Ann Oper Res (2008) 158: 117–131
DOI 10.1007/s10479-007-0244-7

Convexity properties for interior operator games

J.M. Bilbao · C. Chacón · A. Jiménez-Losada · E. Lebrón

Published online: 13 September 2007

Abstract Interior operator games arose by abstracting some properties of several types of cooperative games (for instance: peer group games, big boss games, clan games and information market games). This reason allow us to focus on different problems in the same way. We introduced these games in Bilbao et al. (Ann. Oper. Res. 137:141–160, 2005) by a set system with structure of antimatroid, that determines the feasible coalitions, and a non-negative vector, that represents a payoff distribution over the players. These games, in general, are not convex games. The main goal of this paper is to study under which conditions an interior operator game verifies other convexity properties: 1-convexity, k-convexity ($k \geq 2$) or semiconvexity. But, we will study these properties over structures more general than antimatroids: the interior operator structures. In every case, several characterizations in terms of the gap function and the initial vector are obtained. We also find the family of interior operator structures (particularly antimatroids) where every interior operator game satisfies one of these properties.

Keywords Cooperative game · Antimatroid · Interior operator · Convexity

1 Introduction

A cooperative game describes a situation in which a finite set of n players can generate certain payoffs by cooperation, which are given by a function, named characteristic function. In the following sections we are going to consider a particular case of restricted games, this is a class of games where a certain family $\mathcal{A}$ of coalitions describes the real possibilities of cooperation among the players. To be exact, *interior operator games* are restricted additive games where the feasible coalitions set $\mathcal{A}$ is an *antimatroid*, a known combinatorial structure introduced in 1940 by Dilworth (1940), and the profits that players generate to participate in a coalition are given by a non-negative vector w. The structure of antimatroid defines the

J.M. Bilbao (✉) · C. Chacón · A. Jiménez-Losada · E. Lebrón
Matemática Aplicada II, Escuela Superior de Ingenieros, Camino de los Descubrimientos s/n,
41092 Sevilla, Spain
e-mail: mbilbao@us.es

feasible coalitions based on a certain dependency relationships among the players. Jiménez-Losada (1998) introduced antimatroids in games, defining games on the coalitions of the set system given by an antimatroid. Algaba et al. (2004) have also studied restricted games on antimatroids. A generalization of antimatroids are the *interior operator structures* based on an interior operator.

We introduced interior operator games in Bilbao et al. (2005), showing that our study can be applicable to several known situations: games on permission structures, peer group games, information market games, etc. The worth of a coalition is the maximum worth of the feasible coalitions contained in it. We proved that these games satisfy some used properties in cooperative games: they are monotonic, superadditive and totally balanced. The core of a cooperative game was introduced in Gillies (1953) as the payoff vectors set which are efficient and verify the coalitional rationality principle. A game is convex if its characteristic function is convex. Convex games have many important properties as Driessen (1988) explains in Chap. V. But, in general, the interior operator games are not convex. We found conditions over the initial vector w and the antimatroid $\mathcal{A}$ to be a convex game in Bilbao et al. (2005). These conditions are valid to describe the convexity property on interior operator structures.

Driessen introduced other concepts near to the convexity notion in Driessen (1988). These families of games have good properties too, in the same sense of the convexity. The family of the k-convex games for k between 1 and n is defined making a special convex cover of the games, see Chap. VII in Driessen (1988), where the n-convexity coincides with the convexity. A particular and more interesting case is the 1-convexity (for instance to calculate the Tijs value). Driessen (1988) studied this case and another related concept, the semiconvexity, in Sects. 5 and 6, Chap. III. Since our games are not always convex we now propose to answer similar questions to the above ones about these other convexity properties. When does an interior operator game verify any of these properties? and which are the interior operator structures where all the interior operator games satisfy one of these convexity properties?

The paper is organized as follows. In Sect. 2 we describe some preliminaries about interior operator structures, antimatroids, cooperative games, interior operator games and convex interior operator games. In the Sects. 3, 4 and 5 we explain the main theorems answering the proposed questions about the mentioned convexity properties: 1-convexity, k-convexity and semiconvexity. Last section summarizes the conclusions of the above sections, and states the relationships among the different convexity concepts studied.

2 Interior operator games

Very well-known combinatorial structures are antimatroids, introduced by Dilworth (1940) as particular cases of semimodular lattices. The reader can use (Korte et al. 1991) for more details about this structure.

Let N be a finite set. An *interior operator* on N is a function $int : 2^N \to 2^N$ that verifies the following properties: (I1) $int(S) \subseteq S$ for all $S \subseteq N$, (I2) if $S \subseteq T \subseteq N$ then $int(S) \subseteq int(T)$, (I3) $int(int(S)) = int(S)$, and (I4) $int(N) = N$. The set system defined by an interior operator is $(N, \mathcal{A})$ with the family

$$\mathcal{A} = \{S \subseteq N : int(S) = S\}.$$

We name it as *interior operator structures* and the subsets in $\mathcal{A}$ are *feasible sets*. It is easy to prove that a set system $(N, \mathcal{A})$ with $\mathcal{A} \subseteq 2^N$ is an interior operator structure if and only if

(A1) $\emptyset, N \in \mathcal{A}$, and (A2) if $S, T \in \mathcal{A}$ then $S \cup T \in \mathcal{A}$. In order to give an interior operator for this system we write

$$int(S) = \bigcup_{\{T \in \mathcal{A}: T \subseteq S\}} T.$$

A normal *antimatroid* $(N, \mathcal{A})$ is an interior operator structure adding only one more condition: accessibility, this is if $S \in \mathcal{A}$, and $S \neq \emptyset$ then there is $e \in S$ with $S \setminus \{e\} \in \mathcal{A}$.

Interior operator structures are dual of the closure operator structure. A *closure operator* over a finite set N is a function $\overline{} : 2^N \to 2^N$ satisfying: (C1) $S \subseteq \overline{S}$ for all $S \subseteq N$, (C2) if $S \subseteq T \subseteq N$ then $\overline{S} \subseteq \overline{T}$, (C3) $\overline{\overline{S}} = \overline{S}$, and (C4) $\overline{\emptyset} = \emptyset$. The *closure operator structure* is $(N, \mathcal{L})$ with the family $\mathcal{L} = \{S \subseteq N : \overline{S} = S\}$. A set system $(N, \mathcal{L})$ is a closure operator structure if and only if (L1) $\emptyset, N \in \mathcal{L}$, and (L2) if $S, T \in \mathcal{L}$ then $S \cap T \in \mathcal{A}$. These structures are duals, and there is a relationship between both operators. We remain that if $(N, \mathcal{A})$ is a set system its *dual set system* is the structure $(N, \mathcal{L})$ with $\mathcal{L} = \{S \subseteq N : N \setminus S \in \mathcal{F}\}$. The set system $(N, \mathcal{A})$ is an interior operator structure if and only if its dual set system $(N, \mathcal{L})$ is a closure operator structure. Therefore, for all $S \subseteq N$ the closure operator of $(N, \mathcal{L})$ and the interior operator of $(N, \mathcal{A})$ verify (IC) $N \setminus \overline{S} = int(N \setminus S)$ for all $S \subseteq N$.We name *poset structure* to a set system which is interior operator structure and closure operator structure. In particular, if the interior operator structure is an antimatroid we use *poset antimatroid*.

