting system. In the classical situation, for every cooperative game $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$, the core of v is contained in the Weber set. Moreover, v is *convex* if and only if the Weber set coincides with the core of v. For a game $v : \mathcal{F} \to \mathbb{R}_+$, where $\mathcal{F} \subset 2^N$ is an augmenting system, the inclusion $Core^+(N, v, \mathcal{F}) \subseteq Weber(N, v, \mathcal{F})$ is not true. However, in Section 4 we show that for monotone convex games on augmenting system, the Weber set is contained in the non-negative core which is non-empty. In the last section,

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we show that the superadditivity and a new concept of convexity are hereditary properties with respect to the extended game $\left(N, v^{\mathcal{F}}\right)$. Furthermore, we obtain the following characterization: A monotone game $(N, v, \mathcal{F})$ is convex if and only if $Core^{+}(N, v, \mathcal{F}) = Weber\left(N, v^{\mathcal{F}}\right)$.

2. Augmenting systems

This section is based on Bilbao [3]. It basically recalls preliminaries on augmenting systems and some concepts and results that will be used in the following. Let N be a finite set. A *set system* over N is a pair $(N, \mathcal{F})$ where $\mathcal{F} \subseteq 2^N$ is a family of subsets. The sets belonging to $\mathcal{F}$ are called *feasible*. We will write $S \cup i$ and $S \setminus i$ instead of $S \cup \{i\}$ and $S \setminus \{i\}$ respectively and we will use the symbols $\subset$ and $\subseteq$ to denote strict inclusion and inclusion. We will recall the concept of augmenting system.

Definition 1. An augmenting system is a set system $(N, \mathcal{F})$ with the following properties:

(P1) $\emptyset \in \mathcal{F}$,
(P2) for $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$, we have $S \cup T \in \mathcal{F}$,
(P3) for $S, T \in \mathcal{F}$ with $S \subset T$, there exists $i \in T \setminus S$ such that $S \cup i \in \mathcal{F}$.

Example. The following collections of subsets of $N = \{1, \ldots, n\}$, given by $\mathcal{F} = 2^N$, $\mathcal{F} = \{\emptyset, \{i\}\}$ where $i \in N$, and $\mathcal{F} = \{\emptyset, \{1\}, \ldots, \{n\}\}$, are augmenting systems over N.

Example. Let us consider a communication graph $G = (N, E)$, where N is the set of players and E is the set of edges which represents the bilateral communication between some players. Given a coalition $S \subseteq N$, the set of edges between players in S is denoted by $E(S) = \{ij \in E : i, j \in S\}$. Thus, the set system $(N, \mathcal{F})$ given by

$$\mathcal{F} = \{S \subseteq N : (S, E(S)) \text{ is a connected subgraph of } G\},$$

is an augmenting system.

Example. Gilles et al. [7] showed that the feasible coalition system $(N, \mathcal{F})$ derived from the conjunctive or disjunctive approach contains the empty set, the ground set N, and that it is closed under union. Algaba et al. [1] showed that the coalition systems derived from the conjunctive and disjunctive approach were identified to *poset antimatroids* and *antimatroids with the path property*, respectively. The relationship between antimatroids and augmenting systems given by Bilbao [3] implies that these coalition systems are augmenting systems.

The next property is proved by Algaba, Bilbao, and Slikker [2].

Theorem 2. *An augmenting system $(N, \mathcal{F})$ is the system of connected subgraphs of the graph $G = (N, E)$, where $E = \{S \in \mathcal{F} : |S| = 2\}$ if and only if $\{i\} \in \mathcal{F}$ for all $i \in N$.*

Example. The set system given by $N = \{1, 2, 3, 4\}$ and

$$\mathcal{F} = \{\emptyset, \{1\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, N\}.$$

is an augmenting system. Since $\{1, 4\} \notin \mathcal{F}$ the system $(N, \mathcal{F})$ is not an antimatroid. Moreover, $\{1, 2\} \cap \{2, 4\} = \{2\} \notin \mathcal{F}$ and hence $(N, \mathcal{F})$ is not a convex geometry.

Let $(N, \mathcal{F})$ be a set system and let $S \subseteq N$ be a subset. The maximal non-empty feasible subsets of S are called *components* of S. We denote by $C_{\mathcal{F}}(S)$ the set of the components of a subset $S \subseteq N$. Observe that the set $C_{\mathcal{F}}(S)$ may be the empty set. This set will play a role in the extension of a game restricted by an augmenting system.

Proposition 3. *A set system $(N, \mathcal{F})$ satisfies property (P2) if and only if for any $S \subseteq N$ with $C_{\mathcal{F}}(S) \neq \emptyset$, the components of S form a partition of a subset of S.*

Proof. See Proposition 2.9 in [3]. ∎

3. The Core and the Weber set of $(N, v, \mathcal{F})$

A *cooperative game on the augmenting system* $(N, \mathcal{F})$ is a triple $(N, v, \mathcal{F})$, where $v : \mathcal{F} \to \mathbb{R}_+$ is a function with non-negative real values such that $v(\emptyset) = 0$. Given the game $(N, v, \mathcal{F})$ we define a standard cooperative game $v^{\mathcal{F}} : 2^N \to \mathbb{R}_+$, named the *extension* of $(N, v, \mathcal{F})$, as

$$v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T) \quad \text{for all } S \subseteq N.$$

Note that the family of components $C_{\mathcal{F}}(S)$ forms a partition of a subset of S, and $v^{\mathcal{F}}(S) = v(S)$ for all $S \in \mathcal{F}$. Now, we define the *non-negative core* of the game $(N, v, \mathcal{F})$ and the *core* of $(N, v^{\mathcal{F}})$.