Let $(N, \mathcal{A})$ an interior operator structure and $e \in N$. An *e-path* is a minimal feasible set containing e. The family of e-path is denoted by $A(e)$. Following Theorem 2.1 in Goecke et al. (1986) for poset antimatroids, we characterize poset structures by paths: an interior operator structure $(N, \mathcal{A})$ is a poset structure if and only if every $e \in N$ has just one e-path. Elements $e \in N$ such that $\{e\} \in \mathcal{A}$ are called *atoms* and the set of atoms in $\mathcal{A}$ is $a(\mathcal{A})$. For those $e \in N$ that satisfy $N \setminus \{e\} \in \mathcal{A}$ we will use *coatoms*, and the coatoms form the set $ca(\mathcal{A})$.

A cooperative game (TU-game) is a pair (N, v) where N is a finite set and $v : 2^N \to \mathbb{R}$ is a function with $v(\emptyset) = 0$. Elements of the set $N = \{1, 2, \dots, n\}$ are called *players*, the subsets $S \subseteq N$ *coalitions* and $v(S)$ is the *worth* of S. The number of players in a coalition S is denoted by $|S|$. We will consider games where $v(S)$ is the maximal profit for the players in the coalition S. We denote the set of n-person TU-games as $\Gamma(N)$.

In a cooperative game (N, v), assuming that the grand coalition N will be formed, a *solution concept* prescribes how distribute the profit $v(N)$ among the players. In particular, a *value* is a solution concept that assigns to each game just one payment for each player. That is, a function $\Psi : \Gamma(N) \to \mathbb{R}^N$ where $\Psi(v)$ is the allocation vector that corresponds to the game (N, v). Gillies (1953) introduced the *core* of a cooperative game $v \in \Gamma(N)$ as

$$Core(N, v) = \{x \in \mathbb{R}^N \colon\ x(N) = v(N),\ x(S) \geq v(S) \text{ for all } S \subseteq N\}.$$

Balanced games are the games which have nonempty core.

A cooperative game (N, v) is *convex* if its characteristic function verifies the following condition: $v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$ for all $S, T \subseteq N$.

Let $v \in \Gamma(N)$, the *upper vector* $M^v \in \mathbb{R}^n$ is the vector of the marginal contributions of the players with respect to the grand coalition N, and its components are $M_e^v = v(N) - v(N \setminus \{e\})$ for all $e \in N$. Tijs (1981) proved that the upper vector is an upper bound of the core of a game. The *gap function* $g^v : 2^N \to \mathbb{R}$ given by $g^v(S) = \sum_{e \in S} M_e^v - v(S)$, for all $S \subseteq N$, measures the excesses with respect to this upper vector. The *lower vector* $m^v \in \mathbb{R}^n$

is a lower bound of the core defined by

$$m_e^v = M_e^v - \min\{g^v(S)\colon\ S \subseteq N,\ e \in S\}, \quad \text{for all } e \in N.$$

Let $(N, \mathcal{A})$ be an interior operator structure, $w \in \mathbb{R}_+^N$ a vector such that $w \geq 0$, and (N, w) the additive game where $w(S) = \sum_{e\in S} w_e$ for all nonempty $S \subseteq N$, $w(\emptyset) = 0$. In these conditions, and considering $|N| \geq 2$ throughout this work, we introduce interior operator games. In Bilbao et al. (2005) the authors introduced the particular case of these games on antimatroids.

Definition 1 The interior operator game $(N, w_{\mathcal{A}})$ is the cooperative game $w_{\mathcal{A}} : 2^N \to R$ defined by $w_{\mathcal{A}}(S) = w(int(S))$ for all $S \subseteq N$.

In this game the best feasible coalition for the players of S is its interior, $int(S)$. The game $w_{\mathcal{A}}$ associates to each $S \in \mathcal{A}$ its value in the additive game, $w(S)$. The benefit that a player $e \in N$ can obtain in the game depends on the players of his e-paths. We say that $P_e = \bigcap_{S\in A(e)} S$ is the set of players that *control* to player e.

We use the notation $\overline{\{e\}}$ to indicate the closure of the individual coalition $\{e\}$ in the dual set system $(N, \mathcal{L})$ of the interior operator structure $(N, \mathcal{A})$. This closure set makes possible to represent the players who are controlled by any player $e \in N$. Observe this idea in the next result.

Lemma 2 *Let $(N, \mathcal{A})$ be an antimatroid. Then, for all $e \in N$ and all $S \subseteq N$, the following statements hold*:

(a) $e \in \overline{S} \iff S \cap T \neq \emptyset$ *for all* $T \in A(e)$.
(b) $\overline{\{e\}} = \{e' \in N\colon\ e \in P_{e'}\}$. *That is,* $\overline{\{e\}}$ *is the set of players whose profits in the game* $(N, w_{\mathcal{A}})$ *are controlled by the player* e.
(c) $e' \in \overline{\{e\}} \iff \overline{\{e, e'\}} = \overline{\{e\}}$.

Proof (a) Property (IC) implies that $e \in \overline{S}$ if and only if $e \notin int(N \setminus S)$. Since e-paths are the minimal feasible coalitions containing e, by definition of path, we have $e \in int(N \setminus S)$ if and only if there exists $T \in A(e)$ such that $T \subseteq int(N \setminus S)$. Hence, we deduce that $e \in \overline{S}$ if and only if for all $T \in A(e)$ it holds $T \nsubseteq int(N \setminus S)$. Finally, we conclude that $e \in \overline{S}$ if and only for all $T \in A(e)$ it holds $T \nsubseteq N \setminus S$, since if $T \in A(e)$ were contained in $N \setminus S$ so $int(T) = T \subseteq int(N \setminus S)$.

(b) By the above section, taking $S = \{e\}$ it follows that: $e' \in \overline{\{e\}}$ if and only if for all $T \in A(e')$ it holds that $\{e\} \cap T \neq \emptyset$. That is, $e' \in \overline{\{e\}}$ if and only if for all $T \in A(e')$ it holds $e \in T$. Therefore, $\overline{\{e\}} = \{e' \in N\colon\ e \in P_{e'}\}$ because $P_{e'} = \bigcap_{T\in A(e')} T$.

(c) By Property (C1) it is obvious that if $\overline{\{e, e'\}} = \overline{\{e\}}$, so $e' \in \overline{\{e\}}$. To derive the converse statement it is sufficient to prove that $\overline{\{e, e'\}} \subseteq \overline{\{e\}}$ when $e' \in \overline{\{e\}}$. Consider $e^* \in \overline{\{e', e\}}$. Then it follows from the first section of this lemma that $T \cap \{e', e\} \neq \emptyset$ for all $T \in A(e^*)$. That is, all e^*-path contains player e or player e'. Now we will deduce that all e^*-path contains player e because of $e' \in \overline{\{e\}}$, so we will conclude that $e^* \in \overline{\{e\}}$. Let $T \in A(e^*)$ such that $e' \in T$, then there exists $T' \in A(e')$ such that $T' \subseteq T$, therefore $e \in P_{e'} \subseteq T' \subseteq T$. □

In Bilbao et al. (2005, Theorem 3 and Corollary 4) we found conditions to be convex an interior operator game on antimatroids, and the set of antimatroids where every interior operator game is convex. It is easy to extend these results to interior operator structure in general.

Theorem 3 *An interior operator game* $(N, w_{\mathcal{A}})$ *is convex if and only if* $w_e = 0$ *for all* $e \in N$ *with* $|A(e)| \geq 2$.

Theorem 4 *An interior operator structure is a poset structure if and only if all the interior operator games defined on it are convex.*

Using the above theorem we know that the interior operator games corresponding to Games on permission structures (in the Conjunctive case) (Gilles et al. 1992), Peer-group games (Brânzei et al. 2002), Big-boss games (Muto et al. 1987) (as Driessen proved in (1991) or Modified Clan games (Bilbao et al. 2005, Example 6) are convex games for all $w \in \mathbb{R}_+^N$, because they are defined on poset antimatroids. But, for instance, Information market games (Muto et al. 1986) are not always convex games.

Now, to study the other convexity properties first we need to calculate the gap function. We find a formula to calculate the upper vector of an interior operator game by the property (IC),

$$\begin{aligned} M_e^{w_{\mathcal{A}}} &= w_{\mathcal{A}}(N) - w_{\mathcal{A}}(N \setminus \{e\}) = w(N) - w\big(int(N \setminus \{e\})\big) \\ &= w\big(N \setminus int(N \setminus \{e\})\big) = w\big(\overline{\{e\}}\big). \end{aligned}$$

We show this formula in the next lemma.

Lemma 5 *Let* $(N, w_{\mathcal{A}})$ *be an interior operator game. The upper vector of this game is* $M_e^{w_{\mathcal{A}}} = w(\overline{\{e\}})$, *for all* $e \in N$.

The upper vector is a basic tool to define the gap function. If $(N, w_{\mathcal{A}})$ is an interior operator game. In this work, as we said above, we are interested in studying some interior operator games which can be defined by the gap function. Next we will state a result that we will use frequently.

Lemma 6 *Let* $(N, w_{\mathcal{A}})$ *be an interior operator game. For all* $S \subseteq N$,

$$g^{w_{\mathcal{A}}}(S) = \sum_{e \in S} w\big(\overline{\{e\}} \setminus \{e\}\big) + w(S \setminus int(S)) \geq 0.$$

Therefore

(a) *If* $S \in \mathcal{A}$ *then* $g^{w_{\mathcal{A}}}(S) = \sum_{e \in S} w(\overline{\{e\}} \setminus \{e\})$, *and* $g^{w_{\mathcal{A}}}(S) \leq g^{w_{\mathcal{A}}}(N)$.
(b) *If* $S \notin \mathcal{A}$ *then* $g^{w_{\mathcal{A}}}(S) = g^{w_{\mathcal{A}}}(int(S)) + \sum_{e \in S \setminus int(S)} w(\overline{\{e\}})$.

Proof Let $S \subseteq N$. From the definition and the above lemma we obtain the main claim,

$$\begin{aligned} g^{w_{\mathcal{A}}}(S) &= \sum_{e \in S} M_e^{w_{\mathcal{A}}} - w_{\mathcal{A}}(S) = \sum_{e \in S} w\big(\overline{\{e\}}\big) - w(int(S)) \\ &= \sum_{e \in S} w\big(\overline{\{e\}}\big) - w(S) + w(S \setminus int(S)) \\ &= \sum_{e \in S} w\big(\overline{\{e\}} \setminus \{e\}\big) + w(S \setminus int(S)) \geq 0, \end{aligned}$$

since $w_e \geq 0$ for all $e \in N$.

If $S \in \mathcal{A}$ then $int(S) = S$ and hence $g^{w_{\mathcal{A}}}(S) = \sum_{e \in S} w(\overline{\{e\}} \setminus \{e\})$. In particular, as $w_e \geq 0$ for all $e \in N$, $g^{w_{\mathcal{A}}}(S) \leq g^{w_{\mathcal{A}}}(N)$.

Otherwise, in order to prove (b), if $S \notin \mathcal{A}$

$$\begin{aligned} g^{w_{\mathcal{A}}}(S) &= \sum_{e \in S} w\big(\overline{\{e\}} \setminus \{e\}\big) + w(S \setminus int(S)) \\ &= \sum_{e \in int(S)} w\big(\overline{\{e\}} \setminus \{e\}\big) + \sum_{e \in S \setminus int(S)} w\big(\overline{\{e\}}\big) \\ &= g^{w_{\mathcal{A}}}(int(S)) + \sum_{e \in S \setminus int(S)} w\big(\overline{\{e\}}\big). \end{aligned}$$

□

Lemma 6 expresses that the behavior of the gap function $g^{w_{\mathcal{A}}}$ is fully independent from the weights of the atoms. Let $S \subseteq N$ be a coalition. Notice that if $e \in a(\mathcal{A}) \cap S$ then $e \in int(S)$ and, by Lemma 2(b) and because $P_e = \{e\}$, $e \notin \overline{\{e'\}} \setminus \{e'\}$ for all $e' \in S$. Part (a) expresses the monotonicity of the gap function on $\mathcal{A}$, that is $g^{w_{\mathcal{A}}}(S) \leq g^{w_{\mathcal{A}}}(T)$ if $S \subseteq T$ and $S, T \in \mathcal{A}$. Part (b) expresses the gaps of non-feasible coalitions in that it holds $g^{w_{\mathcal{A}}}(S) \geq g^{w_{\mathcal{A}}}(int(S))$ for all $S \in 2^N \setminus \mathcal{A}$. Particularly, if $e \in a(\mathcal{A})$ then $g^{w_{\mathcal{A}}}(\{e\}) = w(\overline{\{e\}} \setminus \{e\})$ and if $e \in N \setminus a(\mathcal{A})$ then $g^{w_{\mathcal{A}}}(\{e\}) = w(\overline{\{e\}})$ because $int(\{e\}) = \emptyset$.

3 1-convex interior operator games

Driessen introduced the concept of 1-convex game in (Driessen 1988, Sect. 5, Chap. III). A cooperative game (N, v) is said to be an 1-*convex* game if its gap function satisfies $0 \leq g^v(N) \leq g^v(S)$ for all $S \subseteq N$, $S \neq \emptyset$.

Observe that if $\mathcal{A} = 2^N$ then the game $(N, w_{\mathcal{A}})$ is 1-convex for any vector $w \in \mathbb{R}^n_+$. In fact, if $\mathcal{A} = 2^N$, the game $(N, w_{\mathcal{A}})$ is the additive game (N, w) and since $\overline{\{e\}} = \{e\}$ for all $e \in N$, it holds

$$\begin{aligned} g^w(S) &= \sum_{e \in S} M_e^w - w(S) = \sum_{e \in S} w\big(\overline{\{e\}}\big) - w(S) \\ &= \sum_{e \in S} w(\{e\}) - w(S) = 0, \qquad \text{for all } S \subseteq N,\ S \neq \emptyset, \end{aligned}$$

so the game $(N, w_{\mathcal{A}})$ is 1-convex.

Furthermore, any interior operator game $(N, w_{\mathcal{A}})$ is 1-convex when $|N| = 2$ because the gap function of any nonempty coalition is constant.

For the full class of interior operator games the following statements concerning its gap function are equivalents to each other due the previous lemmas.

Proposition 7 *Let $(N, w_{\mathcal{A}})$ an interior operator game. Then the following statements are equivalents*:

(a) $g^{w_{\mathcal{A}}}(S) = 0$ *for all* $S \in \mathcal{A}$.
(b) $g^{w_{\mathcal{A}}}(N) = 0$.
(c) $\sum_{e \in N} g^{w_{\mathcal{A}}}(\{e\}) = w(N \setminus a(\mathcal{A}))$.
(d) $M_e^{w_{\mathcal{A}}} = w_e$ *for each* $e \in N$.
(e) *For all* $e \in N$ *such that* $|P_e| \geq 2$ *it holds* $w_e = 0$.