Definition 4. Let $(N, v, \mathcal{F})$ be a game on the augmenting system $(N, \mathcal{F})$. Then

$$Core^{+}(N, v, \mathcal{F}) = \left\{x \in \mathbb{R}_{+}^{N} : x(N) = v^{\mathcal{F}}(N), x(S) \geq v(S) \quad \text{for all } S \in \mathcal{F}\right\}$$

and

$$Core\left(N, v^{\mathcal{F}}\right) = \left\{x \in \mathbb{R}^{N} : x(N) = v^{\mathcal{F}}(N), x(S) \geq v^{\mathcal{F}}(S) \quad \text{for all } S \subseteq N\right\},$$

where we denote $x(S) = \sum_{i \in S} x_i$, and $x(\emptyset) = 0$.

Proposition 5. *The non-negative core of the game $(N, v, \mathcal{F})$ coincides with the core of its extension $(N, v^{\mathcal{F}})$.*

Proof. Since $Core\left(N, v^{\mathcal{F}}\right) \subseteq Core^{+}(N, v, \mathcal{F})$, we have $Core\,(N, v, \mathcal{F}) = \emptyset$ implies $Core\left(N, v^{\mathcal{F}}\right) = \emptyset$. Then we can suppose that $Core^{+}(N, v, \mathcal{F}) \neq \emptyset$. Next, we prove that for all $x \in Core^{+}(N, v, \mathcal{F})$ we have $x \in Core(N, v^{\mathcal{F}})$. Since $x \geq 0$, we obtain

$$\begin{aligned} x(S) &= \sum_{T \in C_{\mathcal{F}}(S)} x(T) + x\left(S \setminus \bigcup_{T \in C_{\mathcal{F}}(S)} T\right) \\ &\geq \sum_{T \in C_{\mathcal{F}}(S)} x(T) \geq \sum_{T \in C_{\mathcal{F}}(S)} v(T) = v^{\mathcal{F}}(S) \end{aligned}$$

for all $S \subseteq N$. ■

Proposition 6. *The non-negative core of the game $(N, v, \mathcal{F})$ is a polyhedron contained in the convex cone $\mathbb{R}_{+}^{N} = \{x \in \mathbb{R}^{N} : x_i \geq 0$ for all $i \in N\}$. Moreover, $Core^{+}(N, v, \mathcal{F})$ is a bounded polyhedron or polytope.*

Proof. Since $Core^{+}(N, v, \mathcal{F})$ is given by a finite set of inequalities and

$$|x_i| = x_i \leq \sum_{i \in N} x_i = v^{\mathcal{F}}(N)$$

for every component of each $x \in Core^{+}(N, v, \mathcal{F})$, we obtain the property. ■

Let us consider a standard cooperative game $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$, and a total ordering of the elements of N, given by $i_1 < i_2 < \cdots < i_n$. Then we obtain the following chain of coalitions

$$\emptyset \subset C_1 \subset \cdots \subset C_{n-1} \subset C_n = N$$

where $C_k = \{i_1, \ldots, i_k\} \subseteq N$, for all $k = 1, \ldots, n$. The *marginal worth vector* $a^C \in \mathbb{R}^n$ with respect to the above total ordering in the game (N, v) is given by $a_{i_k}^C = v(C_k) - v(C_{k-1})$, for all $k = 1, \ldots, n$.

The *Weber set* of the game (N, v) is the convex hull of the marginal worth vectors, i.e., $Weber(N, v) = \text{conv}\{a^C : C$ is a total ordering of $N\}$. It is easy to show that

$$\sum_{j=1}^{k} a_{i_j}^C = v(C_k) \quad \text{for all } k = 1, \ldots, n.$$

In our model, we can consider the compatible orderings of an augmenting system $(N, \mathcal{F})$ with $N \in \mathcal{F}$, as the total orderings of N, $i_1 < i_2 < \cdots < i_n$ such that the set $\left\{i_1, \ldots, i_j\right\} \in \mathcal{F}$ for all $j = 1, \ldots, n$. A compatible ordering of $(N, \mathcal{F})$ corresponds exactly to a chain of length n in $\mathcal{F}$ and we denote by $Ch(\mathcal{F})$ the set of all the chains of length n of $\mathcal{F}$.

Definition 7. The Weber set of a game $(N, v, \mathcal{F})$ is given by

$$Weber(N, v, \mathcal{F}) = \text{conv}\{a^C : C \in Ch(\mathcal{F})\}.$$

The next property follows from the definition of a^C.

Proposition 8. *Let $(N, \mathcal{F})$ be an augmenting system with $N \in \mathcal{F}$. If $(N, v, \mathcal{F})$ is a game and $C \in Ch(\mathcal{F})$ then*

$$\sum_{i \in S} a_i^C = v(S) \quad \textit{for all } S \in C.$$

Weber [13] showed that any game $v : 2^N \to \mathbb{R}$ satisfies $Core\,(N, v) \subseteq Weber\,(N, v)$. This property may not hold for a game $v : \mathcal{F} \to \mathbb{R}_{+}$ in case $\mathcal{F} \subset 2^N$.