Proof Clearly, part (a) yields (b). The monotonicity of the gap function on $\mathcal{A}$ given by Lemma 6 implies the converse. Using the same lemma we have

$$g^{w_{\mathcal{A}}}(\{e\}) = \begin{cases} w(\overline{\{e\}} \setminus \{e\}), & \text{if } e \in a(\mathcal{A}), \\ w(\overline{\{e\}}), & \text{if } e \notin a(\mathcal{A}). \end{cases}$$

and $g^{w_{\mathcal{A}}}(N) = \sum_{e\in N} w(\overline{\{e\}} \setminus \{e\})$. Then

$$\begin{aligned}\sum_{e\in N} g^{w_{\mathcal{A}}}(\{e\}) &= \sum_{e\in a(\mathcal{A})} w\big(\overline{\{e\}} \setminus \{e\}\big) + \sum_{e\in N\setminus a(\mathcal{A})} w\big(\overline{\{e\}}\big) \\ &= \sum_{e\in N} w\big(\overline{\{e\}} \setminus \{e\}\big) + w(N \setminus a(\mathcal{A})) \\ &= g^{w_{\mathcal{A}}}(N) + w(N \setminus a(\mathcal{A})).\end{aligned}$$

Therefore we obtain that (b) is equivalent to (c). Since $w_e \geq 0$ for all $e \in N$ and (b) $g^{w_{\mathcal{A}}}(N) = \sum_{e\in N} w(\overline{\{e\}}\setminus\{e\}) = 0$ we get $w(\overline{\{e\}}\setminus\{e\}) = 0$ if and only if $M_e^{w_{\mathcal{A}}} = w(\overline{\{e\}}) = w_e$ by using Lemma 5. Part (d) and (b) are also equivalents because $g^{w_{\mathcal{A}}}(N) = \sum_{e\in N} w(\overline{\{e\}} \setminus \{e\})$. Finally, by Lemma 2(b), a player $e \in N$ verifies $|P_e| \geq 2$ if and only if there exists other player $e' \in P_e \setminus \{e\}$ if and only if $e \in \overline{\{e'\}} \setminus \{e'\}$. Hence $w(\overline{\{e\}} \setminus \{e\}) = 0$ for every $e \in N$ if and only if $w_e = 0$ for all $e \in N$ such that $|P_e| \geq 2$, and $g^{w_{\mathcal{A}}}(N) = 0$ is equivalent to (e). □

The next theorem shows that these conditions determine whether or not an interior operator game is 1-convex.

Theorem 8 *Let $(N, w_{\mathcal{A}})$ be an interior operator game where $|N| > 2$. The game $(N, w_{\mathcal{A}})$ is 1-convex if and only if $(N, w_{\mathcal{A}})$ satisfies any condition in the above proposition.*

Proof If we suppose that $(N, w_{\mathcal{A}})$ is an interior operator game verifying condition (b) in Proposition 7, then $g^{w_{\mathcal{A}}}(N) = 0$, and Lemma 6 implies that $g^{w_{\mathcal{A}}}(S) \geq 0 = g^{w_{\mathcal{A}}}(N)$ for all $S \subseteq N$. Hence $(N, w_{\mathcal{A}})$ is an 1-convex game.

Conversely, let $(N, w_{\mathcal{A}})$ be an 1-convex game. We show the condition (e) of Proposition 7, that is, for all $e \in N$ such that $|P_e| \geq 2$ it holds $w_e = 0$. We consider some $e_1 \in N$ such that $|P_{e_1}| \geq 2$ and distinguish two cases.

(i) If $|P_{e_1}| > 2$ then there are two players $e_2 \neq e_3$ such that $e_2, e_3 \in P_{e_1} \setminus \{e_1\}$ and hence $e_1 \in \overline{\{e_2\}} \cap \overline{\{e_3\}}$. Since $g^{w_{\mathcal{A}}}(N) \leq g^{w_{\mathcal{A}}}(\{e_1\})$ by the 1-convexity, which by Lemma 6(b) implies that

$$\sum_{e\in N} w\big(\overline{\{e\}} \setminus \{e\}\big) \leq w\big(\overline{\{e_1\}}\big) - w_{e_1} + w_{e_1},$$

and we obtain

$$\sum_{e\in N\setminus e_1} w\big(\overline{\{e\}} \setminus \{e\}\big) \leq w_{e_1}.$$

Since $e_1 \in \overline{\{e_2\}}$, we get

$$\sum_{e\in N\setminus\{e_1,e_2\}} w\big(\overline{\{e\}} \setminus \{e\}\big) + w\big(\overline{\{e_2\}} \setminus \{e_1, e_2\}\big) \leq 0,$$

which implies that $w(\overline{\{e_3\}} \setminus \{e_3\}) \leq 0$. Furthermore, $e_1 \in \overline{\{e_3\}}$ implies that $w_{e_1} = 0$.

(ii) If $|P_{e_1}| = 2$ then $P_{e_1} \setminus \{e_1\} = \{e_2\}$ and hence $e_1 \in \overline{\{e_2\}}$. Since $|N| > 2$ there exists a player e_3 such that $e_3 \notin P_{e_1}$ and we distinguish two cases. If $e_2 \in P_{e_3}$ then $e_3 \in \overline{\{e_2\}}$ and we use $g^{w_\mathcal{A}}(N) \leq g^{w_\mathcal{A}}(\{e_3\})$ to obtain

$$\sum_{e \in N \setminus e_3} w\big(\overline{\{e\}} \setminus \{e\}\big) \leq w_{e_3}.$$

Since $e_3 \in \overline{\{e_2\}}$, we get $w(\overline{\{e_2\}} \setminus \{e_2, e_3\}) \leq 0$, and $e_1 \in \overline{\{e_2\}}$ implies that $w_{e_1} = 0$.

If $e_2 \notin P_{e_3}$ then there exist a feasible set $S \in A(e_3)$ such that $e_2 \notin S$. Then $g^{w_\mathcal{A}}(N) \leq g^{w_\mathcal{A}}(S)$ implies

$$\sum_{e \in N} w\big(\overline{\{e\}} \setminus \{e\}\big) \leq \sum_{e \in S} w\big(\overline{\{e\}} \setminus \{e\}\big),$$

and hence

$$\sum_{e \in N \setminus S} w\big(\overline{\{e\}} \setminus \{e\}\big) \leq 0.$$

Since $e_2 \notin S$, we obtain $w(\overline{\{e_2\}} \setminus \{e_2\}) \leq 0$, and $e_1 \in \overline{\{e_2\}}$ implies that $w_{e_1} = 0$. □

Conditions (a) and (b) in Proposition 7 establish when an interior operator game is 1-convex in the gap function sense. We have showed that in the operator interior games the most important matter is how to distribute the contributions of the non-atoms players. Condition (c) proves that the 1-convexity condition for these games is equivalent to the fact that the sum of every player losses -when every player decides not cooperate with anyone- be equal to the sum of the benefits contributed by non-atoms players. Condition (d) establishes when an interior operator game is 1-convex in the upper vector sense. Finally, last condition says that an interior operator game is 1-convex if and only if the weights of the controlled players in the structure are zero.

Next we introduce another class of interior operator structures. We will see that any interior operator game defined on these antimatroids is 1-convex.

Definition 9 It is said that an interior operator structure $(N, \mathcal{A})$ is coatomic if $ca(\mathcal{A}) = N$.

Notice that, if an interior operator structure $(N, \mathcal{A})$ is coatomic, all the individual coalitions are feasible in the dual structure by (IC), that is $\overline{\{e\}} = \{e\}$ for each $e \in N$. Actually, in these structures no player controls another one in the sense mentioned in Lemma 2(b).

Theorem 10 *An interior operator structure* $(N, \mathcal{A})$ *is coatomic if and only if for all* $w \in \mathbb{R}^N_+$ *the interior operator game* $(N, w_\mathcal{A})$ *is* 1*-convex.*

Proof If $(N, \mathcal{A})$ is a coatomic interior operator structure, then $\overline{\{e\}} = \{e\}$ for all $e \in N$. So, $M_e^{w_\mathcal{A}} = w(\overline{\{e\}}) = w_e$ for all $e \in N$ and any $w \in \mathbb{R}^N_+$. Therefore, by Theorem 8 and Proposition 7(d), we have that $(N, w_\mathcal{A})$ is 1-convex.

In order to prove the converse implication, suppose that for all $w \in \mathbb{R}^N_+$ the interior operator game $(N, w_\mathcal{A})$ is 1-convex but $(N, \mathcal{A})$ is not coatomic. Then there exists some player $e \in N$ such that $N \setminus \{e\} \notin \mathcal{A}$. Thus it is possible to find a player $e' \in \overline{\{e\}} \setminus \{e\}$. So that player $e' \in N$ verifies $|P_{e'}| \geq 2$, by Lemma 2(b). In these conditions, any interior operator game

$(N, w_{\mathcal{A}})$ defined by a vector $w \in \mathbb{R}_+^N$ such that $w_{e'} > 0$ is not 1-convex because by Theorem 8 that game does not verify the condition (e) in Proposition 7, but it is in contradiction to the hypothesis. □

An example of coatomic operator interior structure, in particular an antimatroid, is the system of the information market games defined by Muto et al. (1986). We introduced a particular case of information market game in (Bilbao et al. 2005). Let N be a set of firms with the firm 1 as a patent holder and let $(r_T)_{T\subseteq N}$ the collection of nonnegative profits in submarkets, the characteristic function is for each $S \subseteq N$, $v(S) = \sum_{\{T\subseteq N: T\cap S\neq\emptyset\}} r_T$ when $1 \in S$ and $v(S) = 0$ if $1 \notin S$. The k-convexity for these games was studied by Driessen (1995). If we take $r_T = 0$ when $|T| \geq 2$ then we obtain the big boss game, and we proposed this case with several patent holder, a set $I \subseteq N$. These information market games are interior operator games with a vector $(r_{\{i\}})_{i\in N}$, a set of informed players I and the function

$$v(S) = \begin{cases} \sum_{i\in S} r_{\{i\}} & \text{if } S\cap I \neq \emptyset, \\ 0 & \text{if } S \cap I = \emptyset, \end{cases}$$

for each coalition S. The family of feasible coalitions

$$\mathcal{A} = \{S \subseteq N\colon\ S \cap I \neq \emptyset\} \cup \{\emptyset\}$$

is a coatomic antimatroid. In particular big-boss games are 1-convex games as Driessen proved in Driessen (1991).

4 k-convex interior operator games, for $k \geq 2$

In the previous section we have analyzed the notion of 1-convexity for interior operator games, but that is a particular case of the following, introduced by Driessen (1988, Chap. VII). Let $k \in \mathbb{N}$. It is said that a cooperative game (N, v) is *k-convex* if the following four conditions hold (see Theorem 2.2, p. 179 in Driessen 1988):

(K1) $g^v(S) \geq g^v(N)$, for all $S \subseteq N$ such that $|S| \geq k$.
(K2) $g^v(N) \geq g^v(S)$, for all $S \subseteq N$ such that $|S| = k-1$.
(K3) $g^v(S\cup\{e\}) - g^v(S) \geq g^v(T\cup\{e\}) - g^v(T)$, for all $S \subseteq T \subseteq N\setminus\{e\}$ such that $|T| \leq k-2$.
(K4) $g^v(S\cup\{e\}) - g^v(S) \geq g^v(N) - g^v(T)$, for all $S \subseteq T \subseteq N\setminus\{e\}$ such that $|T| = k-1$.

Using conditions (K2) and (K4) we obtain next other property:

(K5) $g^v(N) \geq g^v(\{e\})$ for all $e \in N$.

Let $e \in N$. First observe that inequality holds if $k = 2$ due to the condition (K2). So, let us suppose $3 \leq k \leq n-2$. Let T be any coalition with $|T| = k-1$ and $e \in T$. Writing $T = \{e_1, e_2, \ldots, e_{k-1}\}$, with $e_1 = e$, and considering $e' \notin T$ and $S_p = \{e_1, \ldots, e_p\}$ for all $p = 1, \ldots, k-2$, we can apply the condition (K4) to S_p, $T' = T\setminus\{e_{p+1}\}\cup\{e'\}$ and e_{p+1} obtaining

$$g^v(S_{p+1}) - g^v(S_p) \geq g^{w_{\mathcal{A}}}(N) - g^{w_{\mathcal{A}}v}(T') \geq 0,$$

where the last inequality is due to (K2). Therefore $g^{w_{\mathcal{A}}}(S_{p+1}) \geq g^{w_{\mathcal{A}}}(S_p)$ and then $g^{w_{\mathcal{A}}}(T) \geq g^{w_{\mathcal{A}}}(\{e\})$. Again using the condition (K2) on T we now obtain $g^{w_{\mathcal{A}}}(\{e\}) \leq g^{w_{\mathcal{A}}}(N)$.

We now focus our attention on the cases $2 \leq k \leq n-2$ because it is known (Driessen 1988, p. 175) that, for any $k \geq n-1$, a game is k-convex if and only if the game itself is convex.

Proposition 11 *Let $(N, w_{\mathcal{A}})$ an interior operator game. Then the following statements are equivalents*:

(a) $g^{w_{\mathcal{A}}}(S) = 0$ *for all* $S \subseteq N$.
(b) $g^{w_{\mathcal{A}}}(\{e\}) = 0$ *for all* $e \notin a(\mathcal{A})$.
(c) $\sum_{e \in N} g^{w_{\mathcal{A}}}(\{e\}) = 0$.
(d) $M_e^{w_{\mathcal{A}}} = w_e$ *for each* $e \in a(\mathcal{A})$ *and* $M_e^{w_{\mathcal{A}}} = 0$ *otherwise*.
(e) *For all* $e \notin a(\mathcal{A})$ *it holds* $w_e = 0$.

Proof It is trivial that part (a) implies (c). The claim $M_e^{w_{\mathcal{A}}} = g^{w_{\mathcal{A}}}(\{e\}) + w_e$ if $e \in a(\mathcal{A})$ and $M_e^{w_{\mathcal{A}}} = g^{w_{\mathcal{A}}}(\{e\})$ if $e \notin a(\mathcal{A})$ is true by Lemmas 5 and 6. Hence, conditions (c) and (d) are equivalents. It is also trivial that (c) implies (b). If $e \notin a(\mathcal{A})$ and $g^{w_{\mathcal{A}}}(\{e\}) = 0$ then, using Lemma 6(b), $g^{w_{\mathcal{A}}}(\{e\}) = w(\overline{\{e\}}) = 0$ but $e \in \overline{\{e\}}$. We have then that (b) implies (e). In other to obtain (e) from (a) we also use Lemma 6. This lemma proves that the gap function is fully independent from the weights of the atoms and then only depends of the weights of the non-atoms players. □

The next theorem shows that these conditions determine whether or not an interior operator game is k-convex with $2 \leq k \leq n-2$.