Example. Let $N = \{1, 2, 3\}$ and let $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, N\}$. We define the game $v : \mathcal{F} \to \mathbb{R}_+$ by $v(\emptyset) = v(\{1\}) = v(\{2\}) = 0$, $v(\{1, 2\}) = 1$ and $v(\{1, 3\}) = v(N) = 2$. Notice that

$$
\begin{aligned}
Core^+(N, v, \mathcal{F}) &= \left\{x \in \mathbb{R}^3_+ : x(N) = 2, x_1 + x_2 \geq 1, x_1 + x_3 \geq 2\right\} \\
&= \left\{x \in \mathbb{R}^3_+ : x(N) = 2, x_1 \geq 1, x_2 = 0, x_3 \leq 1\right\} \\
&= \text{conv}\{(1, 0, 1), (2, 0, 0)\}.
\end{aligned}
$$

There are three chains of length n in $\mathcal{F}$ given by

$$
\begin{aligned}
&C_1 : \emptyset \subset \{1\} \subset \{1, 2\} \subset \{1, 2, 3\}, \\
&C_2 : \emptyset \subset \{1\} \subset \{1, 3\} \subset \{1, 2, 3\}, \\
&C_3 : \emptyset \subset \{2\} \subset \{1, 2\} \subset \{1, 2, 3\}.
\end{aligned}
$$

The marginal worth vectors are

$$
\begin{aligned}
a^{C_1} &= (v(\{1\}) - v(\emptyset),\ v(\{1, 2\}) - v(\{1\}),\ v(N) - v(\{1, 2\})) = (0, 1, 1), \\
a^{C_2} &= (v(\{1\}) - v(\emptyset),\ v(N) - v(\{1, 3\}),\ v(\{1, 3\}) - v(\{1\})) = (0, 0, 2), \\
a^{C_3} &= (v(\{1, 2\}) - v(\{2\}),\ v(\{2\}) - v(\emptyset),\ v(N) - v(\{1, 2\})) = (1, 0, 1),
\end{aligned}
$$

and hence we obtain

$$Weber(N, v, \mathcal{F}) = \text{conv}\{(1, 0, 1), (0, 1, 1), (0, 0, 2)\}.$$

Then we have that

$$Core^+(N, v, \mathcal{F}) \not\subseteq Weber(N, v, \mathcal{F}) \quad \text{and} \quad Weber(N, v, \mathcal{F}) \not\subseteq Core^+(N, v, \mathcal{F}).$$

4. Monotone convex games

A cooperative game $(N, v, \mathcal{F})$ is *monotone* if for all $S, T \in \mathcal{F}$ with $S \subseteq T$, we have $v(S) \leq v(T)$. In general, the extension $\left(N, v^{\mathcal{F}}\right)$ of a monotone game $(N, v, \mathcal{F})$ is not necessarily monotone as the next example shows.

Example. Let $N = \{1, 2, 3, 4\}$ and we consider the augmenting system

$$\mathcal{F} = \{\emptyset, \{1\}, \{4\}, \{1, 2\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, N\}.$$

The game $v : \mathcal{F} \to \mathbb{R}_+$ defined by $v(S) = 1$ for every non-empty $S \in \mathcal{F}$, and $v(\emptyset) = 0$ is monotone. The extension $v^{\mathcal{F}}$ yields

$$v^{\mathcal{F}}(\{1, 4\}) = v(\{1\}) + v(\{4\}) = 2 > 1 = v^{\mathcal{F}}(N).$$

Shapley [12] introduces the notion of *convexity* for cooperative games $v : 2^N \to \mathbb{R}$ such that $v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$, for all $S, T \subseteq N$. Ichiishi [9] obtained the following characterization of convex games: (N, v) is convex if and only if $Core(N, v) = Weber(N, v)$.

Let us consider a game $v : \mathcal{F} \to \mathbb{R}_+$, where $\mathcal{F} \subseteq 2^N$. In this restricted game, the above convexity inequalities could be applied to feasible coalitions $S, T \in \mathcal{F}$ such that $S \cap T \in \mathcal{F}$ and $S \cup T \in \mathcal{F}$. However, they are not sufficient to ensure that the extended game $v^{\mathcal{F}} : 2^N \to \mathbb{R}_+$ be convex.

Example. Let $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ be an augmenting system and the restricted game $v : \mathcal{F} \to \mathbb{R}_+$ is given by $v(\emptyset) = v(\{1\}) = v(\{2\}) = 0$, and $v(\{1, 3\}) = v(\{2, 3\}) = v(\{1, 2, 3\}) = 1$. This restricted game v is monotone and satisfies the convexity inequalities, but its extended game $v^{\mathcal{F}}$ is not convex because

$$v^{\mathcal{F}}(\{1, 3\}) + v^{\mathcal{F}}(\{2, 3\}) = 2 > 1 = v^{\mathcal{F}}(\{3\}) + v^{\mathcal{F}}(\{1, 2, 3\}).$$

Then we introduce the following concept of convexity for restricted games.