Theorem 12 *Let $(N, w_{\mathcal{A}})$ be an interior operator game and let $k \in \mathbb{N}$ such that $2 \leq k \leq n-2$. The game $(N, w_{\mathcal{A}})$ is k-convex if and only if $(N, w_{\mathcal{A}})$ satisfies one of the conditions of the above proposition.*

Proof It is trivial to obtain by definition that if the gap function of a cooperative game is null then this game is k-convex.

To prove the only if part we assume the k-convexity of the interior operator game $(N, w_{\mathcal{A}})$ and will get condition (e) in Proposition 11, that is $w_e = 0$ for all $e \notin a(\mathcal{A})$. The proof contains three parts.

First we suppose $e \in N \setminus a(\mathcal{A})$ with $|P_e| \geq 2$ and such that there exists $e' \in P_e \setminus \{e\}$ with $|\overline{\{e'\}}| \leq n-k$. That is, consider a non-atom player whose benefit is under control of other player that also controls the benefits of $n-k$ players at most. Since $N \setminus \overline{\{e'\}}$ is a feasible coalition which has at least k elements we can apply the condition (K1), $g^{w_{\mathcal{A}}}(N \setminus \overline{\{e'\}}) \geq g^{w_{\mathcal{A}}}(N)$, and Lemma 6(a). We obtain

$$\sum_{e'' \in N \setminus \overline{\{e'\}}} w\big(\overline{\{e''\}} \setminus \{e''\}\big) \geq \sum_{e'' \in N} w\big(\overline{\{e''\}} \setminus \{e''\}\big),$$

and, simplifying, $\sum_{e'' \in \overline{\{e'\}}} w(\overline{\{e''\}} \setminus \{e''\}) \leq 0$. Particularly, $w(\overline{\{e'\}} \setminus \{e'\}) = 0$ and so we conclude $w_e = 0$ using that $e \in \overline{\{e'\}} \setminus \{e'\}$ by Lemma 2(b).

Second we consider $e \in N \setminus a(\mathcal{A})$ with $|P_e| \geq 2$ such that for all $e' \in P_e \setminus \{e\}$ it holds $|\overline{\{e'\}}| \geq n-k+1$. That is, suppose a non-atom player whose benefit is under control of players that control the benefits of $n-k+1$ players at least. If $e' \in P_e \setminus \{e\}$ Lemma 2(b) implies $e \in \overline{\{e'\}} \setminus \{e'\}$. Hence, the coalition $S = N \setminus \{e, e'\}$ contains $n-2$ members, that is at least k players. Moreover, by (IC) and Lemma 2(c) it holds that

$$int(S) = int(N \setminus \{e, e'\}) = N \setminus \overline{\{e, e'\}} = N \setminus \overline{\{e'\}},$$

and then $S \setminus int(S) = \overline{\{e'\}} \setminus \{e, e'\}$. We are going to apply condition (K1) to $N \setminus \{e, e'\}$, $g^{w_{\mathcal{A}}}(N \setminus \{e, e'\}) \geq g^{w_{\mathcal{A}}}(N)$. Lemma 6 holds

$$\begin{aligned} g^{w_{\mathcal{A}}}(N \setminus \{e, e'\}) &= \sum_{e'' \in N \setminus \{e, e'\}} w\big(\overline{\{e''\}} \setminus \{e''\}\big) + w\big(\overline{\{e'\}} \setminus \{e, e'\}\big) \\ &= \sum_{e'' \in N \setminus \{e\}} w\big(\overline{\{e''\}} \setminus \{e''\}\big) - w_e \\ &\geq \sum_{e'' \in N} w\big(\overline{\{e''\}} \setminus \{e''\}\big). \end{aligned}$$

The inequality reduces to $-w_e \geq w(\overline{\{e\}} \setminus \{e\})$, then $w(\overline{\{e\}}) = 0$, and therefore $w_e = 0$.

Finally we take $e \in N \setminus a(\mathcal{A})$ such that $|P_e| = 1$. That is, consider a non-atom player whose benefit is not controlled by other players. Since we have proved that $w_e = 0$ for all $e \in N \setminus a(\mathcal{A})$ such that $|P_e| \geq 2$, it is easy to verify that $w(\overline{\{e'\}} \setminus \{e'\}) = 0$ for all $e' \in N$. In fact, Lemma 2(b) implies that $\overline{\{e'\}} \setminus \{e'\} \subseteq \{e'' \in N\colon |P_{e''}| \geq 2\}$ for all $e' \in N$ and so, $w(\overline{\{e'\}} \setminus \{e'\}) = 0$ for all $e' \in N$. By (K5) $g^{w_{\mathcal{A}}}(\{e\}) \leq g^{w_{\mathcal{A}}}(N)$ and then using Lemma 6

$$w\big(\overline{\{e\}}\big) \leq \sum_{e' \in N} w\big(\overline{\{e'\}} \setminus \{e'\}\big) = 0,$$

and we have $w_e = 0$ since w is non-negative. □

Actually, after the proof of this theorem, to obtain the conditions of the Proposition 11 is only necessary the following property about the gap function of an interior operator game $(N, w_{\mathcal{A}})$:

$$g^{w_{\mathcal{A}}}(S) \geq g^{w_{\mathcal{A}}}(N) \geq g^{w_{\mathcal{A}}}(\{e\}), \quad \text{for all } S \subseteq N \text{ such that } |S| \geq k \text{ and } e \in N.$$

There only exists one interior operator structure where all the interior operator games are k-convex, for any $2 \leq k \leq n-2$, as states the next result.

Theorem 13 *Let $(N, \mathcal{A})$ be an interior operator game, and $2 \leq k \leq n-2$. Then, for all $w \in \mathbb{R}^N_+$ the interior operator game $(N, w_{\mathcal{A}})$ is k-convex if and only if $\mathcal{A} = 2^N$.*

Proof It is known that all additive games are k-convex, and we also know that if $\mathcal{A} = 2^N$ then any game $(N, w_{\mathcal{A}})$ is additive. On the other hand, if $\mathcal{A} \neq 2^N$ there exists a non-atom player $e \in N \setminus a(\mathcal{A})$. So, every interior operator game $(N, w_{\mathcal{A}})$ given by any vector $w \in \mathbb{R}^N_+$ with $w_e > 0$ is not k-convex, when $2 \leq k \leq n-2$, using Theorem 13. □

5 Semiconvex interior operator games

Thinking about the concept of 1-convexity, where the gap function reached the minimum on the grand coalition, Driessen (1988, Sect. 6, Chap. III) introduced the definition of semiconvex games. It is said that a cooperative game (N, v) is *semiconvex* if it holds

$0 \leq g^v(\{e\}) \leq g^v(S)$, for all $e \in N$ and all $S \subseteq N$ such that $e \in S$. It is known that convexity implies semiconvexity (Driessen 1988, Proposition 1.3, p. 115).

Theorem 14 *Let $(N, w_{\mathcal{A}})$ be an interior operator game. Then, the following statements are equivalent*:

(a) $(N, w_{\mathcal{A}})$ *is semiconvex.*
(b) $w_e \leq g^{w_{\mathcal{A}}}(T \setminus \{e\})$ *for all* $e \notin a(\mathcal{A})$ *and every e-path* $T \in A(e)$.
(c) $g^{w_{\mathcal{A}}}(\{e\}) \leq g^{w_{\mathcal{A}}}(T)$, *for all* $e \notin a(\mathcal{A})$ *and every e-path* $T \in A(e)$.
(d) $m_e^{w_{\mathcal{A}}} = w(int\{e\})$ *for all* $e \in N$.