Definition 9. A cooperative game $(N, v, \mathcal{F})$ is said to be convex if for all $S, T \in \mathcal{F}$ with $S \cup T \in \mathcal{F}$, we have

$$v(S) + v(T) \leq \sum_{C \in C_{\mathcal{F}}(S \cap T)} v(C) + v(S \cup T).$$

Note that if $\mathcal{F} = 2^N$ then the above inequality is the classical convexity.

Theorem 10. *Let $(N, \mathcal{F})$ be an augmenting system such that $N \in \mathcal{F}$. If the game $v : \mathcal{F} \to \mathbb{R}_+$ is monotone convex then $Core^+(N, v, \mathcal{F}) \neq \emptyset$.*

Proof. Since $\emptyset, N \in \mathcal{F}$, property (P3) of the augmenting systems implies the existence of a chain of length n of feasible coalitions

$$\emptyset \subset C_1 \subset \cdots \subset C_{n-1} \subset C_n = N$$

where $C_j = \{i_1, \ldots, i_j\} \in \mathcal{F}$ for all $j = 1, \ldots, n$. We define the corresponding marginal worth vector $x \in \mathbb{R}^N_+$ as follows:

$$x_{i_1} = v(C_1), \qquad x_{i_j} = v(C_j) - v(C_{j-1}), \quad j = 2, \ldots, n.$$

Note that

$$x(C_j) = \sum_{k=1}^{j} x_{i_k} = v(C_j)$$

for all $j = 1, \ldots, n$ and $x(N) = v(N)$. Since v is monotone we obtain $x_{i_j} \geq 0$ for all $j = 1, \ldots, n$.

We will show that $x(S) \geq v(S)$ for any $S \in \mathcal{F}$. For this, we consider $S \in \mathcal{F}$ such that $S \neq C_j$ for all $j = 1, \ldots, n$ and suppose that S is a minimal, with respect to the inclusion, feasible coalition such that $x(S) < v(S)$.

Since $\emptyset \subset C_1 \subset \cdots \subset C_{n-1} \subset C_n = N$ is a chain of length n, there exists $j \in \{1, \ldots, n\}$ such that $S \subset C_j$ and $S \nsubseteq C_{j-1}$. Then $i_j \in S$ and hence $S \cup C_{j-1} = C_j \in \mathcal{F}$. By applying the convexity of v to the feasible coalitions S and C_{j-1} we obtain

$$v(S) + v(C_{j-1}) \leq \sum_{T \in C_{\mathcal{F}}(S \cap C_{j-1})} v(T) + v(C_j).$$

Note that S is minimal feasible coalition such that $x(S) < v(S)$ and for each $T \in C_{\mathcal{F}}(S \cap C_{j-1})$ we have $T \subset S$. Thus $x(T) \geq v(T)$ for all $T \in C_{\mathcal{F}}(S \cap C_{j-1})$. Since $x \geq 0$, we deduce

$$\begin{aligned} x(S) + x(C_{j-1}) &= x(S \cup C_{j-1}) + x(S \cap C_{j-1}) \\ &= v(C_j) + x(S \cap C_{j-1}) \\ &\geq v(C_j) + \sum_{T \in C_{\mathcal{F}}(S \cap C_{j-1})} x(T) \\ &\geq v(C_j) + \sum_{T \in C_{\mathcal{F}}(S \cap C_{j-1})} v(T) \\ &\geq v(S) + v(C_{j-1}). \end{aligned}$$

By using $x(C_{j-1}) = v(C_{j-1})$ we conclude that $x(S) \geq v(S)$, which is a contradiction. ■

Corollary 11. *Let $(N, \mathcal{F})$ be an augmenting system such that $N \in \mathcal{F}$. If the game $(N, v, \mathcal{F})$ is monotone convex then*

$$Weber(N, v, \mathcal{F}) \subseteq Core^+(N, v, \mathcal{F}).$$

Moreover, any marginal worth vector is a vertex of $Core^+(N, v, \mathcal{F})$.

Proof. For all $C \in Ch(\mathcal{F})$ the marginal worth vector a^C belongs to the convex set $Core^+(N, v, \mathcal{F})$. Thus $\operatorname{conv}\{a^C : C \in Ch(\mathcal{F})\} \subseteq Core^+(N, v, \mathcal{F})$. Given $a^C \in Core^+(N, v, \mathcal{F})$, we have $a^C(C_j) = v(C_j)$ for all feasible coalition belonging to chain C of length n. Then we obtain n equations which are linearly independent and therefore its solution a^C is a vertex of the polytope $Core^+(N, v, \mathcal{F})$. ■

Definition 12. A game $(N, v, \mathcal{F})$ is superadditive if for all $S, T \in \mathcal{F}$ with $S \cap T = \emptyset$ and $S \cup T \in \mathcal{F}$, we have $v(S) + v(T) \leq v(S \cup T)$.

We observe that every convex game is superadditive.

Proposition 13. *Let $(N, \mathcal{F})$ be an augmenting system such that $N \in \mathcal{F}$. If $Weber(N, v, \mathcal{F}) \subseteq Core^+(N, v, \mathcal{F})$ then the game $(N, v, \mathcal{F})$ is superadditive.*

Proof. Let $S, T \in \mathcal{F}$ with $S \cap T = \emptyset$ and $S \cup T \in \mathcal{F}$. Then there exists a chain C which contains $S \cup T$ and we take $x = a^C \in Core(N, v, \mathcal{F})$. Thus we conclude that $v(S \cup T) = x(S \cup T) = x(S) + x(T) \geq v(S) + v(T)$. ■

5. Hereditary properties of $(N, v, \mathcal{F})$

In this section, we show that the superadditivity and the convexity of a monotone game $(N, v, \mathcal{F})$ imply the corresponding property of the extended game $\left(N, v^{\mathcal{F}}\right)$.