Proof It is obvious that (a) implies (c). Next we will prove that (c) implies (a). Let $e \in N$. If $S \in \mathcal{A}$ is such that $e \in S$, there exists $T \in A(e)$ with $T \subseteq S$. Then, by the monotonicity of the gap function on $\mathcal{A}$ in Lemma 6(a), $g^{w_{\mathcal{A}}}(\{e\}) \leq g^{w_{\mathcal{A}}}(T) \leq g^{w_{\mathcal{A}}}(S)$. If $S \notin \mathcal{A}$ but $e \in int(S)$ we can apply the above result to $int(S)$ and deduce by Lemma 6(b) that $g^{w_{\mathcal{A}}}(\{e\}) \leq g^{w_{\mathcal{A}}}(int(S)) \leq g^{w_{\mathcal{A}}}(S)$. Let now $S \notin \mathcal{A}$ with $e \in S \setminus int(S)$. In this case $e \notin a(\mathcal{A})$ and, therefore, applying Lemma 6(b) to S and the individual coalition $\{e\}$ we have

$$g^{w_{\mathcal{A}}}(\{e\}) = w\big(\overline{\{e\}}\big) \leq g^{w_{\mathcal{A}}}(int(S)) + \sum_{e' \in S \setminus int(S)} w\big(\overline{\{e'\}}\big) = g^{w_{\mathcal{A}}}(S).$$

Second, we prove that (b) and (c) are equivalent. Let $e \notin a(\mathcal{A})$ and $T \in A(e)$. If $e \notin ca(\mathcal{A})$ we obtain that equivalence directly because, by using Lemma 6, $g^{w_{\mathcal{A}}}(\{e\}) \leq g^{w_{\mathcal{A}}}(T)$ holds if and only if

$$w\big(\overline{\{e\}}\big) \leq \sum_{e' \in T} w\big(\overline{\{e'\}} \setminus \{e'\}\big), \qquad w_e \leq \sum_{e' \in T \setminus \{e\}} w\big(\overline{\{e'\}} \setminus \{e'\}\big).$$

Now take $e \in ca(\mathcal{A})$, $\overline{\{e\}} = \{e\}$. In this case we affirm that $g^{w_{\mathcal{A}}}(T) = g^{w_{\mathcal{A}}}(T \setminus \{e\})$ because we use Lemma 6(a) and $w(\overline{\{e\}} \setminus \{e\}) = 0$. Therefore $g^{w_{\mathcal{A}}}(\{e\}) \leq g^{w_{\mathcal{A}}}(T)$ is equivalent to $g^{w_{\mathcal{A}}}(\{e\}) \leq g^{w_{\mathcal{A}}}(T \setminus \{e\})$ and Lemma 6 implies that

$$w_e \leq \sum_{e' \in T \setminus \{e\}} w\big(\overline{\{e'\}} \setminus \{e'\}\big) = g^{w_{\mathcal{A}}}(T \setminus \{e\}),$$

because $\overline{\{e\}} = \{e\}$ and $T \setminus \{e\} \in \mathcal{A}$.

Finally, the equivalence between (a) and (d). The game is semiconvex if and only if for each player $e \in N$

$$\min\{g^{w_{\mathcal{A}}}(S) : S \subseteq N,\ e \in S\} = g^{w_{\mathcal{A}}}(\{e\}),$$

if and only if, using Lemmas 5 and 6,

$$m_e^{w_{\mathcal{A}}} = w\big(\overline{\{e\}}\big) - g^{w_{\mathcal{A}}}(\{e\}) = \left\{ \begin{matrix} w_e & \text{if } e \in a(\mathcal{A}), \\ 0 & \text{otherwise.} \end{matrix} \right\} = w(int\{e\})$$

for all $e \in N$. □

We introduce the following family of interior operator structures.

Definition 15 An interior operator structure $(N, \mathcal{A})$ is said a control interior operator structure if for every $e \in N \setminus a(\mathcal{A})$ it holds $|P_e| \geq 2$.

In an interior operator structure we can observe two classes of non-atom players: the *controlled players* (that is, players $e \in N \setminus a(\mathcal{A})$ with $|P_e| \geq 2$) and the noncontrollable *players* (that is, players $e \in N \setminus a(\mathcal{A})$ with $|P_e| = 1$). In a control interior operator structure each player either he is an atom or he is a controlled player. Control interior operator structures and coatomic interior operator structures are dual in the following sense: in the first there are not controlled players and in the second one there are not noncontrollable players. In particular poset structures are control interior operator structures, but there are control interior operator structures which are not poset. This last situation happens, for instance, if we take the antimatroid $N = \{1, 2, 3, 4\}$ and $\mathcal{A} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, N\}$.

Now, we will find the family of interior operator structures where every interior operator game is a semiconvex game.

Theorem 16 *An interior operator structure* $(N, \mathcal{A})$ *is a control interior operator structure if and only if for all* $w \in \mathbb{R}_+^N$ *the interior operator game* $(N, w_{\mathcal{A}})$ *is semiconvex.*

Proof Let $(N, \mathcal{A})$ be a control interior operator structure, $w \in \mathbb{R}_+^N$, $e \notin a(\mathcal{A})$ and $T \in A(e)$. Then, there exists $e' \in P_e \setminus \{e\}$ or equivalently there exists e' such that $e \in \overline{\{e'\}} \setminus \{e'\}$ and $e' \in T$. So,

$$w_e \leq w\big(\overline{\{e'\}} \setminus \{e'\}\big) \leq \sum_{e'' \in T \setminus e} w\big(\overline{\{e''\}} \setminus \{e''\}\big),$$

and we conclude that $(N, w_{\mathcal{A}})$ is a semiconvex game by Theorem 14.

On the other hand, if $(N, \mathcal{A})$ is not a control antimatroid then there exists a player $e' \notin a(\mathcal{A})$ such that $P_{e'} = \{e'\}$. That is, e' does not belong to the closure of any other player. So, taking the vector $w \in \mathbb{R}_+^N$ defined by

$$w_e = \begin{cases} 1 & \text{if } e = e', \\ 0 & \text{if } e \neq e', \end{cases}$$

we obtain the following conclusion

$$w_{e'} = 1 > 0 = \sum_{e \in T \setminus e'} w\big(\overline{\{e\}} \setminus \{e\}\big).$$

Thus, Theorem 14 implies that $(N, w_{\mathcal{A}})$ is not a semiconvex game. □

Conclusions

We now analyze the relationships among the different concepts of convexity which we have seen before for the interior operator games.

Fixed an interior operator structure $(N, \mathcal{A})$, we use the next notation to denote some games defined on it: $\mathcal{C}_{(N,\mathcal{A})}$ is the set of convex interior operator games, $\mathcal{C}^k_{(N,\mathcal{A})}$ for $1 \leq k \leq n-1$ is the set of k-convex interior operator games, and $\mathcal{C}^s_{(N,\mathcal{A})}$ is the set of semiconvex interior operator games. It is known that $\mathcal{C}_{(N,\mathcal{A})} = \mathcal{C}^n_{(N,\mathcal{A})} = \mathcal{C}^{n-1}_{(N,\mathcal{A})}$ and $\mathcal{C}_{(N,\mathcal{A})} \subset \mathcal{C}^s_{(N,\mathcal{A})}$. In general $\mathcal{C}_{(N,\mathcal{A})} \neq \mathcal{C}^s_{(N,\mathcal{A})}$ because there are semiconvex games $(N, w_{\mathcal{A}})$ which are not convex. In fact, by Theorems 4 and 16 we know that if $(N, \mathcal{A})$ is a control interior operator structure which is not poset structure, then every interior operator game is semiconvex but it is possible to select one non convex.

According to Theorem 12 we have $\mathcal{C}^2_{(N,\mathcal{A})} = \mathcal{C}^3_{(N,\mathcal{A})} = \cdots = \mathcal{C}^{n-2}_{(N,\mathcal{A})}$. In addition, this same theorem guarantees that if $(N, w_{\mathcal{A}}) \in \mathcal{C}^k_{(N,\mathcal{A})}$ for any k such that $2 \le k \le n-2$, then $w_e = 0$ for all $e \in N \setminus a(\mathcal{A})$. So, for any k such that $2 \le k \le n-2$, in view of above results it is deduced:

1. $\mathcal{C}^k_{(N,\mathcal{A})} \subset \mathcal{C}_{(N,\mathcal{A})}$, since every player $e \in N$ with $|A(e)| \ge 2$ is non-atom (see Theorems 3 and 12). Furthermore, by Theorems 4 and 13 we know that the inclusion is strict because there are poset structures different to 2^N.
2. $\mathcal{C}^k_{(N,\mathcal{A})} \subset \mathcal{C}^1_{(N,\mathcal{A})}$, since every player $e \in N$ with $|P_e| \ge 2$ is non-atom (see Theorems 8 and 12). Furthermore, by Theorems 10 and 13 we know that the inclusion is strict because there are coatomic interior operator structures different to 2^N.
3. $\mathcal{C}^k_{(N,\mathcal{A})} \subset \mathcal{C}^s_{(N,\mathcal{A})}$, since every player $e \in N$ such that exists $e' \in N \setminus \{e\}$ with $e \in \overline{\{e'\}}$ is non-atom and then

$$0 = w_e = \sum_{e' \in T \setminus \{e\}} w\big(\overline{\{e'\}} \setminus \{e'\}\big),$$

(see Theorems 12 and 14). Furthermore, by Theorems 13 and 16 we know that the inclusion is strict because there are control interior operator structures different to 2^N.

We now check that no relation of inclusion is verified between the sets $\mathcal{C}^s_{(N,\mathcal{A})}$ and $\mathcal{C}^1_{(N,\mathcal{A})}$. Theorems 13 and 16, and the fact that there are coatomic interior operator structures which are not control interior operator structures, as for example the antimatroid where $N = \{1, 2, 3\}$ and

$$\mathcal{A} = \big\{\{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, N\big\},$$

implies that $\mathcal{C}^s_{(N,\mathcal{A})} \not\supseteq \mathcal{C}^1_{(N,\mathcal{A})}$.

The same theorems and the fact that there are control interior operator structures which are not coatomic interior operator structures, as for example the antimatroid given by $N = \{1, 2, 3, 4\}$ and

$$\mathcal{A} = \big\{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, N\big\},$$

assures that $\mathcal{C}^s_{(N,\mathcal{A})} \not\subseteq \mathcal{C}^1_{(N,\mathcal{A})}$.

Finally, notice that $\mathcal{C}^1_{(N,\mathcal{A})} \cap \mathcal{C}^s_{(N,\mathcal{A})} = \mathcal{C}^k_{(N,\mathcal{A})}$ for every k such that $2 \le k \le n-2$. In fact, it is enough to prove the inclusion $\mathcal{C}^1_{(N,\mathcal{A})} \cap \mathcal{C}^s_{(N,\mathcal{A})} \subseteq \mathcal{C}^k_{(N,\mathcal{A})}$. Let $(N, w_{\mathcal{A}})$ be a semiconvex and 1-convex interior operator game. By Theorem 8 it follows that $g^{w_{\mathcal{A}}}(S) = 0$ for all $S \in \mathcal{A}$. In particular, for all $e \in N$, if $T \in A(e)$ it holds $\sum_{e' \in T \setminus \{e\}} w(\overline{\{e'\}} \setminus \{e'\}) = 0$. Furthermore, by Theorem 14 we have that $w_e \le \sum_{e' \in T \setminus \{e\}} w(\overline{\{e'\}} \setminus \{e'\})$ for all $e \in N \setminus a(\mathcal{A})$ and every $T \in A(e)$. Therefore, $w_e = 0$ for every $e \in N \setminus a(\mathcal{A})$ and by using Theorem 12 we deduce that the game $(N, w_{\mathcal{A}})$ is k-convex.

Acknowledgements This research has been partially supported by the Spanish Ministry of Education and Science and the European Regional Development Fund, under grant SEJ2006-00706, and by the FQM237 grant of the Andalusian Government. The authors are grateful for the constructive and detailed comments and suggestions made by an anonymous referee which lead to a considerable improvement of our paper.

References

Algaba, E., Bilbao, J. M., van den Brink, R., & Jiménez-Losada, A. (2004). Cooperative games on antimatroids. *Discrete Mathematics*, *282*, 1–15.

Bilbao, J. M., Jiménez-Losada, A., Lebrón, E., & Chacón, C. (2005). Values for interior operator games. *Annals of Operations Research*, *137*, 141–160.

Brânzei, R., Fragnelli, V., & Tijs, S. (2002). Tree-connected peer group situations and peer group games. *Mathematical Methods of Operations Research*, *55*, 93–106.

Dilworth, R. P. (1940). Lattices with unique irreducible decompositions. *Annals of Mathematics*, *41*, 771–777.

Driessen, T. S. H. (1988). *Cooperative games. Solutions and applications*. Dordrecht: Kluwer Academic.

Driessen, T. S. H. (1991). k-Convexity of big boss games and clan games. *Methods of Operations Research*, *64*, 267–275.

Driessen, T. S. H. (1995). A bankruptcy problem and an information trading problem: applications to k-convex games. *ZOR—Mathematical Methods and Models of Operations Research*, *41*, 313–324.

Gillies, D. B. (1953). *Some theorems on n-person games*. Ph.D. thesis, Princeton University Press, Princeton.

Gilles, R. P., Owen, G., & van den Brink, R. (1992). Games with permission structures: the conjunctive approach. *International Journal of Game Theory*, *20*, 277–293.

Goecke, O., Korte, B., & Lovász, L. (1986). Examples and algorithmic properties of greedoids. In B. Simeone (Ed.), *Combinatorial optimization*. Berlin: Springer.

Jiménez-Losada, A. (1998). *Valores para juegos sobre estructuras combinatorias*. Ph.D. thesis, University of Seville, Spain.

Korte, B., Lóvasz, L., & Schrader, R. (1991). *Greedoids*. Berlin: Springer.

Muto, S., Potters, J., & Tijs, S. (1986). Information market games. *International Journal of Game Theory*, *18*, 209–226.

Muto, S., Nakayama, M., Potters, J., & Tijs, S. (1987). On big boss games. *Economic Studies Quarterly*, *39*, 303–321.

Tijs, S. (1981). Bounds for the core and the τ-value. In O. Moeschlin & D. Pallaschke (Eds.), *Game theory and mathematical economics* (pp. 123–132). Amsterdam: North-Holland.