Theorem 14. *Let $(N, \mathcal{F})$ be an augmenting system such that $N \in \mathcal{F}$. If $(N, v, \mathcal{F})$ is a monotone superadditive game then the game $\left(N, v^{\mathcal{F}}\right)$ is superadditive and monotone.*

Proof. Suppose that $(N, v^{\mathcal{F}})$ is not superadditive. Then there exist coalitions $S, T \subseteq N$ with $S \cap T = \emptyset$ and components

$$C_{\mathcal{F}}(S) = \{S_1, \ldots, S_n\}, \qquad C_{\mathcal{F}}(T) = \{T_1, \ldots, T_m\},$$

such that $v^{\mathcal{F}}(S) + v^{\mathcal{F}}(T) > v^{\mathcal{F}}(S \cup T)$ and that n is minimal among those choices of coalitions for which the inequality holds.

If $n \geq 2$ we consider the coalitions $S \setminus S_n$ and S_n. Note that $|C_{\mathcal{F}}(S_n)| = 1$ and $|C_{\mathcal{F}}(S \setminus S_n)| = n - 1$, since otherwise S_n is not a maximal feasible subset of S. Then we have

$$v^{\mathcal{F}}(S \setminus S_n) + v^{\mathcal{F}}(S_n \cup T) \leq v^{\mathcal{F}}(S \cup T),$$
$$v^{\mathcal{F}}(S_n) + v^{\mathcal{F}}(T) \leq v^{\mathcal{F}}(S_n \cup T).$$

Thus, we obtain

$$v^{\mathcal{F}}(S \cup T) \geq v^{\mathcal{F}}(S \setminus S_n) + v^{\mathcal{F}}(S_n) + v^{\mathcal{F}}(T) = v^{\mathcal{F}}(S) + v^{\mathcal{F}}(T),$$

which is a contradiction. Hence we conclude that $n = 1$.

Now $C_{\mathcal{F}}(S) = \{S_1\}$, $C_{\mathcal{F}}(T) = \{T_1, \ldots, T_m\}$, such that $v^{\mathcal{F}}(S) + v^{\mathcal{F}}(T) > v^{\mathcal{F}}(S \cup T)$ and we assume that m is as small as possible among all such counterexamples. We next consider the following three cases.

(i) The set $C_{\mathcal{F}}(S \cup T) = \left\{S_1', T_1', \ldots, T_m'\right\}$ where $S_1 \subseteq S_1'$ and $T_i \subseteq T_i'$ for all $i = 1, \ldots, m$. Since v is monotone,

$$\begin{aligned} v^{\mathcal{F}}(S) + v^{\mathcal{F}}(T) &= v(S_1) + v(T_1) + \cdots + v(T_m) \\ &\leq v\left(S_1'\right) + v\left(T_1'\right) + \cdots + v\left(T_m'\right) \\ &= v^{\mathcal{F}}(S \cup T), \end{aligned}$$

and this gives a contradiction.

(ii) The set $C_{\mathcal{F}}(S \cup T) = \left\{C, T_{p+1}', \ldots, T_m'\right\}$ where $1 \leq p < m$, $T_i \subseteq T_i'$ for all $i = p+1, \ldots, m$ and $S_1 \cup T_1 \cup \cdots \cup T_p \subseteq C$. Then

$$v^{\mathcal{F}}(S) + v^{\mathcal{F}}(T) = v^{\mathcal{F}}(S) + v^{\mathcal{F}}\left(T \setminus \bigcup_{i=p+1}^{m} T_i\right) + v^{\mathcal{F}}\left(\bigcup_{i=p+1}^{m} T_i\right).$$

Since $|C_{\mathcal{F}}(S)| = 1$ and $\left|C_{\mathcal{F}}\left(T \setminus \bigcup_{i=p+1}^{m} T_i\right)\right| = p < m$, we obtain

$$v^{\mathcal{F}}(S) + v^{\mathcal{F}}\left(T \setminus \bigcup_{i=p+1}^{m} T_i\right) \leq v^{\mathcal{F}}\left((S \cup T) \setminus \bigcup_{i=p+1}^{m} T_i\right).$$

Moreover, using $\left|C_{\mathcal{F}}\left((S \cup T) \setminus \bigcup_{i=p+1}^{m} T_i\right)\right| = 1$ and $\left|C_{\mathcal{F}}\left(\bigcup_{i=p+1}^{m} T_i\right)\right| = m - p < m$, we deduce that

$$v^{\mathcal{F}}\left((S \cup T) \setminus \bigcup_{i=p+1}^{m} T_i\right) + v^{\mathcal{F}}\left(\bigcup_{i=p+1}^{m} T_i\right) \leq v^{\mathcal{F}}(S \cup T).$$

Therefore, $v^{\mathcal{F}}(S) + v^{\mathcal{F}}(T) \leq v^{\mathcal{F}}(S \cup T)$ and this is a contradiction.

(iii) The set $C_{\mathcal{F}}(S \cup T) = \{C^*\}$ where $S_1 \cup T_1 \cup \cdots \cup T_m \subseteq C^*$. Since $S_1 \subset C^*$ and S_1 is a maximal feasible subset of S there exists a chain of feasible coalitions

$$S_1 = S_1^0 \subset S_1^1 \subset \cdots \subset S_1^k \subset \cdots \subset C^*,$$

such that $S_1^k \setminus S_1^{k-1} = \left\{t^k\right\} \subseteq T$ for all $k = 1, \ldots, |C^* \setminus S_1|$. Moreover, $T_1 \cup \cdots \cup T_m \subseteq C^*$ and we may select the first coalition S_1^p in the chain such that $S_1^p \cap T_j \neq \emptyset$ for some $j \in \{1, \ldots, m\}$. It follows that $S_1^p \cup T_j \in \mathcal{F}$. Note that $S_1^p \cap T_j = \{t^p\}$ and hence $S_1^p \setminus \{t^p\} \in \mathcal{F}$ satisfying

$$\left(S_1^p \setminus \left\{t^p\right\}\right) \cup T_j = S_1^p \cup T_j \text{ and } \left(S_1^p \setminus \left\{t^p\right\}\right) \cap T_j = \emptyset.$$

By using the superadditivity of v, we have

$$v\left(S_1^p \setminus \left\{t^p\right\}\right) + v\left(T_j\right) \leq v\left(S_1^p \cup T_j\right) = v^{\mathcal{F}}\left(S_1^p \cup T_j\right).$$

Therefore,

$$\begin{aligned} v^{\mathcal{F}}(S) + v^{\mathcal{F}}(T) &= v(S_1) + v\left(T_j\right) + v^{\mathcal{F}}\left(T \setminus T_j\right) \\ &\leq v\left(S_1^p \setminus \left\{t^p\right\}\right) + v\left(T_j\right) + v^{\mathcal{F}}\left(T \setminus T_j\right) \\ &\leq v^{\mathcal{F}}\left(S_1^p \cup T_j\right) + v^{\mathcal{F}}\left(T \setminus T_j\right). \end{aligned}$$

Since $\left|C_{\mathcal{F}}(S_1^p \cup T_j)\right| = 1$ and $\left|C_{\mathcal{F}}\left(T \setminus T_j\right)\right| = m - 1 < m$, we obtain

$$\begin{aligned} v^{\mathcal{F}}\left(S_1^p \cup T_j\right) + v^{\mathcal{F}}\left(T \setminus T_j\right) &\leq v^{\mathcal{F}}\left(S_1^p \cup T\right) \\ &= v^{\mathcal{F}}\left(S_1 \cup T\right) = v^{\mathcal{F}}\left(S \cup T\right), \end{aligned}$$

and this contradiction completes the proof of the superadditivity of $v^{\mathcal{F}}$.

Finally, we will show that $v^{\mathcal{F}}$ is monotone. Let $S, T \subseteq N$ with $S \subseteq T$. Since $v^{\mathcal{F}}$ is superadditive and $v^{\mathcal{F}}(T \setminus S) \geq 0$, we conclude that $v^{\mathcal{F}}(S) \leq v^{\mathcal{F}}(S) + v^{\mathcal{F}}(T \setminus S) \leq v^{\mathcal{F}}(T)$. ■

Remark 15. We present in the Appendix an alternative proof of Theorem 14 proposed by an anonymous reviewer.

Theorem 16. *Let $(N, \mathcal{F})$ be an augmenting system such that $N \in \mathcal{F}$. If $(N, v, \mathcal{F})$ is a monotone convex game then the game $\left(N, v^{\mathcal{F}}\right)$ is convex.*

Proof. We remark first that $\left(N, v^{\mathcal{F}}\right)$ is superadditive and monotone. Next, suppose that $\left(N, v^{\mathcal{F}}\right)$ is not convex. Then there exist coalitions $S, T \subseteq N$ with components

$$C_{\mathcal{F}}(S) = \{S_1, \ldots, S_n\}, \qquad C_{\mathcal{F}}(T) = \{T_1, \ldots, T_m\},$$

such that $v^{\mathcal{F}}(S) + v^{\mathcal{F}}(T) > v^{\mathcal{F}}(S \cap T) + v^{\mathcal{F}}(S \cup T)$ and that n is minimal among all pairs of coalitions satisfying the above inequality.

Suppose that $n \geq 2$ and let $S \setminus S_n$ and S_n, where $|C_{\mathcal{F}}(S_n)| = 1$ and $|C_{\mathcal{F}}(S \setminus S_n)| = n - 1$. Then we have

$$\begin{aligned} &v^{\mathcal{F}}(S \setminus S_n) + v^{\mathcal{F}}(S_n \cup T) \leq v^{\mathcal{F}}((S \setminus S_n) \cap T) + v^{\mathcal{F}}(S \cup T), \\ &v^{\mathcal{F}}(S_n) + v^{\mathcal{F}}(T) \leq v^{\mathcal{F}}(S_n \cap T) + v^{\mathcal{F}}(S_n \cup T). \end{aligned}$$

We note that

$$\begin{aligned} &((S \setminus S_n) \cap T) \cup (S_n \cap T) = ((S \setminus S_n) \cup S_n) \cap T = S \cap T, \\ &((S \setminus S_n) \cap T) \cap (S_n \cap T) = \emptyset. \end{aligned}$$

The superadditivity of $v^{\mathcal{F}}$ implies

$$v^{\mathcal{F}}(S \cap T) \geq v^{\mathcal{F}}((S \setminus S_n) \cap T) + v^{\mathcal{F}}(S_n \cap T).$$

Therefore,

$$\begin{aligned} v^{\mathcal{F}}(S \cap T) &\geq v^{\mathcal{F}}(S \setminus S_n) + v^{\mathcal{F}}(S_n \cup T) - v^{\mathcal{F}}(S \cup T) + v^{\mathcal{F}}(S_n) + v^{\mathcal{F}}(T) - v^{\mathcal{F}}(S_n \cup T) \\ &= v^{\mathcal{F}}(S) + v^{\mathcal{F}}(T) - v^{\mathcal{F}}(S \cup T), \end{aligned}$$

and this contradicts the hypothesis that $n \geq 2$.

We thus have $C_{\mathcal{F}}(S) = \{S_1\}$ and we assume that m is as small as possible. There are two possible cases:

(i) The intersection $S_1 \cap \left(\bigcup_{j=1}^m T_j\right) = \emptyset$ so that $S_1 \cap T_j = \emptyset$ for all $j = 1, \ldots, m$. Since $v^{\mathcal{F}}$ is superadditive and monotone, we obtain

$$\begin{aligned} v^{\mathcal{F}}(S) + v^{\mathcal{F}}(T) &= v^{\mathcal{F}}(S_1) + v^{\mathcal{F}}\left(\bigcup_{j=1}^m T_j\right) \\ &\leq v^{\mathcal{F}}\left(S_1 \cup T_1 \cup \cdots \cup T_m\right) \\ &\leq v^{\mathcal{F}}(S \cup T), \end{aligned}$$

which is a contradiction.

(ii) There exists at least $p \in \{1, \ldots, m\}$ such that $S_1 \cap T_p \neq \emptyset$. Since $S_1 \cap T_p \neq \emptyset$, property (P2) implies that $S_1 \cup T_p \in \mathcal{F}$ and applying the convexity of v,

$$\begin{aligned} v^{\mathcal{F}}(S) + v^{\mathcal{F}}(T) &= v\left(S_1\right) + v\left(T_p\right) + v^{\mathcal{F}}\left(T \setminus T_p\right) \\ &\leq \sum_{C \in C_{\mathcal{F}}(S_1 \cap T_p)} v\left(C\right) + v\left(S_1 \cup T_p\right) + v^{\mathcal{F}}\left(T \setminus T_p\right) \\ &= v^{\mathcal{F}}(S_1 \cap T_p) + v^{\mathcal{F}}\left(S_1 \cup T_p\right) + v^{\mathcal{F}}\left(T \setminus T_p\right). \end{aligned}$$

By using the monotonicity of $v^{\mathcal{F}}$, we obtain

$$v^{\mathcal{F}}(S) + v^{\mathcal{F}}(T) \leq v^{\mathcal{F}}(S \cap T_p) + v^{\mathcal{F}}\left(S \cup T_p\right) + v^{\mathcal{F}}\left(T \setminus T_p\right).$$

Since $\left|C_{\mathcal{F}}(S \cup T_p)\right| = 1$ and $\left|C_{\mathcal{F}}(T \setminus T_p)\right| = m - 1 < m$, we have

$$v^{\mathcal{F}}\left(S \cup T_p\right) + v^{\mathcal{F}}\left(T \setminus T_p\right) \leq v^{\mathcal{F}}\left(\left(S \cup T_p\right) \cap \left(T \setminus T_p\right)\right) + v^{\mathcal{F}}(S \cup T) = v^{\mathcal{F}}\left(S \cap \left(T \setminus T_p\right)\right) + v^{\mathcal{F}}(S \cup T).$$

From the superadditivity of $v^{\mathcal{F}}$, we deduce that

$$v^{\mathcal{F}}(S \cap T_p) + v^{\mathcal{F}}\left(S \cap \left(T \setminus T_p\right)\right) \leq v^{\mathcal{F}}(S \cap T).$$

Therefore,

$$v^{\mathcal{F}}(S) + v^{\mathcal{F}}(T) \leq v^{\mathcal{F}}(S \cap T) + v^{\mathcal{F}}(S \cup T),$$

and this contradiction completes the proof. ■

Corollary 17. *Let $(N, \mathcal{F})$ be an augmenting system such that $N \in \mathcal{F}$ and let $(N, v, \mathcal{F})$ be a monotone game. Then $(N, v, \mathcal{F})$ is convex if and only if $\left(N, v^{\mathcal{F}}\right)$ is convex.*

Proof. It is sufficient to show that the convexity of $\left(N, v^{\mathcal{F}}\right)$ implies that $(N, v, \mathcal{F})$ is convex. Let us consider $S, T \in \mathcal{F}$ with $S \cup T \in \mathcal{F}$. Then

$$v^{\mathcal{F}}(S) + v^{\mathcal{F}}(T) \leq v^{\mathcal{F}}(S \cap T) + v^{\mathcal{F}}(S \cup T),$$

which implies

$$v(S) + v(T) \leq \sum_{C \in C_{\mathcal{F}}(S \cap T)} v(C) + v(S \cup T),$$

and we obtain the result. ■

By using Proposition 5 and the classical characterization of convex games, we obtain the following characterization of the convexity for games on augmenting systems.

Theorem 18. *Let $(N, \mathcal{F})$ be an augmenting system such that $N \in \mathcal{F}$ and let $(N, v, \mathcal{F})$ be a monotone game. Then $(N, v, \mathcal{F})$ is convex if and only if $Core^{+}(N, v, \mathcal{F}) = Weber\left(N, v^{\mathcal{F}}\right)$.*

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Appendix

An anonymous reviewer propose the following proof of Theorem 14. In our opinion it is a nice and transparent demonstration.

Proof. We split up the proof into two parts. First, we prove that monotonicity and superadditivity of a restricted game $(N, v, \mathcal{F})$ imply that $(N, v, \mathcal{F})$ is *proper*, that is, $v(S) \geq v(S_1) + \cdots + v(S_k)$ for all $S \in \mathcal{F}$, and pairwise disjoint subsets $S_1, \ldots, S_k \in \mathcal{F}$ of S.

Suppose on the contrary that

$$v(S) < \sum_{j=1}^{k} v\left(S_j\right), \tag{1}$$

where $S \in \mathcal{F}$, and $S_1, \ldots, S_k \in \mathcal{F}$ are pairwise disjoint subsets of S. We may choose these feasible coalitions such that $k + \left|S'\right|$ is minimal among those choices of feasible coalitions for which (1) holds, with $S' = S \setminus \bigcup_{j=1}^{k} S_j$. Since $(N, \mathcal{F})$ is an augmenting system and $S_1 \subset S$, it follows from property (P3) that there exists $i \in S \setminus S_1$ such that $S_1 \cup i \in \mathcal{F}$. If $i \in S'$ then due to the choice of S and $S_1, \ldots, S_k$ we must have

$$v(S) \geq v(S_1 \cup i) + \sum_{j=2}^{k} v\left(S_j\right) = \sum_{j=1}^{k} v\left(S_j\right) + v(S_1 \cup i) - v(S_1) > v(S) + v(S_1 \cup i) - v(S_1),$$

and hence $v(S_1) > v(S_1 \cup i)$. Applying monotonicity we arrive at a contradiction, and we therefore conclude that $i \notin S'$, i.e., there exists an index $r \neq 1$ with $i \in S_r$.

Let S'' denote $(S_1 \cup i) \cup S_r = S_1 \cup S_r$, which is feasible according to property (P2). Again, due to the choice of S and $S_1, \ldots, S_k$ we must have

$$v(S) \geq v(S'') + \sum_{j=2, j\neq r}^{k} v(S_j) > v(S) + v(S'') - v(S_1) - v(S_r).$$

Therefore, $v(S_1 \cup S_r) = v(S'') < v(S_1) + v(S_r)$ and this contradicts the superadditivity of $(N, v, \mathcal{F})$. We conclude that $(N, v, \mathcal{F})$ is proper.

Next, we will show that the extended game $(N, v^{\mathcal{F}})$ is superadditive. Let $S, T \subseteq N$ be disjoint coalitions. Let $S_1, \ldots, S_s \in \mathcal{F}$, $T_1, \ldots, T_t \in \mathcal{F}$, and $U_1, \ldots, U_m \in \mathcal{F}$ denote the components of S, T, and $S \cup T$. Observe that $S_k \subseteq U_j$ or $S_k \cap U_j = \emptyset$ for each combination of components S_k and U_j since otherwise (P2) implies $U_j \subset S_k \cup U_j \in \mathcal{F}$ contradicting the maximality of U_j in $S \cup T$. The same holds for the components of T and U.

Since the restricted game $(N, v, \mathcal{F})$ is proper and the components of S and T are pairwise disjoint, we have for each $j = 1, \ldots, m$,

$$v(U_j) \geq \sum_{k=1}^{s} v(U_j \cap S_k) + \sum_{r=1}^{t} v(U_j \cap T_r).$$

For each index $k = 1, \ldots, s$ we have $S_k \subseteq U_q$ for exactly one index $q \in \{1, \ldots, m\}$, and this holds also for the components of T. Therefore, we have

$$v(S_k) = \sum_{j=1}^{m} v(U_j \cap S_k) \quad \text{for each } k = 1, \ldots, s,$$

$$v(T_r) = \sum_{j=1}^{m} v(U_j \cap T_r) \quad \text{for each } r = 1, \ldots, t.$$

We now deduce that

$$\begin{aligned} v^{\mathcal{F}}(S \cup T) &= \sum_{j=1}^{m} v(U_j) \geq \sum_{j=1}^{m} \left(\sum_{k=1}^{s} v(U_j \cap S_k) + \sum_{r=1}^{t} v(U_j \cap T_r) \right) \\ &= \sum_{k=1}^{s} \sum_{j=1}^{m} v(U_j \cap S_k) + \sum_{r=1}^{t} \sum_{j=1}^{m} v(U_j \cap T_r) \\ &= \sum_{k=1}^{s} v(S_k) + \sum_{r=1}^{t} v(T_r) \\ &= v^{\mathcal{F}}(S) + v^{\mathcal{F}}(T). \end{aligned}$$

This shows the superadditivity of $(N, v^{\mathcal{F}})$. ■

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